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EXCITATION OF NUCLEAR ROTATIONAL STATES BY THE ELECTRIC FIELD OF IMPINGING PARTICLES

BY

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1. Introduction.

When heavy nuclei are bombarded with charged particles with an energy appreciably below the Coulomb barrier the short range nuclear forces cannot operate. The collisions may, however, still give rise to nuclear excitations produced by the electric field of the impinging particles. Such reactions are especially simple to interpret, and the cross-sections can be expressed in terms of the same nuclear properties that determine the transition probabilities for electromagnetic radiative processes.*†§

It has been suggested** that the method of Coulomb excitation should provide a powerful tool for the study of nuclear collective properties. According to the nuclear model which describes the dynamics of the nucleus in terms of the coupled motion of individual particles and surface oscillations***, low-lying excited states of collective type are expected for nuclei possessing large deformations. These states exhibit a rotational spectrum, and should reveal themselves by their especially large cross-sections for Coulomb excitation.

In order to test some of these predictions we have undertaken an investigation of the Coulomb excitation effects produced

* The mechanism of nuclear excitation by the Coulomb field of bombarding particles has been discussed by several authors: Weisskopf (1938); Ramsey (1951); Mullin and Guth (1951); Huby and Newns (1951); Breit, Hull, and Gluckstern (1952); Ter-Martirosyan (1952); Bohr and Mottelson (1953).

† Nuclear excitations with cross-sections too large to be explained by barrier penetration have been observed on a number of occasions, and the possibility of attributing the effects to Coulomb excitation has been discussed. Cf., e. g., Barnes and Aradine (1939); Risser, Lark-Horowitz and Smith (1940).

§ While the present work was being prepared for publication, we have learned that γ-rays resulting from Coulomb excitation of heavy nuclei have recently been observed by C. L. McClelland and C. Goodman. We are indebted to Professor Goodman for sending us a manuscript of their work in advance of publication.

** A. Bohr and B. Mottelson. (Cf., e. g., Report of the Internatiєnal Physics Conference, Copenhagen, June 1952).

*** A. Bohr (1952); Bohr and Mottelson (1953); the latter paper will be referred to in the following as B.-M.

by the protons from a 2 MeV electrostatic accelerator*. We here report the results obtained by the bombardment of tantalum and tungsten.

In § 2, a brief summary is given of some of the relevant aspects of the theory of Coulomb excitation. A description of the nuclear rotational spectrum appears in § 3. The experimental arrangement is described in § 4. In § 5, the excitation cross-sections for the first excited state in Ta^{181} are given and compared with the theory of Coulomb excitation; the nuclear data obtained are discussed in relation to the theory of rotational states. The excitation of the second rotational state in Ta is described in § 6. In § 7 the Coulomb excitation of the first excited states of the even-A isotopes of W is reported. The yields of the characteristic K X-rays, which are also excited by the protons, are discussed in § 8. A survey of main conclusions is contained in § 9.

2. Theory of Coulomb Excitation**.

When the energy of the bombarding particles is low enough to exclude penetration through the Coulomb barrier, the parameter

$$\varkappa = 2\,\frac{Z_1 Z_2 e^2}{\hbar v} \tag{1}$$

is large compared to unity and the trajectory may be described by means of classical mechanics (N. Bohr, 1948). In expression (1), the charge numbers of the projectile and the target nucleus are denoted by Z_1 and Z_2, respectively, while v is the relative velocity.

* In previous experiments performed with the electrostatic accelerator of the California Institute of Technology (Day and Huus, 1952), a strong γ-ray had been observed to arise from the Ta target backing. The present experiments were undertaken after the recognition that this γ-ray of 137 keV resulted from Coulomb excitation.

** The present formulation is based on the work of Ter-Martirosyan (1952). Cf. also B.-M., Appendix VI, who, in connection with a review of this work, have especially discussed the relationship between Coulomb excitation and electromagnetic transitions, and the applications to the study of nuclear collective properties. We here follow the presentation given in the latter reference.

The effect of the projectile on the target nucleus can then be described in terms of the time varying electric potential, given by

$$V(t) = \sum_{p=1}^{Z_2} \frac{Z_1 e^2}{|\vec{r}_p - \vec{r}(t)|}, \tag{2}$$

where $\vec{r}_p$ are the coordinates of the protons in the target nucleus, and where $\vec{r}(t)$ gives the classical trajectory of the projectile considered as a point charge. Since the probability for exciting the nucleus in any single collision is small, the excitation process may be treated by quantum-mechanical perturbation theory.

The collective nuclear excitations are produced by the electric quadrupole component of (2) and, for the total cross-section for excitation of a given level, one obtains

$$\sigma = \frac{2\pi^2}{25} \frac{1}{Z_2^2 e^2} \left(\frac{mv}{\hbar}\right)^2 B_e(2)\, g_2(\xi), \tag{3}$$

where m is the reduced mass. The quantity $B_e(2)$ is a constant containing the nuclear matrix element; this quantity also determines the electric quadrupole ($E2$) decay probability for the inverse transition. Theoretical values of B for rotational excitations are quoted in the following paragraph. The last factor $g_2(\xi)$ can be expressed in terms of integrals over the trajectories of the bombarding particles and depends on the parameter

$$\xi = \frac{\Delta E}{2E} \frac{Z_1 Z_2 e^2}{\hbar v}, \tag{4}$$

where E is the bombarding energy and ΔE the nuclear excitation energy. The parameter ξ thus represents the ratio of the collision time to the nuclear period; for small ξ, the function g_2 approaches the constant value 1.13 while, for large ξ, the collisions become adiabatic, resulting in an exponentially decreasing g_2. The function $g_2(\xi)$ has been evaluated numerically (cf. forthcoming publication by A. WINTHER, whose results for g_2 are also given in B.-M., Appendix VI).

The angular distribution of the emitted radiation can also be expressed in terms of integrals over the trajectories, which have been calculated numerically (cf. forthcoming publication by K. ALDER and A. WINTHER).

3. Nuclear Rotational States.

The coupled particle-surface model predicts the occurrence of low-lying collective excitations for the strongly deformed nuclei encountered in regions removed from closed shells (B.-M., Chapter VI). These states exhibit a spectrum of rotational character, and also reveal themselves by their very large $E2$ transition probabilities.

In even-even nuclei, the spectrum is given by

$$E = \frac{\hbar^2}{2\Im} I(I+1) \qquad \underset{\text{even parity}}{I = 0, 2, 4 \cdots} \tag{5}$$

where the effective moment of inertia $\Im$ is proportional to the square of the nuclear deformation, and is expected to vary slowly with the atomic number A. States of this type have recently been identified by the regularity of the spectrum, the systematic dependence of the energies on A, and the lifetimes which are often more than a hundred times shorter than expected for single-particle transitions (Bohr and Mottelson, 1952, 1953a, 1953b; cf. also Ford, 1953 and Asaro and Perlman, 1953).

In odd-A nuclei, the rotational spectrum is given by

$$E = \frac{\hbar^2}{2\Im}(I(I+1) - I_0(I_0+1)) \qquad \underset{\text{same parity as ground state}}{I = I_0, I_0+1, I_0+2 \cdots} \tag{6}$$

where I_0 is the ground state spin*. Since $\Im$ is expected in general to vary slowly with A, the rotational excitation energies in odd-A nuclei can be related to those in even-even nuclei.

Due to the very large $E2$ transition probabilities for these rotational states, the method of Coulomb excitation is especially suited for their identification. The excitation cross-section (3) depends on the reduced transition probability $B_e(2)$ which for the rotational excitations $I_0 \rightarrow I_0 + 1$ and $I_0 \rightarrow I_0 + 2$ is given by (B.-M., § VIIc.ii and Appendix VI)

* An additional term in the rotational energy, resulting in a less regular spectrum, occurs when the angular momentum of the particles along the nuclear axis equals $1/2\,\hbar$ (cf. B.-M., § VIc.iii).

$$B_e(2) = \frac{15}{16\pi} e^2 Q_0^2 \frac{I_0}{(I_0+1)(I_0+2)} \qquad I_0 \to I_0+1 \tag{7}$$

and

$$B_e(2) = \frac{15}{8\pi} e^2 Q_0^2 \frac{1}{(2I_0+3)(I_0+2)} \qquad I_0 \to I_0+2. \tag{8}$$

The latter formula also refers to the excitation of the $(2+)$ first excited state in even-even nuclei.

The transition probabilities (7) and (8) are expressed in terms of the intrinsic nuclear quadrupole moment Q_0, measured with respect to the nuclear axis (B.-M., Chapter V). The spectroscopically measured quadrupole moment Q is related to Q_0 by

$$Q = \frac{I_0}{I_0+1} \frac{2I_0-1}{2I_0+3} Q_0. \tag{9}$$

Thus, the measurement of the excitation cross-sections, just as of the corresponding $E2$ decay probability, provides a measure of the nuclear deformation which can be directly compared with the spectroscopic data (cf. B.-M., Table XXVII).

The rotational character of the collective excitation spectrum represents a limiting situation realized when the zero-point oscillations of the nuclear surface are negligible compared with the total deformation. These zero-point oscillations give rise to deviations from the expressions given in this section. In the region of closed shells, where the deformations are small, an entirely different spectrum results (B.-M., § VIc.i and ii).

4. Experimental Arrangement.

The 2 MeV electrostatic generator was used to produce a separated beam of protons, which passed through the system of stops shown in Fig. 1. The guard ring (G in the figure) was kept at a negative voltage of 100 volts in order to avoid the influence of secondary electrons on the current measurements.

The target holder was mounted in a horizontal tube in such a way that a number of different targets could be inserted into

the beam and set at an arbitrary angle. An aluminum target holder and target tube were employed to make the absorption small. Aluminum, however, radiates strongly when bombarded

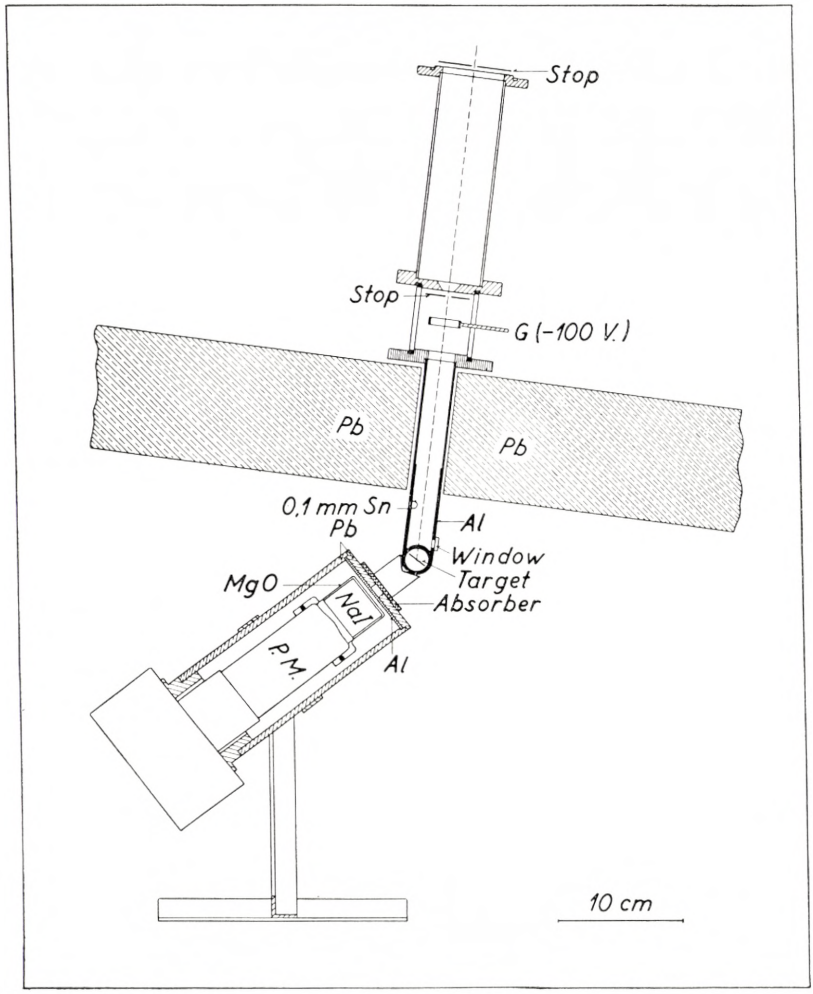

Fig. 1. Experimental arrangement.

with protons, and the target tube was, therefore, lined with a 0.1 mm tin foil in order to reduce the effect from the scattered protons. Tin is good for this purpose because its closed proton shell structure implies a low cross-section for Coulomb excitation; moreover, its nuclear charge is great enough that capture radia-

tion is unimportant and small enough that the characteristic X-radiation does not interfere with the present experiments.

A 10 cm thick layer of lead, 50 cm × 50 cm, was placed above the counter in order to shield it from the hard X-rays from the acceleration tube. These X-rays would produce a background of soft, secondary radiation. The counter was further mounted in a 5 mm lead tube to absorb the scattered radiation. On the top of this tube various absorbers could be placed for the purpose of filtering the radiation from the target. The distance between the counter and the target tube could be fixed by means of two lucite rods when measuring angular distributions.

The γ-rays were detected by a cylindrical NaI (Tl) crystal, 4 cm in diameter and 3 cm high. The crystal was mounted on a RCA 5819 photo multiplier tube by means of a glass cap which was filled with Nujol mineral oil and covered on the outside with a layer of MgO.

The pulse spectrum was measured with a single-channel analyzer. The linearity of the equipment was checked by a relay-pulser connected to the preamplifier, and energy calibrations were made by means of the annihilation radiation from Na^{22} and by means of the characteristic K X-rays emitted by the bombarded targets.

5. Excitation of First State in Ta.

For a detailed study of the Coulomb excitation process, Ta is a particularly suited target, since it is known to have a large nuclear quadrupole moment, and occurs in a region of the elements where the trends of the rotational states seem well established. Moreover, Ta possesses only a single isotope ($_{73}Ta^{181}$).

A piece of Ta metal of 0.1 mm thickness (i. e. thick to protons of 2 MeV) was used as a target. The γ-ray spectrum shown in Fig. 2a was obtained with 1.75 MeV protons; the measurements were made at an angle of 80° with respect to the beam. The strong peak is due to the K X-radiation from the Ta atoms, which results mainly from the ejection of K-electrons by the bombarding particles (cf. CHADWICK, 1913; BOTHE and FRÄNZ,

1928; HENNEBERG, 1933 LIVINGSTON, GENEVESE and KONOPINSKI, 1937), and partly also from the internal conversion of the nuclear excitation. The weak peak on the low energy side corresponds to the K-radiation from the Sn-lining of the target tube.

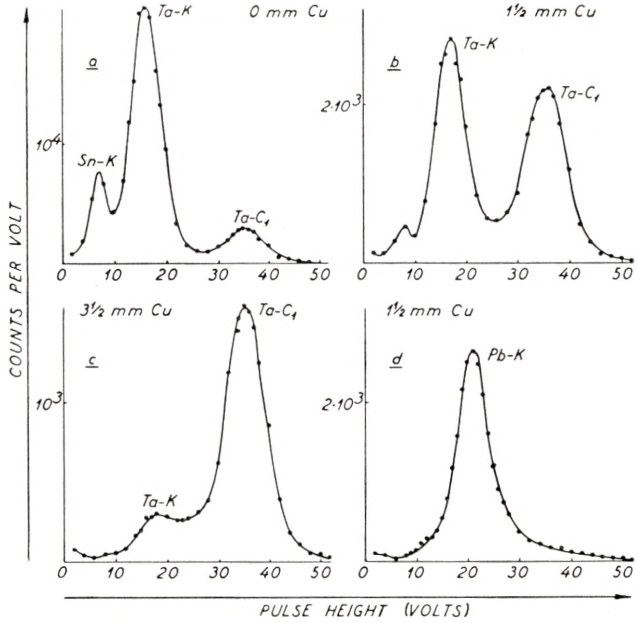

Fig. 2. Pulse spectra measured at a proton energy of 1.75 MeV. The Ta-C_1 peak is due to Coulomb excitation of the first excited state of Ta, and the K-peaks to the characteristic X-radiation from the atoms. Figs. *a*, *b*, and *c* show the spectra which are obtained for tantalum when various absorbers are used; Fig. *d* shows the spectrum for lead with a 1.5 mm Cu absorber.

The high energy peak (C_1 in the figure) is due to the Coulomb excitation of the Ta-nuclei. The relative intensity of the C_1-peak with respect to the Ta K-peak can be increased by the insertion of absorbers, since the absorption coefficients are strongly energy dependent in this region. This is illustrated in Figs. 2 *a*, *b* and *c*. It was convenient, for most of the quantitative measurements, to use an absorber of 1.5 mm Cu, which makes the two peaks about equally strong.

The nuclear origin of the C_1-peak was checked by measurements on Pb (Fig. 2 *d*), which has a similar X-ray spectrum to that of Ta, but whose closed-shell nuclear structure implies high

excitation energies and correspondingly small cross-sections for Coulomb excitation under the present conditions.

The cross-section for formation of a compound nucleus in the bombardment of Ta by 1.75 MeV protons is expected to be of the order of 10^{-38} cm^2 (cf., e.g., BLATT and WEISSKOPF, 1952, p. 352), which is many orders of magnitude too small to explain the observed yield. It is therefore strongly indicated that the excitations result from the influence of the electric field of the protons.

The energy of the C_1-peak was found to be 135 ± 5 keV from the pulse size. An independent determination of the energy was made by absorption measurements which yielded 140 ± 7 keV, and also confirmed the monochromatic character of the radiation. As an average, we adopt the value 137 ± 5 keV for the energy of the C_1-line. This value is in good agreement with the value of 136 keV found for the first excited level of $_{73}Ta^{181}$ in other experiments (cf. GOLDHABER and HILL, 1952).

The angular distribution of the C_1-radiation was measured at bombarding energies of 1.25 MeV and 1.75 MeV in the region from 0° to 80°. When corrections were made for the absorption in the target material, approximately isotropic distributions were found, in agreement with theoretical expectations for the transitions in question.

The thick target yield for the C_1-radiation was measured at 80° with respect to the beam for bombarding energies from 1 to 2.2 MeV. The total yield of γ-quanta per proton, obtained by assuming isotropic distribution, is shown in Fig. 3. The solid curve represents the theoretical energy dependence for $E2$ Coulomb excitation obtained from (3), using the numerically calculated g_2 function. Since the energy loss ΔE in the nuclear excitation process is not quite negligible compared with the bombarding energies used, the theoretical curve has been calculated for an effective energy equal to the incident energy minus $1/2\ \Delta E$. The stopping power for Ta is taken from the semi-empirical relation given by LINDHARD and SCHARFF (1952).

The close agreement between the energy dependence of the experimental yield of the C_1-radiation and that given by the theory confirms the interpretation of the observed effects in terms of Coulomb excitation by the electric quadrupole field of the protons.

For a bombarding energy of 2 MeV, a cross-section for γ-emission of approximately 0.5 millibarns (mb) was obtained. In order to derive the total excitation cross-section, it is necessary to take into account the de-excitation by internal conversion. The decay following the $E2$ excitation is expected to be $E2$ or $M1$, both of which yield total conversion coefficients of about 2 (Rose et al., 1951; Goldhaber and Sunyar, 1951), which is also consistent with the experimentally measured electron yield (Fan,

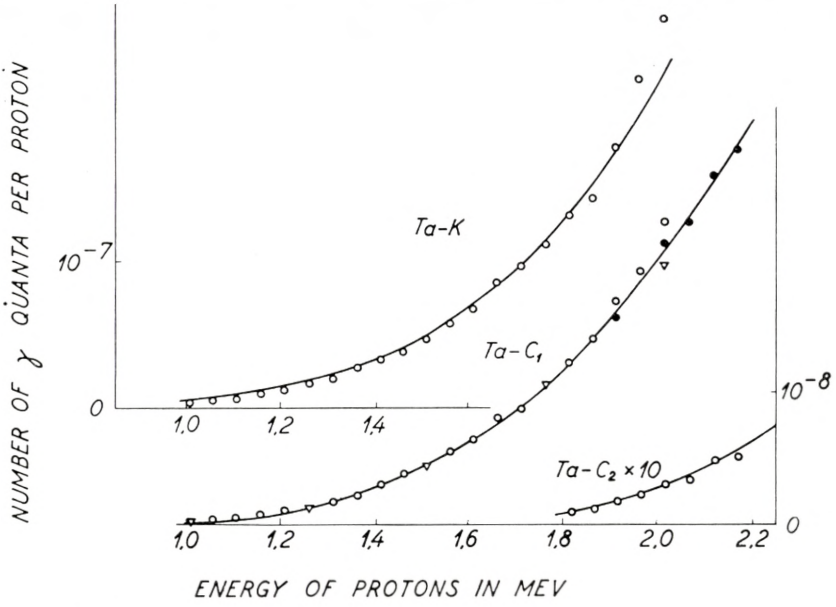

Fig. 3. Total yields for the Ta-K, Ta-C_1, and Ta-C_2 peaks. The theoretical curves are adjusted to fit the measured yields at a proton energy of about 1.8 MeV.

1952). Employing the value 2 for the total conversion coefficient, an excitation cross-section of approximately 1.6 mb is found for protons of 2 MeV.

From the measured cross-section, one derives from (3) the reduced transition probability $B_e(2)$, and the value obtained is more than a hundred times that expected for the excitation of a single particle in the nucleus (cf. B.-M. § VIIb.i for estimates of $B_e(2)$ for single-particle transitions). Thus, the observed large cross-section directly indicates a collective excitation and suggests a rotational interpretation.

The rotational spectrum (6) for an odd-A nucleus depends on the ground state spin, which for $_{73}Ta^{181}$ is known to be 7/2 (cf. MACK, 1950). The first rotational excitation should thus have $I = 9/2$ and an energy of $9\hbar^2/2\Im$ which is 3/2 times the rotational energy expected for the first excited (2+) state in an even-even nucleus with the same value of $\Im$ (cf. (5)). The energy of 93 keV for the first excited state in $_{72}Hf^{180}$ (cf. SCHARFF-GOLDHABER, 1953) thus implies an energy of about 140 keV for the 9/2 state in Ta, in good agreement with the 137 keV γ-ray produced by Coulomb excitation*.

Assuming the rotational interpretation of the 137 keV state, one derives from the empirically determined $B_e(2)$, by means of (7), the intrinsic nuclear quadrupole moment $Q_0 \simeq 7 \times 10^{-24}\,\mathrm{cm}^2$. This may be compared with the value $Q_0 \simeq 14 \times 10^{-24}\,\mathrm{cm}^2$. derived by means of (9) from the quadrupole moment Q determined from the atomic hyperfine structure (BROWN and TOMBOULIAN, 1952). Another estimate of deformations for nuclei in this region may be obtained from measured lifetimes of rotational states in even-even nuclei, which yield $Q_0 \simeq 8 \times 10^{-24}\,\mathrm{cm}^2$ (cf. B.-M., Table XXVII). The three estimates of Q_0 are of the same order of magnitude; it does not seem excluded that the differences may be attributed to experimental uncertainties.

6. Excitation of Second Rotational State in Ta.

The rotational interpretation of the first excited state in Ta implies the existence of a second rotational state with spin 11/2, and an energy of 20/9 times that of the first state. Since this state may also be excited in an $E2$ transition of collective type, it is expected to have an appreciable cross-section for Coulomb excitation. However, the yield of the higher energy γ-ray is expected to be considerably smaller than that for the 137 keV γ-line for several reasons: partly, the value of $B_e(2)$ for the former transition is about four times smaller than for the latter (cf. (7) and

* A level scheme for Ta^{181} has been given (GOLDHABER and HILL, 1952) according to which the first excited state of 136 keV is assigned a spin of 7/2. However, it does not seem inconsistent with the data to identify this level with the one observed in the Coulomb excitation process, and to assign it a spin of 9/2.

(8)); partly, the higher value of ξ implies smaller values of $g_2(\xi)$; finally, the de-excitation of the higher state may proceed either by a cascade through the 137 keV state or by a cross-over transition to the ground state, of which only the latter could be detected with the available resolution.

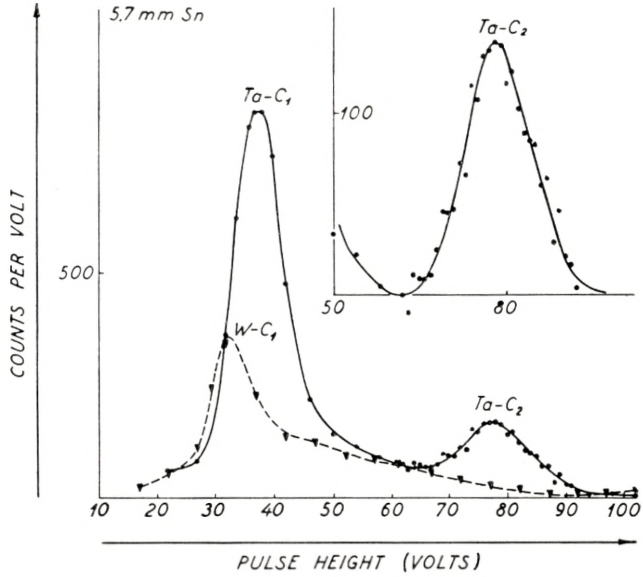

Fig. 4. Pulse spectra for tantalum and tungsten, measured at a proton energy of 2.1 MeV and with a 5.7 mm Sn-absorber. In the insert, the difference between the yields for Ta and W is plotted in the region of the Ta-C_2 line, which corresponds to the second excited state in Ta. No other γ-lines were found in Ta.

The importance of this level as a test of the rotational spectrum led us to undertake a careful search for higher energy γ-rays. In order to determine the background radiation, measurements were also made with targets of the neighbouring element W, since, in the relevant energy range, W is not expected to give rise to appreciable γ-radiation resulting from Coulomb excitation (cf. the following paragraph).

The comparison between the γ-spectra obtained with Ta and W targets, for a proton energy of 2.1 MeV, is shown in Fig. 4 and clearly reveals a higher energy line in Ta (the Ta-C_2 peak).

In order to exclude the possibility of effects arising from the coincidence of two C_1-quanta or from the distortion of the back-

ground due to absorbers, the spectrum was measured with a number of different strong absorbers. The interpretation of the yield from W-targets as representing the background for higher energies was further supported by the observation that the Ta and W yields in all cases coincided at a point somewhat above the Ta-C_1 peak.

Subtracting the background from the observed Ta-spectrum, the Ta-C_2 peak shown in the insert to Fig. 4 was obtained. The energy of the Ta-C_2 line was determined to be 300 ± 10 keV, which corresponds to a ratio of 2.2 for the C_2 and C_1 energies. This ratio is in very good agreement with the ratio 20/9 predicted for the rotational spectrum.

The observed yield for the C_2-line, which is plotted in Fig. 3, also agrees with the theoretical energy-dependence for $E2$ Coulomb excitation, shown by the solid line. The cross-section for the emission of the 300 keV γ-ray was found to be about 0.02 mb for a proton energy of 2 MeV.

From (3) and (8), assuming the value of Q_0 deduced from the excitation cross-section of the 137 keV line, one calculates a cross-section of 0.10 mb for the excitation of the 300 keV level. The conversion coefficient for the 300 keV γ-line is expected to be only of the order 0.05—0.10, since this transition is of $E2$ type. The comparison between the excitation cross-section and the γ-yield therefore indicates that about 20 % of the de-excitations take place via the cross-over transition to the ground state.

The relatively large probability for cross-over transitions is characteristic of the rotational spectrum, arising from the strong enhancement of the $E2$ radiation (cf. B.-M. § VII c.ii).

From the branching ratio one can obtain an estimate of the $M1$ transition probability for the $\Delta I = 1$ radiative transition within the rotational family. This transition probability can also be estimated from the static magnetic moment of the ground state (B.-M. § VII c.ii). The value obtained in this way is somewhat smaller than that indicated by the branching ratio, but the estimate is very sensitive to the value of the magnetic moment. A precision measurement of this moment would thus be of interest.

7. Coulomb Excitation of W.

In even-even nuclei with large deformations, the first excited (2+) states should show up strongly in the Coulomb excitation process. As a first example, $_{74}W$ was selected since it consists predominantly of even isotopes, and since the excitation energies in this region indicated that it would be possible to resolve the nuclear radiation from the K X-rays.

The W-spectrum is shown in Fig. 5 and clearly exhibits the well separated peaks resulting from the X-radiation (W-K) and the Coulomb excitation (W-C_1). The W-C_1 peak was observed

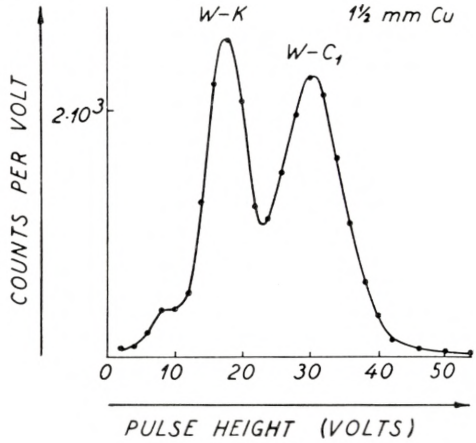

Fig. 5. Pulse spectrum for tungsten, measured at a proton energy of 1.75 MeV and with a 1.5 mm Cu absorber.

to be composite, presumably arising from the different isotopes, since its position and shape could be altered by the use of absorbers.

As an average energy of the W-C_1 lines, we obtained about 115 keV, which is consistent with the known excitation energies of 102 keV for W^{180} and 123 keV for W^{186} (cf. SCHARFF-GOLDHABER, 1953).

The cross-section for γ-emission at a proton energy of 2 MeV is found to be about 1.5 mb, assuming the theoretical energy dependence (3). Using an average total internal conversion

coefficient of 2 for these $E2$ transitions, one obtains from (8) a deformation of $Q_0 \simeq 7 \times 10^{-24}$ cm^2. which is just of the same magnitude as that observed in the excitation of the neighbouring element Ta. This value of Q_0 is also in agreement with that derived from the measured lifetimes of excited states of even-even nuclei in this region.

8. Yield of K X-rays.

As a by-product of these investigations, we obtained the yield of X-rays resulting from the ejection of K-electrons by protons on Ta. The results are shown in Fig. 3. The X-ray yield has been corrected for internal conversion of the Ta-C_1 line, which amounts to about 20 % of the total X-ray yield. The solid line represents the theoretical energy dependence (HENNEBERG, 1933), which is seen to agree rather well with the measurements. The absolute yield is found to be about a factor of two smaller than predicted by the theory, but it is not yet clear to what extent this discrepancy is to be attributed to experimental uncertainties or to the approximations involved in the theory. A comparison of the Ta-K and Pb-K yields for a proton energy of 1.75 MeV was found to be in good agreement with the theoretical Z-dependence.

9. Conclusions.

One may summarize as the main conclusions of these investigations:

1) The feasibility of nuclear excitation by the electric field of bombarding particles has been confirmed.

2) Over the energy region investigated the theoretical energy dependence of the cross-sections has been verified.

3) The first two rotational levels in Ta have been found at the predicted energies.

4) The large cross-sections for excitation of the levels in Ta and W confirm the collective character of the excitation.

5) The nuclear deformations deduced from the measured

cross-sections are of the same magnitude as those derived from spectroscopic evidence and measured lifetimes.

The use of the electric field of charged particles to excite nuclear levels provides the possibility of studying the spectra of a wide variety of nuclei. Preliminary experiments have also exhibited the effect in other elements, and further investigations of the Coulomb excitation processes are in progress.

This work has been performed at the Institute for Theoretical Physics, University of Copenhagen. We wish to thank Professor NIELS BOHR for his continued interest in the experiments and for the good working facilities provided at the Institute. We are also very grateful for the many stimulating discussions on the subject, which we have had with A. BOHR and B. MOTTELSON, in close co-operation with whom the theoretical parts of this paper have been written. To A. WINTHER and K. ALDER we are indebted for making results of their calculations available to us in advance of publication. Our thanks are further due B. MADSEN for developing the electronic equipment used in the investigations, and J. BJERREGAARD for aid in the experiments.

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ÜBER DEN ZUSAMMENHANG ZWISCHEN DER LAMINAREN STRÖMUNG DER REINEN GASE DURCH ROHRE UND DEM SELBSTDIFFUSIONSKOEFFIZIENTEN

VON

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§ 1. Martin Knudsen[1] hat 1909 seine grundlegenden Untersuchungen über die stationäre Strömung reiner Gase durch kreisförmige, zylindrische Rohre veröffentlicht; in dieser Untersuchung sind zum ersten Mal die Gesetze für die reine Molekularströmung theoretisch behandelt und durch experimentelle Resultate bestätigt worden; gleichzeitig wurde für die Gase: Wasserstoff, Sauerstoff und Kohlensäure, der Übergang der Molekularströmung in die Poiseuille'sche, laminare Strömung experimentell untersucht, wodurch das Strömungsbild in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, experimentell festgelegt wurde. R bezeichnet den Radius des kreisförmigen Rohres, während λ die mittlere freie Weglänge des reinen Gases ist.

Die experimentellen Ergebnisse sind von Martin Knudsen durch die semi-empirische Formel:

$$Q_t = \left[a\bar{p} + b \cdot \frac{1 + c_1 \cdot \bar{p}}{1 + c_2 \cdot \bar{p}}\right](p_1 - p_2), \tag{1}$$

oder

$$Q_t = \left[a\bar{p} + \frac{b}{1 + c_2 \cdot \bar{p}} + \frac{b \cdot c_1 \cdot \bar{p}}{1 + c_2 \cdot \bar{p}}\right](p_1 - p_2), \tag{1 a}$$

wiedergegeben worden. —

In dieser Formel bezeichnen:

Q_t: die in der Zeiteinheit durchströmende Gasmenge, gemessen durch das Produkt von Volumen, V, und Druck, $\bar{p}$, bei der Temperatur der Rohrwand, t °C.

p_1 und p_2: die Drucke des Gases an den Enden des kreisförmigen, zylindrischen Rohres mit der Länge, L; ferner gilt:

$$p_1 > p_2, \quad \bar{p} = \frac{1}{2}(p_1 + p_2), \quad \frac{(p_1 - p_2)}{\bar{p}} \ll 1, \quad \frac{R}{L} \ll 1.$$

[1] Martin Knudsen: Ann. d. Physik 4, B 28, S. 75, 1909.

a: Der Koeffizient des POISEUILLE'schen Ausdruckes, also:

$$a = \frac{\pi}{8} \cdot \frac{1}{\eta} \cdot \frac{R^4}{L},$$

η: Der Koeffizient der inneren Reibung des reinen Gases bei der Temperatur, t °C.

b: Der Koeffizient der reinen Molekularströmung, also:

$$b = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{{}_1\varrho}} \cdot \frac{R^3}{L},$$

${}_1\varrho$: das spezifische Gewicht des reinen Gases bei dem Druck, $p = 1$ Bar., und der Temperatur, t °C.

Alle Einheiten sind absolute Einheiten.

Die in der Zeiteinheit durchströmende Masse des Gases, G, in Gramm gemessen, wird also bestimmt durch:

$${}_1\varrho \cdot \frac{Q_t}{p_1 - p_2} = {}_1\varrho \cdot T_1 = G,$$

wenn die Druckdifferenz, $p_1 - p_2 = 1$ Bar.

c_1 und c_2 in der Formel (1) sind druckunabhängige Konstanten. Unter Berücksichtigung der Messgenauigkeit der verschiedenen Messreihen und nach Ausgleichung hat MARTIN KNUDSEN aus allen Beobachtungen für die untersuchten Gase die folgenden Mittelwerte abgeleitet:

$$c_1 = 2{,}00 \cdot \frac{\sqrt{{}_1\varrho}}{\eta} \cdot R \text{ und } c_2 = 2{,}47 \cdot \frac{\sqrt{{}_1\varrho}}{\eta} \cdot R.$$

Nach den Berechnungen von S. CHAPMAN[1] und unter Verwendung der gewöhnlichen Terminologie[2] ist:

$$\eta = 0{,}499 \cdot Nm\,\bar{\Omega}\lambda \simeq \frac{1}{2} \cdot Nm\bar{\Omega}\lambda,$$

1) S. CHAPMAN & T. G. COWLING: The mat. theory of non-uniform gases. (Cambr. Univ. Press) 1939.

2) Vgl.: J. H. A. TER HEERDT: Dissertatie, Utrecht, 1923.

und also:

$$p\lambda = \sqrt{\frac{\pi}{2}} \cdot \frac{\eta}{\sqrt{{}_1\varrho}} = {}_1\lambda = {}_1\lambda_0 \left(\frac{273{,}1 + t}{273{,}1}\right)^{1+n}.$$

In diesen Formeln sind: m die Masse des Moleküls, N die Anzahl der Moleküle per cm³ und $\overline{\Omega}$ die mittlere Geschwindigkeit[1] der Moleküle bei der Temperatur, t, während ${}_1\lambda$ die mittlere freie Weglänge im Gase beim Druck 1 Bar. bezeichnet.

Wir werden nun die KNUDSEN'sche Formel (1) näher betrachten und in diese die CHAPMAN'sche freie Weglänge, λ, einführen; hieraus ergibt sich:

$$\frac{Q_t}{p_1 - p_2} = T_1 = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{{}_1\varrho}} \cdot \frac{R^3}{L}\left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{1 + 2{,}00\sqrt{\frac{\pi}{2}} \cdot \frac{R}{\lambda}}{1 + 2{,}47\sqrt{\frac{\pi}{2}} \cdot \frac{R}{\lambda}}\right],$$

oder

$$\frac{Q_t}{p_1 - p_2} = T_1 = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{{}_1\varrho}} \cdot \frac{R^3}{L}\left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{1 + 2{,}508 \cdot \frac{R}{\lambda}}{1 + 3{,}097 \cdot \frac{R}{\lambda}}\right], \qquad (2\,a)$$

woraus erhellt, dass T_1 in dem ganzen Druckgebiet eine eindeutige Funktion von $\frac{R}{\lambda}$ ist. Wir erhalten auch aus (2 a) durch Umschreibung:

$$\frac{Q_t}{p_1 - p_2} = T_1 = \frac{\pi}{8} \cdot \frac{1}{\eta} \cdot \frac{R^4}{L} \cdot \bar{p}\left[1 + \frac{16}{3\pi} \cdot \frac{1 + 2{,}508 \cdot \frac{R}{\lambda}}{1 + 3{,}097 \cdot \frac{R}{\lambda}} \cdot 4\frac{\lambda}{R}\right]. \qquad (2\,b)$$

Aus (2 a) und (2 b) ergeben sich die folgenden Grenzwerte:

Für $\frac{R}{\lambda} \doteq 0$:

$$T_{1 \to 0} = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{{}_1\varrho}} \cdot \frac{R^3}{L}\left[1 - 0{,}442 \cdot \frac{R}{\lambda} + \cdots\right],$$

[1] Der Mittelwert einer Grösse wird durch einen horizontalen Strich über der Grösse angedeutet.

und für $\frac{R}{\lambda} \to \infty$:

$$T_{1 \to \infty} = \frac{\pi}{8} \cdot \frac{1}{\eta} \cdot \frac{R^4}{L} \cdot \bar{p} \left[1 + k_2 \cdot 4 \frac{\lambda}{R} \right],$$

wo

$$k_2 = \frac{16}{3\pi} \cdot \frac{2{,}508}{3{,}097} = \underline{1{,}375} = \text{ca. } \frac{4}{3}.$$

Für $R/\lambda \to \infty$ erhalten wir also den bekannten Ausdruck für die POISEUILLE'sche Strömung mit Gleitungskorrektion.

Der Gleitungskoeffizient: $\zeta = k_2 \cdot \lambda$, wird demnach aus den Messungen von KNUDSEN bestimmt zu:

$$\zeta = k_2 \cdot \lambda = 1{,}375 \cdot \lambda, \text{ übereinstimmend mit: } c_1/c_2 = 0{,}81.$$

Wenn $c_1/c_2 = 0{,}785$, ein Wert, welchen MARTIN KNUDSEN[1] auch für CO_2 gefunden hat, wird $\zeta = k_2 \cdot \lambda = 1{,}333 \cdot \lambda = \frac{4}{3} \cdot \lambda$.

Aus der Formel (2), die das Beobachtungsmaterial von MARTIN KNUDSEN ziemlich befriedigend wiedergibt, erhellt, dass T_1 einen Minimumwert, $T_{1,\,\text{min.}}$, hat für den Wert $(R/\lambda)_{\text{min.}}$. Auf bekannte Weise gelangt man, wenn:

$$\alpha' = 2{,}47 \sqrt{\frac{\pi}{2}} = 3{,}097 \text{ und } \beta' = 2{,}00 \sqrt{\frac{\pi}{2}} = 2{,}508,$$

zu:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = \frac{1}{\alpha'} \left[\frac{8}{\sqrt{3\pi}} \sqrt{\alpha' - \beta'} - 1 \right] = 0{,}323$$

und

$$T_{1,\,\text{min.}} = 0{,}952 \cdot T_{1,0}, \text{ oder } \frac{T_{1,\,\text{min.}}}{T_{1,0}} = 0{,}952.$$

Wird T_1 als eine Funktion von R/λ graphisch dargestellt, sieht man, dass das Minimum ziemlich flach ist, so dass man leicht eine gute graphische Bestimmung der Grösse, $T_{1,\,\text{min.}}$, erhalten kann, während es schwieriger ist, durch eine graphische Darstellung die genaue Lage des Minimums, bezw. $(R/\lambda)_{\text{min.}}$, zu bestimmen; diesen Wert kann man also am besten mit Hilfe einer

[1] Vgl. TER HEERDT, loc. cit. Seite 86.

guten Interpolationsformel für das Beobachtungsmaterial bestimmen. —

Die von MARTIN KNUDSEN aus seinem Versuchsmaterial bestimmten Werte von $T_{1,\,\min.}$ und $(R/\lambda)_{\min.}$ sind innerhalb der Messgenauigkeit in befriedigender Übereinstimmung mit den aus Formel (2) berechneten Werten, wenn in Betracht gezogen wird, dass MARTIN KNUDSEN in dieser Arbeit nicht die CHAPMAN'sche freie Weglänge, λ, sondern die O. E. MEYER'sche freie Weglänge, $\lambda_{\text{Mey.}}$, verwendet hat; wie bekannt, ist:

$$\lambda_{\text{Meyer}} = 1{,}615 \cdot \lambda_{\text{Chapman}}.$$

Eine einwandfreie molekular-kinetische Erklärung für die Entstehung des Minimums, bezw. dessen gaskinetische Berechnung, und die theoretische Deutung der KNUDSEN'schen semiempirischen Formel ist mit Ausnahme der Grenzwerte für $R/\lambda = 0$ (die reine Molekularströmung) und für $R/\lambda \to \infty$ (die laminare POISEUILLE-Strömung mit Gleitungskorrektion) bis jetzt nicht gegeben worden[1]; ich werde im folgenden u. a. diese Probleme näher behandeln. —

§ 2. Aus der Formel (2) folgt, dass:

$$S = T_1 \sqrt{1\varrho} \cdot \frac{1}{2\sqrt{2\pi}} \cdot \frac{L}{R^3} = \frac{2}{3}\left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{1 + 2{,}00 \cdot \sqrt{\frac{\pi}{2} \cdot \frac{R}{\lambda}}}{1 + 2{,}47 \cdot \sqrt{\frac{\pi}{2} \cdot \frac{R}{\lambda}}}\right],$$

oder:

$$S = \frac{2}{3} \cdot f\left(\frac{R}{\lambda}\right). \qquad (3)$$

Dr. E. MELKONIAN[2] hat aus sämtlichen Beobachtungen von MARTIN KNUDSEN, unter Verwendung der CHAPMAN'schen mittleren freien Weglänge, λ, die Grösse, S, für jede Beobachtung

1) MARTIN KNUDSEN: The kinetic Theory of Gases, Methuen & Co., Ltd., London, Seite 24, 1934.

2) E. MELKONIAN: Vgl. W. G. POLLARD und R. D. PRESENT: Phys. Rev. Vol. 73, 1948, Seite 762, bezw. Zeichnung, Seite 771 u. 772. Ferner: E. MELKONIAN: Manhattan Project Report M-1485, 11. Jan., 1945.

Auch bei dieser Gelegenheit möchte ich Dr. MELKONIAN für seine Mitteilungen, wie auch für die Überlassung der Tabelle I, meinen herzlichen Dank aussprechen.

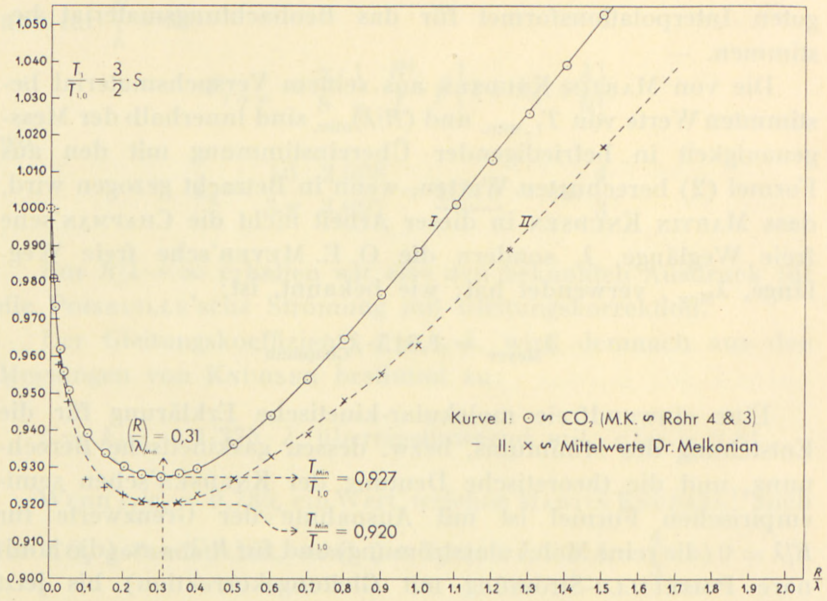


Fig. 1.

berechnet; die hierdurch erhaltenen Werte hat er für eine graphische Darstellung des ganzen Materials verwendet und die wahrscheinlichste, mittlere Kurve gezogen. In dieser Weise kann die Funktion $S = \frac{2}{3} \cdot f\left(\frac{R}{\lambda}\right)$ mit $\frac{R}{\lambda}$ als Abszisse dargestellt werden.

Melkonian hat aus dieser graphischen, ausgeglichenen Kurve die folgenden, korrespondierenden Werte bestimmt:

Tabelle I.

$\frac{R}{\lambda}$	$S = \frac{2}{3} \cdot f\left(\frac{R}{\lambda}\right)$	$\frac{R}{\lambda}$	$S = \frac{2}{3} \cdot f\left(\frac{R}{\lambda}\right)$	$\frac{R}{\lambda}$	$S = \frac{2}{3} \cdot f\left(\frac{R}{\lambda}\right)$
0·0000	0.667	1·0	0·642	7.0	1·18
0·0001	.665	1·5	·678	10.0	1·44
0·001	.658	2·0	·717	20.0	2·44
0·01	.654	2·5	·763	50.0	5·34
0·05	.632	3·0	·805	100.0	10·4
0·10	.618	3·5	·853	200.0	20.6
0·20	.615	4.0	·903	400.0	39.9
0·50	.618	5.0	·996		

Dr. MELKONIAN hat mir mitgeteilt, dass für $R/\lambda < 3$ alle observierten Werte weniger als 5 % von der mittleren Kurve abweichen; für $R/\lambda > 3$ liegen die Beobachtungen viel näher an der mittleren Kurve. —

Verwenden wir die Werte der Tabelle I für eine graphische Darstellung, gelangt man leicht zu (vgl. Fig. 1, Kurve II):

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}310 \text{ und } T_{1,\,\text{min.}} = 0{,}920 \cdot T_{1,0},$$

und ausserdem $k_2 = 1{,}33 = \text{ca. } \frac{4}{3}$, und dementsprechend: $c_1/c_2 = 0{,}785$, also zu Werten, die innerhalb der Messgenauigkeit mit den Werten von KNUDSEN übereinstimmen.

Untersucht man die obenstehenden Mittelwerte von MELKONIAN näher, z. B. mit Hilfe einer graphischen Darstellung, stellt es sich heraus, dass diese Mittelwerte bei grösseren Werten von R/λ, d. h. in dem Gebiete der laminaren POISEUILLE-Strömung mit Gleitungskorrektion, nicht so regelmässig liegen, dass eine gute Bestimmung des Gleitungskoeffizienten vorgenommen werden kann. Diese Unregelmässigkeiten hängen wahrscheinlich damit zusammen, dass die KNUDSEN'schen Messreihen, insbesondere bei grösseren Werten von R/λ, bezw. bei grösserem Mitteldruck, $\bar{p}$, nicht gleichwertig sind; man hätte wahrscheinlich in der Mittelwertbildung, wenn möglich, den verschiedenen Messreihen verschiedenes Gewicht beilegen müssen. —

MARTIN KNUDSEN hat selber hierauf hingewiesen (loc. cit. S. 118 und S. 90) und hervorgehoben, dass die Messungen mit dem Rohr No. 4 (Bündel aus 24 Rohren) die besten sind; insbesondere gilt dies für die Messungen mit CO_2, auch bei höherem Mitteldruck, weil die Druckmessungen in diesem Falle, wegen der kleineren Durchströmungsgeschwindigkeit von CO_2, mit befriedigender Präzision ausgeführt werden konnten. Es erhellt weiter, dass diese Messungen mit dem Rohr No. 4 und die Messungen für CO_2 mit dem Rohr Nr. 3 (Länge ca. 12 cm, $R = \text{ca. } 0{,}01415$ cm) gute Übereinstimmung zeigen, insbesondere für $R/\lambda < 0{,}6$, so dass angenommen werden darf, dass dieses Material gut brauchbar ist. —

Martin Knudsen hat auch für die CO_2-Messungen mit Rohr Nr. 4 und Rohr Nr. 3 die druckunabhängigen Konstanten in seiner semi-empirischen Formel durch Ausgleichung nach der Methode der kleinsten Quadrate bestimmt.

Nach Neuberechnung und Kontrolle des Knudsen'schen Versuchsmaterials mit Rohr Nr. 4, bezw. Rohr Nr. 3, für Kohlensäure und mit Hilfe der semi-empirischen Formel und graphischer Darstellungen haben sich mir die in der Tabelle II angegebenen Werte für S als die wahrscheinlichsten ergeben (vgl. Fig. 1, Kurve I.):

Tabelle II.

$\frac{R}{\lambda}$	$S=\frac{2}{3}\cdot f\left(\frac{R}{\lambda}\right)$	$\frac{R}{\lambda}$	$S=\frac{2}{3}\cdot f\left(\frac{R}{\lambda}\right)$	$\frac{R}{\lambda}$	$S=\frac{2}{3}\cdot f\left(\frac{R}{\lambda}\right)$
0·00	0.6667	0.80	0.6430	8.00	1.330
·01	.6540	0.90	.6505	9.00	1.428
·05	.6350	1.00	.6590	10.0	1.525
·10	.6260	1.50	.7020	20.0	2.506
·20	.6205	2.00	.7480	50.0	5.451
·30	.6180	3.00	.8430	100.0	10.359
·40	.6195	4.00	0.9395	200.0	20.175
·50	.6235	5.00	1.037	400.0	39.810
·60	.6295	6.00	1.134		
·70	.6360	7.00	1.232		

Hieraus ergibt sich (für Rohr Nr. 4):

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}310, \quad T_{1,\,\text{min.}} = 0{,}925 \cdot T_{1,0}$$

und ausserdem $c_1/c_2 = 0{,}812 \pm 0{,}06$, woraus:

$$k_2 = 1{,}378,$$

während man auf diese Weise aus den Werten für CO_2 mit dem Rohr Nr. 3 erhält:

$$c_1/c_2 = 0{,}785 \text{ und dementsprechend: } k_2 = 1{,}333 = \frac{4}{3}.$$

In diesen ersten Messungen von Martin Knudsen waren die gesättigten Quecksilberdämpfe im Versuchsapparat leider nicht entfernt, bezw. ausgefroren, so dass es nicht ganz ausgeschlossen

erscheint, dass die durchströmende Gasmenge, G, bei den allerniedrigsten Werten von R/λ hierdurch beeinflusst worden ist, weil in diesem Gebiet der Einfluss des gesättigten Quecksilberdampfes bei der Temperatur des Strömungsrohres — ca. 25° C. — gegenüber dem Mitteldruck, $\bar{p}$, des reinen Gases, nicht ganz vernachlässigt werden kann. Zur Orientierung kann erwähnt werden, dass der Quecksilberdampfdruck bei 25° C. ca. 0,00181 mm Hg oder ca. 2,4 Bar. beträgt. Es ist also möglich, dass in einzelnen Messreihen die gemessenen Werte von T_1 in der Nähe der reinen Molekularströmung, d. h. $R/\lambda \doteq 0$, zu niedrig ausgefallen sind. Bei dem Rohr Nr. 4 ist es aber ausgeschlossen, dass die Lage und die Grösse des Minimums hierdurch nennenswert beeinflusst worden sein können, weil der Mitteldruck $\bar{p}$ von CO_2 bei dem Minimum $\left(\frac{R}{\lambda} = 0.31\right)$ ca. 0,33 mm Hg oder ca. 450 Bar. betrug.

Die Frage nach dem Einfluss der Quecksilberdämpfe ist aber sowohl von Martin Knudsen als auch von W. Gaede in gesonderten Experimenten näher untersucht worden.

Martin Knudsen[1] wollte in erster Linie untersuchen, ob die Anwesenheit von Quecksilberdampf[2] in seinen ersten Untersuchungen die Lage und die Grösse des Minimums beeinflusst hatte, während Gaede[3] auch in Verbindung mit seinen Strömungsresultaten für einen rechtangulären Spalt festzustellen wünschte, ob der beobachtete Grenzwert, $T_{1,0}$ für $\frac{R}{\lambda} = 0$, für kreisförmige, zylindrische Rohre, genau mit der theoretischen Formel von Martin Knudsen, bezw. von M. von Smoluchowski[4], übereinstimmte.

Martin Knudsen verwendete bei seiner Präzisionsuntersuchung (1911) Wasserstoff; die Länge des Strömungsrohres war: $L = 29{,}81$ cm und der Radius $R = 9{,}729.10^{-3}$ cm, also $\frac{2R}{L} = 0{\cdot}000653$; die Temperatur des Strömungsrohres wurde durch ein Wasserbad auf 26,25° C gehalten. Das Strömungsrohr war an beiden Enden glatt abgeschnitten, und der Queck-

1) Martin Knudsen: Ann. d. Phys. 4, 35, S. 389, 1911.
2) Willard J. Fischer: Phys. Rev. XXXI, No. 5, S. 586, 1910.
3) W. Gaede: Ann. d. Phys. 4, 41, S. 289, 1913.
4) M. von Smoluchowski: Ann. d. Phys. 33, S. 1559, 1910.

silberdampf und andere eventuelle Dämpfe im Apparat wurden während der Messungen, deren jede ca. 48 Stunden dauerte, auf einwandfreie Weise durch flüssige Luft ausgefroren.

Die Resultate sind in der Tabelle III wiedergegeben:

Tabelle III.

$\frac{1}{2}(p_1 + p_2) = \bar{p}$ (Bar.)	$\frac{R}{\lambda}$	$T_{1,\text{ beob.}}$	$T_{1,\text{ berech.}}$
0	0	0·01160*	0·01148
32.26	0.0323	0·01152	0·01134
133.5	0.105	0·01117	0·01124
	0.322	0·01080*	0·01094 ~ Min.
520.2	0.402	0·01099	0·01095
970,2	0.750	0·01127	0·01123

Die mit * versehenen Werte sind interpoliert.

In Kolonne 1 ist der mittlere Druck $\bar{p}$ angegeben; in Kolonne 2 das Verhältnis R/λ, und in den Kolonnen 3 und 4 die beobachteten und berechneten Werte nach der KNUDSEN'schen semi-empirischen Formel (2). In der Kolonne 3 sind ferner die graphisch interpolierten Werte für $R/\lambda = 0$ und $R/\lambda = 0{,}322$ angegeben. Es erhellt hieraus, dass die Lage und die Grösse des Minimums bestimmt sind durch:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}322 \text{ und } T_{1,\text{ min.}} = 0{,}935 \cdot T_{1,0},$$

also in befriedigender Übereinstimmung mit den früheren Beobachtungsresultaten bei den Messreihen, worin der Quecksilberdampf nicht ausgefroren war.

In Verbindung mit den Resultaten, die sich W. GAEDE aus seinen Strömungsuntersuchungen mit einem kurzen, rechtangulären Spalt ergeben hatten, schien es ihm zweifelhaft, ob ein konstanter, oberer Grenzwert für $R/\lambda = 0$ vorhanden sei, und er hielt es für möglich, dass die Konstanz der Werte von T_1 bei den niedrigsten

Drucken in den KNUDSEN'schen Beobachtungsreihen vielleicht der Anwesenheit von Quecksilberdampf oder anderen Dämpfen zugeschrieben werden müsse.

GAEDE hat darum die Strömungsversuche KNUDSEN's mit kreisförmigen, zylindrischen Rohren für $R/\lambda \geqq 0$ und unter Verwendung von sehr reinem Wasserstoff und Stickstoff wiederholt und auf diese Messungen, bezw. auf das Ausfrieren des Quecksilberdampfes, auf die Herstellung eines konstanten und sehr hohen Vacuums, auf die Messung der niedrigsten Drucke u. s. w. alle Sorgfalt und Präzision verwendet.

Die Strömungskapillare hatte eine Länge, L, von 5,67 cm und einen Radius, R, von 0,02064 cm im Mittel, also $2\,R/L$ = ca. 0,0073. Die konischen Erweiterungen an den Enden der Kapillare wurden genau ausgemessen; sie erhöhten den Strömungswiderstand der Kapillare um 3 %, so dass die beobachteten Werte von T_1 um 3 % erhöht werden müssen, um einen Vergleich mit der KNUDSEN'schen Formel für $R/\lambda = 0$ zu ermöglichen. Die Temperatur, t, der Rohrwand war in den Versuchen mit Wasserstoff, 18° C., und in den Versuchen mit Stickstoff, 16° C. Die Resultate von GAEDE sind in der Tabelle IV wiedergegeben.

Tabelle IV.

H_2, $(p\lambda)_t = 12{,}19$, $t = 18°$ C.				N_2, $(p\lambda)_t = 6{,}25$, $t = 16°$ C.			
$\bar{p}$ m. m. Hg.	$\bar{p}$ Bar.	R/λ	T_1	$\bar{p}$ m. m. Hg.	$\bar{p}$ Bar.	$\frac{R}{\lambda}$	T_1
0·007	9,31	0·01580	0·538	0·008	10,64	0·0350	0·1431
·003	4,00	·00676	·543	·006	8,00	·0264	·1438
·001	1,33	·00225	·546	·003	4,00	·0132	·1473
·0009	1,20	·00203	·554	·0023	3,06	·0101	·1484
·0008	1,06	·00179	·555	·0012	1,60	·0053	·1497
·0007	0,93	·00157	·552	·0006	0,80	·0026	·1511
·00055	0,73	·00124	·546	·00046	0,60	·0020	·1498
·00050	0,67	·00113	·550	·00034	0,45	·0015	·1516
·00040	0,53	·00090	·551	·00025	0,33	·0011	·1511
·00028	0,37	·00063	·551				
·00026	0,35	·00059	·554				
·00023	0,31	·00052	·547				
·00019	0,25	·00042	·551				

Hieraus ergibt sich für $R/\lambda = 0$:

Für H_2 (Wasserstoff): $T_{1,0}$ (beob.) = 0·551 + 3 % = 0,568,

während man aus der KNUDSEN'schen Formel für die reine Molekularströmung:

$$T_{1,0} = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{1\varrho}} \cdot \frac{R^3}{L},$$

erhält:

$$T_{1,0}\,(\text{ber.}) = 0{,}5684.$$

Für N_2 (Stickstoff) ergibt sich auf dieselbe Weise:

$$T_{1,0}\,(\text{beob.}) = 0{,}1508 + 3\,\% = 0{,}1553,$$

während:

$$T_{1,0}\,(\text{ber.}) = 0{,}1520.$$

Aus dieser Untersuchung erhellt, dass die bei den niedrigsten Werten von R/λ beobachteten Werte von $T_{1\to 0}$ konstant sind, und dass die beobachteten und berechneten Werte befriedigend übereinstimmen.

Nach Auswertung erhalten wir aus der Tabelle IV die folgenden Werte:

Tabelle V.

R/λ	$T_1/T_{1,0}$	$T_1/T_{1,0}$
	H_2	N_2
0·000	1·000	1·000
0·010	0·982	0·983
0·015	0·972	0·975
0·025		0·958
0·035		0·944

In Verbindung mit der theoretischen Berechnung dieser Werte werde ich später auf diese Tabelle zurückkommen. —

Der theoretische Wert von $T_{1,0}$, d. h. die durchströmende Menge in der reinen Molekularströmung, für welche $R/\lambda = 0$ gesetzt werden kann, ist abhängig von dem Zurückwerfungsgesetz der Moleküle an der festen Wand. Unter Voraussetzung der

absoluten Gültigkeit des Cosinusgesetzes hat MARTIN KNUDSEN den theoretischen Ausdruck:

$$T_{1,0} = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{1\varrho}} \cdot \frac{R^3}{L} \quad \text{für} \quad \frac{R}{\lambda} = 0,$$

abgeleitet.

Setzt man aber in Übereinstimmung mit MAXWELL[1] voraus, dass ein Teil, f, der gegen die Wand stossenden Moleküle nach dem Cosinusgesetz von der Wand zurückgeworfen wird, und der übrigbleibende Teil $(1-f)$ spiegelnd von der Wand zurückgeworfen wird, erhält man:

$$T'_{1,0} = \frac{2-f}{f} \cdot T_{1,0} = \frac{2-f}{f} \cdot \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{1\varrho}} \cdot \frac{R^3}{L}.$$

Wäre z. B. der Wert von $(1-f) = 0{,}02$, so hätte sich als beobachteter Wert für H_2 in der KNUDSEN'schen Präzisionsuntersuchung ca. 0,0120 anstatt 0,0115 für $R/\lambda = 0$ ergeben müssen und in dem GAEDE'schen Versuch als $T_{1,0} = 0{,}591$ anstatt $T_{1,0} = 0{,}568$; eine so grosse Abweichung in den Beobachtungen darf aber als ausgeschlossen betrachtet werden.

In dieser Verbindung kann vollständigkeitshalber auch erwähnt werden, dass es nach den theoretischen Untersuchungen von W. GAEDE[2], P. EPSTEIN[3] und P. CLAUSING[4] zu erwarten ist, dass das Cosinusgesetz für die Zurückwerfung der Moleküle von der Wand gelten muss in einem Raum, worin Temperatur- und Druckgleichgewicht herrschen; es kann dann auch angenommen werden, dass das Cosinusgesetz, jedenfalls mit grosser Annäherung, gültig ist in einem durch ein Rohr stationär strömenden Gas, worin nur ein kleiner Druckgradient vorhanden ist, d. h. wenn $\frac{p_1 - p_2}{L} \cdot \frac{1}{\bar{p}} \to 0$.

Am Ende dieser Übersicht über das vorliegende experimentelle Material kann vollständigkeitshalber noch erwähnt werden, dass HIROSHI ADZUMI[5] 1939 eine orientierende Untersuchung

1) Vgl. z. B.: EARLE H. KENNARD: Kinetic Theory of Gases: S. 306, New York, 1938, und: LEONARD B. LOEB: The kinetic Theory of Gases: S. 325, New York, 1934.
2) W. GAEDE: Ann. d. Phys. 4, 41, S. 331, 1913.
3) P. EPSTEIN: Phys. Rev. 23, S. 710, 1924.
4) P. CLAUSING: Ann. d. Phys. 4, S. 533, 1930.
5) HIROSHI ADZUMI: Bulletin of the Chem. Soc. of Japan: Vol. 14, S. 343, 1939.

über das Durchströmen von Wasserstoff durch Metallkapillare durchgeführt hat, hauptsächlich bei niedrigen Drucken.

ADZUMI verwendete Kapillare von Silber, Aluminium, Kupfer und Eisen. Aus diesen Versuchen, die nur als orientierende Experimente betrachtet werden können, weil die Quecksilberdämpfe nicht ausgefroren waren und weil die Messgenauigkeit nicht sehr gross war, geht hervor, dass kein prinzipieller Unterschied zwischen den Resultaten mit Glas- und Metall-kapillaren besteht; dies wäre auch nicht zu erwarten. Untenstehend sind die von mir, nach graphischer Ausgleichung der Versuchsergebnisse von ADZUMI, gefundenen Resultate wiedergegeben:

1° Silberkapillare; $L = 7{,}70$ cm, $R = 0{,}027$ cm,

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}31, \quad T_{1,\,\text{min.}} = 0{,}98 \cdot T_{1,0}$$

2° Aluminiumkapillare; $L = 5{,}40$ cm, $R = 0{,}029$ cm,

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}31, \quad T_{1,\,\text{min.}} = 0{,}92 \cdot T_{1,0}$$

3° Kupferkapillare; $L = 9{,}80$ cm, $R = 0{,}030$ cm,

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}32, \quad T_{1,\,\text{min.}} = 0{,}905 \cdot T_{1,0}$$

4° Eisenkapillare; $L = 9{,}96$ cm, $R = 0{,}029$ cm,

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}32, \quad T_{1,\,\text{min.}} = 0{,}870 \cdot T_{1,0}.$$

Die Temperatur des Strömungsrohres war in diesen Versuchen, worin Wasserstoff verwendet wurde: $t = 15{,}5°$ C.

Hieraus erhellt, dass die Lage des Minimums, bezw. der Wert von: $(R/\lambda)_{\text{min.}}$, mit dem Resultat der KNUDSEN'schen Messungen übereinstimmt; dies gilt auch der Grössenordnung nach für die Grösse des Minimums, weil die Genauigkeit dieser Messungen von ADZUMI nur auf 5 à 10 % geschätzt werden darf.

Von den theoretischen Arbeiten über die Molekularströmung der Gase muss ausser den älteren Arbeiten[1] von MARTIN KNUD-

1) Vgl. u. a.: J. H. A. TER HEERDT, Dissertatie, Utrecht, 1923.

SEN und M. VON SMOLUCHOWSKI über die reine Molekularströmung insbesondere die spätere interessante Arbeit von W. G. POLLARD und R. D. PRESENT[1] erwähnt werden.

POLLARD und PRESENT (im folgenden zitiert als P. und P.) lenken die Aufmerksamkeit darauf, dass die reine Molekularströmung als die untere Grenze, d. h. der Grenzwert für $R/\lambda = 0$, einer Diffusionsströmung in einem Kapillarrohr aufgefasst werden muss; sie nennen diese Diffusionserscheinung »Wanddiffusion«.

Ausgehend von diesem Wanddiffusionszustand berechnen sie den Einfluss der gegenseitigen Zusammenstösse der Moleküle auf die Molekularströmung und erhalten hierdurch den Ausdruck für die von mir in dieser Abhandlung genannte Druckdiffusionsströmung (CHAPMAN), bezw. für reine Gase Selbstdiffusionsströmung.

Sie gewinnen auf diese Weise eine theoretische Formel für die Variation dieser Strömung durch ein Kapillarrohr mit dem Druck $\bar{p}$ des Gases, bezw. für die Variation des Selbstdiffusionskoeffizienten, $D_{11,p}$, mit dem Wert von R/λ. Um numerische Werte zu erhalten, sind die Werte der in der Formel vorkommenden bestimmten Integralen durch numerische Integration für mehrere Werte von $\frac{R}{\lambda}$ berechnet, wodurch die theoretische Kurve ($D_{11,p}$, R/λ) gezeichnet werden könnte (Fig. 2, Seite 771, Abh. P. und P.).

Es erhellt hieraus, dass diese Kurve annäherungsweise mit der laut P. und P. von C. H. BOSANQUET[2] angegebenen Interpolationsformel für $D_{11,p}$ in einem Kapillarrohr:

$$\frac{1}{D_{11,p}} = \frac{1}{D_{11,0}} + \frac{1}{D_{11,p\to\infty}}, \quad \text{übereinstimmt.}$$

Ausserdem untersuchten P. und P. den Einfluss der endlichen Länge, L, des Strömungsrohres auf den Diffusionskoeffizienten, $D_{11,p\to 0}$, also in dem Zustande, charakterisiert durch:

$$R \ll L \ll \lambda \ , \ \text{wo} \ \lambda \to \infty .$$

P. und P. haben aber nicht damit gerechnet, dass die von gegenseitigen Zusammenstössen kommenden Moleküle im In-

1) W. G. POLLARD und R. D. PRESENT. Phys. Rev., Vol. 73, S. 762, 1948.

2) Vgl. P. und P. loc. cit., Seite 770 (C. H. BOSANQUET: British T. A. Report BR — 507, September 27, 1944).

neren des Rohres eine gemeinschaftliche Massengeschwindigkeit (zusätzliche Geschwindigkeit), v_x, in der Richtung des Druckgradienten erhalten haben, wodurch die von mir in dieser Arbeit genannte »Gleitungsströmung« entsteht.

Im Laufe dieser Arbeit werde ich mehrmals auf die Untersuchung von P. und P. zurückkommen und die Resultate von P. und P. mit einigen der von mir erhaltenen Resultate vergleichen, insoweit diese miteinander vergleichbar sind. —

§ 2. Wie bekannt, geht bei grösseren Werten von R/λ, bezw. $\lambda/R \to 0$, die Molekularströmung von KNUDSEN in einem langen Kapillarrohr in die hydrodynamische, laminare Strömung von POISEUILLE über; für diesen Grenzfall wird, wenn die Strömungsgeschwindigkeit des Gases an der Rohrenwand, v_w, gleich Null gesetzt wird, die pro Sekunde durchströmende Gasmenge bestimmt durch die Formel:

$$G(p_1 - p_2) = {}_1\varrho \cdot T_{1,\infty}(p_1 - p_2) = \frac{\pi}{8} \cdot \frac{{}_1\varrho}{\eta} \cdot \frac{R^4}{L} \bar{p}(p_1 - p_2). \quad (4)$$

Ausgehend von dem obengenannten Grenzzustand, bezw. $R/\lambda \to \infty$, muss diese Formel für G, wenn der Wert von R/λ kleiner wird, nach den Untersuchungen von KUNDT und WARBURG[1] für die Gleitung des strömenden Gases an der festen Rohrwand korrigiert werden, d. h. in diesem Falle darf in der Ableitung der Formel (4) für G die Strömungsgeschwindigkeit des Gases an der Wand und parallel mit dieser, v_w, nicht mehr vernachlässigt werden.

Nach KUNDT und WARBURG kann in diesem Zustande für die Geschwindigkeit, v_w, angenommen werden:

$$v_w = \zeta \cdot \left(\frac{dv}{dr}\right)_{\to \text{Wand}} = \zeta \cdot \left(\frac{dv}{dr}\right)_{r \geq R} \cdot \quad (5)$$

$(dv/dr)_{r \geq R}$ ist der Geschwindigkeitsgradient des Gases an der festen Wand in der Richtung der Normale dieser Wand; $(dv/dr)_{r \geq R}$ muss dicht an der Wand, aber im vollen Gase genommen werden, d. h. dort, wo der Geschwindigkeitsgradient nicht mehr durch die an der Wand zurückgeworfenen Moleküle beeinflusst ist, also an der Stelle, wo $(dv/dr)_{r \geq R}$ von dem Zurückwer-

[1] A. KUNDT und E. WARBURG: Pogg. Ann., Bd. 155, S. 337 und 525, 1875.

fungsgesetz der Moleküle an der Wand unabhängig geworden ist; dies wird in einem Abstande von der Wand, der einigen freien Weglängen des Gases entspricht, der Fall sein.

Diese Gleitungsgeschwindigkeit, v_w, muss also aufgefasst werden als eine über den Querschnitt konstante Massengeschwindigkeit oder »zusätzliche Geschwindigkeit« von allen Molekülen im Rohre; die Richtung dieser zusätzlichen Geschwindigkeit, v_w, und des Konzentrationsgradienten im Gase ist dieselbe, und also für ein zirkulares Rohr die Richtung der Achse.

Der Koeffizient, ζ, wird der Gleitungskoeffizient genannt, und hierfür ist von KUNDT und WARBURG experimentell gefunden:

$$\zeta = \frac{\zeta'}{p} = k \cdot \lambda,$$

wo p der Druck ist; k ist eine Konstante in dem Gebiet: $R/\lambda \to \infty$.

Aus den KNUDSEN'schen Messungen mit Kapillarrohren von sehr verschiedenen Dimensionen erhellt, dass der Wert von k von dem Radius des Kapillarrohres, bezw. von der Krümmung der festen Wand, unabhängig ist, so dass angenommen werden darf, — jedenfalls in erster Annäherung —, dass der hieraus bestimmte, experimentelle Wert von k auch für eine plane Oberfläche Gültigkeit hat.

Die gaskinetische Theorie der Gleitung ist im MAXWELL'schen Zustand der Gase mathematisch sehr kompliziert, und bis jetzt besteht hierfür nur die angenäherte Theorie von MAXWELL; diese ist später von R. A. MILLIKAN[1] auf sehr einfache und übersichtliche Weise dargestellt worden. Diese Theorie von MAXWELL ist aber nur eine angenäherte, und solange das Gesetz der Geschwindigkeitsverteilung der Moleküle in den an die feste Wand grenzenden Gasschichten nicht hinreichend bekannt ist, wird es kaum möglich sein, eine exakte und einwandfreie Theorie für die Gleitung aufzustellen[2]; es ist aber einfach zu beweisen, dass in dem MAXWELL'schen Zustande der Gase jedenfalls $k \leqq 2$ sein muss, wenn das Cosinusgesetz für die Zurückwerfung der Moleküle an der Wand Gültigkeit hat.

Dass k von dem Zurückwerfungsgesetz der Moleküle an der

1) R. A. MILLIKAN: Phys. Rev. 21, S. 217, 1923.
2) H. A. KRAMERS: Il Nuovo Cimento: Vol. VI, Ser. IX, S. 297, 1949.

Wand abhängig ist, hat bereits MAXWELL dargetan und er hat, ausgehend von teilweiser Zurückwerfung nach dem Cosinusgesetz und teilweiser, spiegelnder Zurückwerfung, nach seiner obenerwähnten Theorie abgeleitet:[1]

$$\zeta = \frac{2-f}{f}\sqrt{\frac{\pi}{2}}\cdot\frac{\eta}{p\sqrt{1\varrho}} \simeq \frac{2-f}{f}\cdot\lambda_{\text{Chapman.}},$$

und also, wenn das Cosinusgesetz absolute Gültigkeit hat, d. h. $f = 1$:

$$\zeta = \lambda_{\text{Ch.}} \quad \text{oder} \quad k = 1\cdot -$$

Zieht man in Betracht, dass das Cosinusgesetz für die Zurückwerfung der Moleküle an einer festen Wand in der reinen Molekularströmung, praktisch gesprochen, absolute Gültigkeit besitzt, besteht keine Veranlassung anzunehmen, dass in dem MAXWELL'schen Zustande der Gase für *dieselbe Wand* ein anderes Zurückwerfungsgesetz, z. B. teilweise spiegelnde Zurückwerfung, gültig sein sollte, weil der kinetische Zustand an der festen Wand — auch im MAXWELL'schen Zustande der Gase — innerhalb eines Abstandes von der Wand, entsprechend der mittleren freien Weglänge, λ, ganz analog und vergleichbar ist mit dem Zustand an der Wand in der reinen Molekularströmung.

Ergibt sich also im MAXWELL'schen Zustande für k experimentell ein Wert von $k > 1$, darf hieraus nicht abgeleitet werden, dass ein Teil der Moleküle spiegelnd zurückgeworfen wird[2]; dies ist sicher nicht erlaubt, solange eine einwandfreie Theorie der Gleitung in dem Gebiet $R/\lambda \to \infty$ noch nicht vorliegt. MAXWELL selber hat darauf bereits aufmerksam gemacht.

In dieser Verbindung kann die Arbeit von W. P. J. LIGNAC[3] erwähnt werden. LIGNAC hat u. a. eine Übersicht über alle bekannten Messungen und Bestimmungen von k gegeben und dies Material kritisch bearbeitet; ich werde später hierauf in Verbindung mit den Bestimmungen von k nach der kapillaren Durchströmungsmethode näher zurückkommen, weil diese Methode scheinbar im Mittel grössere Werte für k ergibt als die

1) Vgl. EARLE H. KENNARD: loc. cit., pag. 296.
2) Vgl. EARLE H. KENNARD: loc. cit., pag. 299.
3) W. P. J. LIGNAC: Dissertatie, Leiden, 1949.

anderen zur Bestimmung der Gleitung benutzten Methoden, wie z. B. die Ablenkungsmethoden der rotierenden Scheibe oder des rotierenden Aussenzylinders, die Methode der schwingenden Scheibe u. s. w.

Lignac hat aus dem gesamten Material für den Wert von k nach allen Methoden: $k > 1$, gefunden.

Aus dem von ihm veröffentlichen Material scheint weiter hervorzugehen, dass im Mittel: $k = \text{ca. } 1.20$; die Bestimmungen von k nach der Durchströmungsmethode ergeben aber im Allgemeinen: $k = \text{ca. } \frac{4}{3}$.

Wir können also vorläufig am besten für den Zustand: $R/\lambda \to \infty$, den Ausdruck:

$$v_w = \zeta \cdot \left(\frac{dv}{dr}\right)_{r \geqq R} = k_2 \cdot \lambda \cdot \left(\frac{dv}{dr}\right)_{r \geqq R}$$

verwenden, mit dem Zahlenwert von k_2, der aus der Beobachtungsreihe hervorgeht. —

Wenn wir den Einfluss der Gleitungsgeschwindigkeit, v_w, auf die Formel von Poiseuille berechnen wollen, können wir ein Raumelement $2\pi r\, dl \cdot \pi r^2$ (vgl. Fig. 2) im Rohre mit Radius, R, und Länge, L, betrachten. Ist der Zustand stationär, erhalten

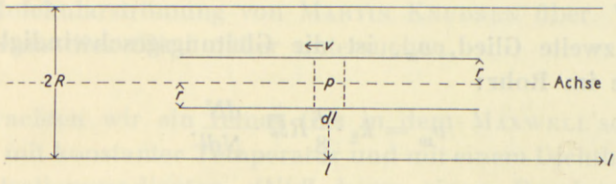


Fig. 2.

wir, weil die Kräfte in der Richtung der Achse in diesem Falle im Gleichgewicht sein müssen, die folgende Gleichgewichtsbedingung:

$$\pi r^2 \cdot \frac{dp}{dl} \cdot dl = 2\pi r\, dl \cdot \eta \frac{dv}{dr},$$

woraus, weil $\frac{dp}{dl}$ als eine Konstante angesehen werden kann, $\frac{dv}{dr} = \frac{r}{2\eta} \cdot \frac{dp}{dl}$, oder nach Integration:

$$v = \frac{1}{4\eta}[(R^2 - r^2) + 2\zeta R] \cdot \frac{dp}{dl}.$$

Da $\eta \simeq \frac{1}{2} Nm\bar{\Omega}\lambda$ und $\frac{dp}{dl} = \frac{\pi}{8} m\bar{\Omega}^2 \cdot \frac{dN}{dl}$, wird:

$$v = v_p + v_w = \frac{\pi}{16} \frac{R^2 - r^2}{\lambda} \bar{\Omega} \frac{dN}{Ndl} + k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl}.$$

Die Strömungsgeschwindigkeit, v, des Gases im Rohre besteht also aus zwei Gliedern; das erste Glied, v_p, ist die Geschwindigkeit, herrührend von der hydrodynamischen, laminaren Strömung, und also:

$$v_p = \frac{\pi}{16} \cdot \frac{R^2 - r^2}{\lambda} \bar{\Omega} \cdot \frac{dN}{Ndl}, \quad \text{woraus sich}$$

für $r = R$, $v_{p,R} = 0$[1], und für $r = 0$ (Achse des Rohres), $v_{p,\text{Achse}} = \frac{\pi}{16} \frac{R^2}{\lambda} \bar{\Omega} \frac{dN}{Ndl} = \frac{1}{4} \frac{R^2}{\eta} \cdot \frac{dp}{dl}$, ergibt, während die mittlere Geschwindigkeit über den Querschnitt, πR^2, wird:

$$v_{p,\text{Mitt.}} = \frac{1}{8} \cdot \frac{R^2}{\eta} \cdot \frac{dp}{dl} = \frac{\pi}{32} \cdot R\bar{\Omega} \cdot \frac{dN}{Ndl} \cdot \frac{R}{\lambda}, —$$

Das zweite Glied, v_w, ist die Gleitungsgeschwindigkeit der Moleküle im Rohr:

$$v_w = k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl}.$$

Nach Einführung des experimentellen Wertes von $k_2 = \text{ca. } \frac{4}{3}$, ergibt sich:

$$v_w = k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl} \simeq \frac{\pi}{6} R\bar{\Omega} \cdot \frac{dN}{Ndl}.$$

Aus dem Wert von v erhält man für G den bekannten Ausdruck:

1) Da $v_p = 0$ für $r = R$ (Wand des Rohres), erhellt hieraus, dass die hydrodynamische Strömung sich auch mit dem Cosinusgesetz für die Zurückwerfung der Moleküle an der festen Wand verträgt. —

$$G(p_1-p_2) = mN\int_0^R v \cdot 2\pi r\, dr = \frac{\pi}{8}\cdot\frac{R^4}{\eta}\,{}_1\varrho\,\bar{p}\left[1+4\frac{\zeta}{R}\right]\frac{dp}{dl},$$

bezw.

$$G(p_1-p_2) = \frac{2\pi}{3}\,m\bar{\Omega}\cdot R^3\frac{dN}{dl}\left[\frac{3\pi}{64}\cdot\frac{R}{\lambda}+\frac{3\pi}{16}k_2\right],$$

oder, wenn $k_2 = \frac{4}{3}$:

$$G(p_1-p_2) = \frac{4}{3}\sqrt{2\pi}\sqrt{{}_1\varrho}\cdot\frac{R^3}{L}(p_1-p_2)\left[\frac{3\pi}{64}\cdot\frac{R}{\lambda}+\frac{\pi}{4}\right].$$

§ 3. Bei näherer Betrachtung von G sehen wir aber, dass diese zweigliedrige Formel noch nicht vollständig ist, weil in diesem Ausdruck für die Strömung von POISEUILLE mit Gleitungskorrektion *noch ein Glied fehlt, nämlich ein Glied, herrührend von der Selbstdiffusion in dem laminar strömenden Gase zufolge des Konzentrationsgradienten,* $\frac{dN}{dl}$.

Dies Glied ist bei grösseren Werten von R/λ, verglichen mit den zwei anderen Gliedern, zahlenmässig ohne wesentliche Bedeutung; bei kleineren Werten von R/λ gewinnt aber diese Selbstdiffusionsströmung überwiegende Bedeutung und geht für $R/\lambda = 0$ in die reine Molekularströmung von MARTIN KNUDSEN über. Wir werden darum dies Glied näher untersuchen.

Betrachten wir ein reines Gas in dem MAXWELL'schen Zustande, mit konstanter Temperatur und mit einem Dichtheits- oder Konzentrationsgradienten, dN/dl, bezw. einem Druckgradienten, dp/dl, und nehmen wir an, dass dieser Druckgradient *keine* Turbulenzströmungen in dem Gase veranlasst, muss eine stationäre Diffusion in der Richtung des Druckgradienten, bezw. Konzentrationsgradienten, von statten gehen. Eine Berechnung nach der einfachen kinetischen Theorie — ausgehend von der Annahme, dass die Moleküle als harte, elastische Kugeln aufgefasst werden können — ergibt für ein *reines* Gas in dem stationären Zustand, dass die durch ein Flächenelement, dS, senkrecht zur Richtung des Konzentrationsgradienten, dN/dl, diffundierende Gasmenge in dem MAXWELL'schen Zustande des Gases wird:

$$m \cdot \frac{1}{3} \bar{\Omega} \lambda \cdot \frac{dN}{dl} \cdot dS = m \cdot D_{11,\infty} \cdot \frac{dN}{dl} \cdot dS,$$

wo $D_{11,\infty}$ den Selbstdiffusionskoeffizienten des reinen Gases in dem MAXWELL'schen Zustande des Gases darstellt.

In einem reinen, ruhenden Gase im MAXWELL'schen Zustande ist dieser theoretische Diffusionszustand selbstverständlich nicht möglich, bezw. stabil, aber anders wird es, wenn man es mit einem *laminar strömenden* Gase zu tun hat.

In diesem Falle, z. B. in der Strömung von POISEUILLE, hat man in jeder zirkularzylindrischen, strömenden Lamelle, $2\pi r dr \cdot L$, einen Konzentrationsgradienten, dN/dl, und in jeder Lamelle muss also, solange keine Turbulenz entsteht und die laminare Strömung zerstört, eine Druckdiffusion von statten gehen, so dass unabhängig von der hydrodynamischen Strömung und der Gleitungsströmung *ausserdem eine Diffusionsströmung* stattfindet.

Ist das strömende Gas ein reines, unvermischtes Gas, wird diese Diffusion eine Selbstdiffusion.

Wir können annehmen, dass diese drei Strömungen, d. h. die hydrodynamische Strömung, die Gleitungsströmung und die Diffusionsströmung, einander nicht beeinflussen, so dass der vollständige Ausdruck für die totale, stationäre Strömung — in dem Zustande: $R/\lambda \to \infty$, d. h. in dem MAXWELL'schen Zustande des Gases, — durch Superposition der drei einzelnen Strömungen gefunden wird und also:

$$G' = G(p_1 - p_2) = \frac{2\pi}{3} m \bar{\Omega} R^3 \frac{dN}{dl} \cdot \left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{3\pi}{16} \cdot k_2 \right] + m \cdot D_{11,\infty} \cdot \frac{dN}{dl} \cdot \pi R^2,$$

oder:

$$G' = G(p_1 - p_2) = \frac{2\pi}{3} m \bar{\Omega} R^3 \frac{dN}{dl} \cdot \left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{3\pi}{16} \cdot k_2 + \frac{\lambda}{2R} \right], \qquad (6)$$

$$\text{wenn } \frac{R}{\lambda} \to \infty \quad \text{und} \quad D_{11,\infty} = \frac{1}{3} \bar{\Omega} \lambda.$$

Der besseren Übersicht wegen werden wir zuerst den obenerwähnten Wert des Selbstdiffusionskoeffizienten nach der einfachen kinetischen Theorie ableiten, indem wir von einem reinen Gase in dem Zustande: $R/\lambda \to \infty$, z. B. einem laminar strömenden

Gase mit einem Konzentrationsgradienten, dN/dl, ausgehen. Durch ein Flächenelement, dS, senkrecht zu dem Konzentrationsgradienten, werden per Sek., der Druckdiffusion wegen, gehen:

$n_r \cdot dS$ Moleküle, die von rechts kommen (Gruppe I)

und

$n_l \cdot dS$ Moleküle, die von links kommen (Gruppe II).

Wir erhalten leicht, da wir von der konstanten Massenströmung absehen können, vgl. Fig. 3:

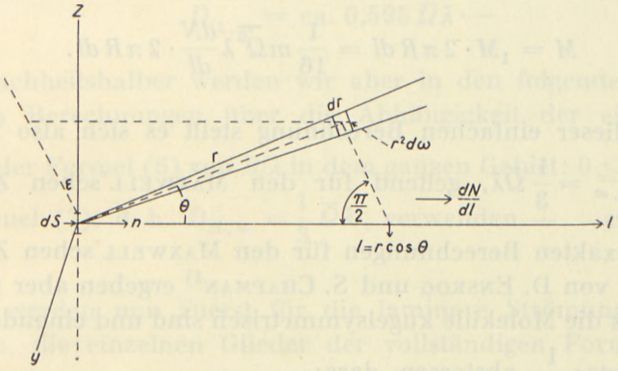


Fig. 3.

$$n_r \cdot dS = dS \cdot \frac{1}{4\pi} \cdot \frac{N_1 \bar{\Omega}}{\lambda} \int \cos\theta \, d\omega \, e^{-\frac{r}{\lambda}} dr$$

und

$$n_l \cdot dS = dS \cdot \frac{1}{4\pi} \cdot \frac{N_2 \bar{\Omega}}{\lambda} \int \cos\theta \, d\omega \, e^{-\frac{r}{\lambda}} dr,$$

wo:

$$N_1 = N_0 + \frac{dN}{dl} \cdot r\cos\theta \quad \text{und} \quad N_2 = N_0 - \frac{dN}{dl} \cdot r\cos\theta.$$

Hieraus erhält man:

$$(n_r - n_l)\, dS = dS \cdot \frac{1}{4\pi} \cdot 2 \cdot \frac{dN}{dl} \cdot \frac{\bar{\Omega}}{\lambda} \cdot \int \cos^2\theta \, d\omega \, e^{-\frac{r}{\lambda}} r \cdot dr$$

und, weil $d\omega = \sin\theta \, d\theta \, d\varepsilon$:

$$\left.\begin{aligned}(n_r - n_l)\,dS &= dS \cdot \frac{1}{2\pi}\,\bar{\Omega}\lambda\,\frac{dN}{dl} \cdot \int_0^{2\pi} d\varepsilon \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta\, d\theta \int_0^{\infty} \frac{r}{\lambda} \cdot e^{-\frac{r}{\lambda}}\, d\left(\frac{r}{\lambda}\right) \\ &= dS \cdot \frac{1}{3}\,\bar{\Omega}\lambda \cdot \frac{dN}{dl}.\end{aligned}\right\} \quad (7)$$

In derselben Weise lässt sich auch die in der Zeiteinheit von einem Rohrstück, $2\pi R\,dl$, empfangene Bewegungsgrösse, herrührend von der Diffusionsströmung, leicht berechnen; diese Bewegungsgrösse in der Richtung des Dichtheitsgradienten wird in dem stationären Zustande:

$$M = {}_1M \cdot 2\pi R\,dl = \frac{1}{16}\,m\overline{\Omega^2}\lambda\,\frac{dN}{dl} \cdot 2\pi R\,dl.$$

Bei dieser einfachen Berechnung stellt es sich also heraus, dass $D_{11,\infty} = \frac{1}{3}\,\bar{\Omega}\lambda$, geltend für den MAXWELL'schen Zustand des Gases.

Die exakten Berechnungen für den MAXWELL'schen Zustand der Gase von D. ENSKOG und S. CHAPMAN[1] ergeben aber für den Fall, dass die Moleküle kugelsymmetrisch sind und einander nach dem Gesetz: $\frac{1}{r^5}$, abstossen, dass:

$$D_{11,\infty} = 1{,}543 \cdot \frac{\eta}{\varrho} = 0{,}772 \cdot \bar{\Omega}\lambda.$$

Sind aber die Moleküle als harte, elastische Kugeln aufzufassen, ergibt sich:

$$D_{11,\infty} = \frac{6}{5} \cdot \frac{\eta}{\varrho} = 1{,}20 \cdot \frac{\frac{1}{2}\,Nm\bar{\Omega}\lambda}{Nm} = 0{,}60 \cdot \bar{\Omega}\lambda.$$

Es ist einfach zu sehen, dass diese Lösung die richtige darstellt, weil es bekannt ist[2], dass die gegenseitige Diffusion der Moleküle der zwei durch dS strömenden Gruppen , I und II , unabhängig ist von den gegenseitigen Zusammenstössen der Moleküle in der Gruppe I bezw. in der Gruppe II. Lassen wir die

1) Vgl. EARLE H. KENNARD, loc. cit., pag. 195.
2) Vgl. EARLE H. KENNARD, loc. cit., pag. 191.

gegenseitigen Zusammenstösse der Moleküle der Gruppe I, bezw. der Gruppe II, ausser Betracht, dann sind nur die Zusammenstösse zwischen den Molekülen der Gruppe I und den Molekülen der Gruppe II von Bedeutung; in diesem Falle erhält man, wie früher von mir abgeleitet[1]:

$$m \cdot (n_r - n_l)\, dS = dS \cdot m \cdot \frac{1}{3} \bar{\Omega} \lambda \frac{dN}{dl} \cdot \frac{1}{S_1},$$

wo $S_1 =$ ca. 0,56[2], woraus:

$$D_{11,\infty} = \frac{1}{3} \bar{\Omega} \lambda \cdot \frac{1}{S_1} \text{ d. h.}$$

$$D_{11,\infty} = \text{ca. } 0{,}595\, \bar{\Omega} \lambda \cdot -$$

Einfachheitshalber werden wir aber in den folgenden angenäherten Berechnungen über die Abhängigkeit der einzelnen Glieder der Formel (6) von R/λ in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, die Formel (7), d. h. $D_{11,\infty} = \frac{1}{3} \bar{\Omega} \lambda$, verwenden. —

Wir werden nun zuerst für die laminare Strömung, wenn $R/\lambda \to \infty$, die einzelnen Glieder der vollständigen Formel (6):

$$G' = G_3' + G_2' + G_1' = \frac{2\pi}{3} m \bar{\Omega} \cdot R^3 \frac{dN}{dl} \left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{3\pi}{16} k_2 + \frac{\lambda}{2R} \right],$$

wo $G(p_1 - p_2) = G'$, näher betrachten und überlegen, wie diese Glieder sich in dem Gebiet, $0 \leqq \frac{R}{\lambda} \leqq \infty$, mit R/λ ändern, um hierdurch eine Annäherungsformel zu erhalten, die mit der semiempirischen Formel von Martin Knudsen vergleichbar ist.

a) Das Diffusionsglied: $G_1' = \pi R^2 \cdot m D_{11,p} \cdot \frac{dN}{dl}$, wird für $\frac{R}{\lambda} \to \infty$:

1) Sophus Weber: D. Kgl. Danske Vid. Selskab; Mat.-fys. Medd. XXIV, No. 4, pag. 50, 1947.

2) W. P. J. Lignac: Dissertatie, Leiden, S. 22, 1949. In dieser Arbeit wird für den Mittelwert von S_1 angegeben:

$$S_1 = S_{LW} = \frac{1}{2} \left[\frac{3}{2} \sqrt{2} - 1 \right] = 0{,}56.$$

$$G'_{1,\infty} = \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \frac{dN}{dl} \cdot \frac{\lambda}{2R}, \quad \text{wenn} \quad D_{11,\infty} = \frac{1}{3}\bar{\Omega}\lambda \text{, bezw.}$$

$$D_{11,\infty} = \frac{1}{3}\bar{\Omega} \cdot {}_1\lambda \cdot \frac{1}{p}, \quad \text{weil} \quad {}_1\lambda = p\lambda \cdot -$$

Für kleine Werte von R/λ ist der Rohrdurchmesser, $2R$, gegenüber λ nicht mehr gross, so dass wir nach den Überlegungen von Martin Knudsen[1] über den Einfluss der festen Wand auf den Wert von λ diesen annähernd durch:

$$\lambda' = \frac{\lambda}{1 + \frac{\lambda}{2R}},$$

ersetzen müssen.

Wir erhalten hieraus:

$$D_{11,p} = \frac{1}{3}\bar{\Omega}\lambda \frac{1}{1 + \frac{\lambda}{2R}}$$

und also:

$$G'_{1,p} = \pi R^2 \cdot m D_{11,p} \cdot \frac{dN}{dl} = \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \cdot \frac{dN}{dl} \cdot \frac{1}{1 + 2R/\lambda} \cdot \quad (8)$$

Der Wert dieses Ausdrucks für $R/\lambda = 0$ ist aber der von Martin Knudsen auf direkte Weise abgeleitete Wert für die durchströmende Gasmenge in der reinen Molekularströmung[2].

Ob der Ausdruck von $D_{11,p}$, der in Übereinstimmung ist mit der Druckabhängigkeit der inneren Reibung und der Wärmeleitfähigkeit der Gase, befriedigende Resultate in dem ganzen Druckgebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, ergeben wird, kann man a priori nicht wissen, auch weil experimentelle Beobachtungen über die Variation des allgemeinen Diffusionskoeffizienten, $D_{1,2}$, mit dem Druck p immer noch fehlen. Es wird sich aber später herausstellen, dass in dem Gebiet: $R/\lambda \geqq 0$, die Variation von $D_{11,p}$ nur annäherungsweise durch diese Formel befriedigend wiedergegeben werden kann (vgl. Seite 106).

1) Martin Knudsen: Ann. d. Phys. 31, S. 222, 1910.

2) Martin Knudsen: Ann. d. Phys. 28, S. 111, 1909. Dieser, in jeder Hinsicht einwandfrei abgeleitete, Wert wurde durch Addition der einzelnen durch den Querschnitt strömenden Moleküle berechnet.

Da wir aus den experimentellen Untersuchungen von Martin Knudsen und W. Gaede wissen, dass der theoretische Wert für $\frac{R}{\lambda} = 0$:

$$G'_{1,0} = \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \frac{dN}{dl} = \frac{4}{3}\sqrt{2\pi}\sqrt{\varrho} \cdot R^3 \cdot \frac{p_1 - p_2}{L},$$

in vollständiger Übereinstimmung mit dem experimentellen Wert für: $R/\lambda \gtrsim 0$, ist, können wir hieraus schliessen, dass für $\frac{R}{\lambda} = 0$, sowohl:

$$G'_{3,0} = 0, \quad \text{als auch:} \quad G'_{2,0} = 0.$$

Die Formel:

$$D_{11,p} = \frac{1}{3}\bar{\Omega}\lambda \cdot \frac{1}{1 + \frac{\lambda}{2R}},$$

kann für kreisförmige, zylindrische Rohre auch geschrieben werden:

$$\frac{1}{D_{11,p}} = \frac{1}{D_{11,0}} + \frac{1}{D_{11,\infty}}, \tag{9}$$

so dass wir für $D_{11,p}$ einen Ausdruck erhalten, der ganz analog ist mit der von mir[1] angegebenen und untersuchten Formel für die Druckabhängigkeit des Wärmeverlustes von einer planen Oberfläche:

$$\frac{1}{q_p} = \frac{1}{q_0} + \frac{1}{q_\infty}.$$

Hierin bezeichnen, wenn $T_1 - T_2 = 1°$ C:

q_p : den Wärmeverlust pro Sek. und cm² bei dem Druck p,
q_0 : den Wärmeverlust pro Sek. und cm² in dem molek. Zust.
q_∞ : den Wärmeverlust pro Sek. und cm² in dem Maxwell'schen Zustand.

Die obenstehende Formel (9) für $D_{11,p}$ ist auch bei Pollard und Present[2] in Verbindung mit theoretischen Betrachtungen von Bosanquet erwähnt; vgl. S. 17.

1) Sophus Weber: D. Kgl. Danske Vid. Selsk., Mat.-fys. Medd. XIX, No. 11, S. 6, 1942.
2) W. G. Pollard und R. D. Present: Phys. Rev., Vol. 73, S. 770, 1948.

b) Wir werden nun das Gleitungsglied G_2' betrachten; dies Gleitungsglied ist im MAXWELL'schen Zustande, $R/\lambda \to \infty$, aufzufassen als davon herrührend, dass alle Moleküle im Rohre eine gemeinschaftliche, konstante Geschwindigkeit, v_w (zusätzliche Geschwindigkeit) in der Richtung der Achse besitzen; diese Gleitungsgeschwindigkeit ist in dem Zustand, $R/\lambda \to \infty$:

$$v_w = v_2 = k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl},$$

wo sich für $\frac{R}{\lambda} \to \infty$, experimentell ergeben hat: $k_2 = \text{ca.}\ \frac{4}{3}$. Die durchströmende Gasmenge der Gleitung wegen wird also für $\frac{R}{\lambda} \doteq \infty$:

$$G'_{2,\infty} = mNv_2 \cdot \pi R^2 = \frac{\pi}{8} \cdot \frac{R^4}{\eta} \cdot {}_1\varrho \cdot p \cdot 4\, k_2 \frac{\lambda}{R} \frac{dp}{dl}.$$

Wenn das Cosinusgesetz gültig ist, und dies erhellt aus der Übereinstimmung zwischen dem experimentellen und theoretischen Wert der reinen Molekularströmung, wissen wir, dass die an der Wand zurückgeworfenen Moleküle im Mittel keine gemeinschaftliche Geschwindigkeit (zusätzliche Geschwindigkeit) in der Richtung der Achse besitzen, so dass nur die Moleküle, die von gegenseitigen Zusammenstössen mit anderen Molekülen kommen, eine Gleitungsgeschwindigkeit besitzen können. Der Einfluss der Wand auf den Ausdruck für G_2' kann also annäherungsweise auf Grund folgender Betrachtung abgeleitet werden:

Betrachten wir eine Längeneinheit des Rohres, wird die Wandoberfläche $2\,\pi R$ und das Volumen πR^2; die Anzahl der gegenseitigen Molekularstösse in dem Volumen, $\pi R^2 \cdot 1$ cm, wird $n_1 = \frac{N\bar{\Omega}}{\lambda} \cdot \pi R^2$, während die ganze Anzahl der Stösse, wenn auch die Stösse gegen die Wand mitgezählt werden, beträgt:

$$n = \frac{N\bar{\Omega}}{\lambda} \cdot \pi R^2 + 2\pi R \cdot \frac{1}{4} N\bar{\Omega},$$

und also:

$$\frac{n_1}{n} = \frac{1}{1 + \frac{\lambda}{2R}}.$$

Wir ersehen hieraus, dass die durchströmende Gasmenge G_2' bei dem Druck, p, da von der totalen Anzahl, n, der Stösse nur n_1 für die Gleitung von Bedeutung ist, wird:

$$G_{2,p}' = \frac{n_1}{n} \cdot G_{2,\infty}' = G_{2,\infty}' \frac{1}{1+\frac{\lambda}{2R}},$$

so dass wir schreiben können:

$$G_{2,p}' = \frac{\pi}{8} \cdot \frac{R^4}{\eta} \cdot {}_1\varrho p \cdot 4 k_2 \frac{\lambda}{R} \cdot \frac{dp}{dl} \frac{1}{1+\frac{\lambda}{2R}}$$

$$= \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \frac{dN}{dl} \cdot \frac{3\pi}{16} k_2 \cdot \frac{\frac{2R}{\lambda}}{1+\frac{2R}{\lambda}},$$

oder, wenn $k_2 = \frac{4}{3}$:

$$G_{2,p}' = \frac{2\pi}{3} m\bar{\Omega} R^3 \frac{dN}{dl} \cdot \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1+\frac{2R}{\lambda}} = G_{1,0}' \cdot \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1+\frac{2R}{\lambda}} \qquad (10)$$

Wenn $\frac{R}{\lambda} = 0$, folgt hieraus: $G_{2,0}' = 0$.

Für $\frac{R}{\lambda} \gtrsim 0$, erhalten wir somit:

$$G_{2,p\to 0}' = G_{1,0}' \cdot \frac{\pi}{4} \cdot \frac{2R}{\lambda} = \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \frac{dN}{dl} \cdot \frac{\pi}{4} \cdot \frac{2R}{\lambda}$$

$$= \frac{4}{3} mN\bar{u}_0 \cdot \pi R^2 \cdot \frac{R}{\lambda}, \quad \text{wo} \quad \bar{u}_0 = \frac{\pi}{4} R\bar{\Omega} \cdot \frac{dN}{Ndl} \cdot -$$

Wir werden später diesen letzten Ausdruck für $\frac{R}{\lambda} \gtrsim 0$ auch auf direkte Weise ableiten.

c) Betrachten wir nun das hydrodynamische Glied, $G_{3,p}'$, erhalten wir nach der hydrodynamischen Ableitung und also jedenfalls gültig für grössere Werte von R/λ:

$$G'_{3,p} = \frac{\pi}{8} \cdot \frac{R^4}{\eta} \cdot {}_1\varrho p \cdot \frac{dp}{dl} = \frac{2\pi}{3} m\bar{\Omega} \cdot R^3 \frac{dN}{dl} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}$$

$$= G'_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}.$$

Hieraus erhellt, dass $G'_{3,p}$ proportional ist mit $\frac{2R}{\lambda}$, bezw. $\bar{p}$; wird dieser Ausdruck auch für kleinere Werte von $\frac{R}{\lambda}$ verwendet, wird

$$\text{für } \frac{R}{\lambda} = 0: \quad G'_{3,0} = 0,$$

wie auch auf Grund der Beobachtungen erwartet werden muss (vgl. S. 29).

Man kann a priori nicht wissen, wie das hydrodynamische Glied, $G'_{3,p}$, in dem Gebiet: $R/\lambda \gtrsim 0$, genau verläuft. Man darf aber annehmen, dass die hydrodynamische Strömung, G'_3, nicht einsetzen kann, solange man es nur mit *vereinzelten*, gegenseitigen Zusammenstössen der Moleküle zu tun hat. Da dies der Fall ist in dem Gebiet: $R/\lambda \gtrsim 0$, muss man auch für: $\frac{R}{\lambda} \gtrsim 0$, $G'_{3,p} \gtrsim 0$ erwarten.

Das Einsetzen der hydrodynamischen Strömung wird wahrscheinlich erst dann stattfinden können, wenn die Anzahl der gegenseitigen Zusammenstösse in einem Raumelement, z. B. in der Mitte des Rohres, eine hinreichende Grösse erreicht hat, wodurch der Anfang eines »Kontinuums« von Molekülen im Rohr ermöglicht wird[1]; bei welchem Wert von R/λ in dem Gebiet, $\frac{R}{\lambda} \gtrsim 0$, dieser Zustand eintreten wird, kann man aber im Voraus nicht wissen.

Es wäre dann auch sehr wohl möglich, dass das Glied, $G'_{3,p}$, in dem Zustande: $R/\lambda \gtrsim 0$, nicht genau proportional mit R/λ ist, sondern eine andere Abhängigkeit von R/λ besitzt; andererseits wird die numerische Grösse des Gliedes:

$$G'_{3,p}/G'_{1,0} = \frac{3\pi}{128} \cdot \frac{2R}{\lambda},$$

1) Vgl. u. a.: J. JEANS: An introduction to the kinetic theory of Gases, chapt. IX, p. 225, 1948 (Cambridge University Press).

in dem Gebiet: $R/\lambda \geqq 0$, mit eins verglichen sehr klein, und also ohne grossen Einfluss auf die Summe aller drei Glieder.

Bevor ich hierauf näher eingehe, werde ich erst die Summe der zwei gaskinetischen Glieder, $G'_{2,p}$ und $G'_{1,p}$, in der dreigliedrigen Formel:

$$G'_p = G'_{3,p} + G'_{2,p} + G'_{1,p},$$

näher betrachten.

Wir erkennen leicht, dass die Summe $(G'_{1,p} + G'_{2,p})$ als Funktion von $\frac{R}{\lambda}$ *kein Minimum* (oder Maximum) hat, weil:

$$G'_{1,p} + G'_{2,p} = G'_{1,0} \cdot \frac{1 + \frac{\pi}{4} x}{1 + x}, \quad \text{wo} \quad x = \frac{2R}{\lambda};$$

dies erhellt u. a. daraus, dass der Differentialkoeffizient:

$$\frac{d}{dx}(G'_{1,p} + G'_{2,p}) = -G'_{1,0} \cdot \frac{1 - \frac{\pi}{4}}{(1+x)^2},$$

in dem ganzen Gebiet: $0 \leqq R/\lambda \leqq \infty$, negativ ist und erst für $x = \infty$ gleich Null wird.

Die Summe: $(G'_{1,p} + G'_{2,p})$, wird also in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, gleichmässig von $G'_{1,0}$ für $\frac{R}{\lambda} = 0$ auf $\frac{\pi}{4} \cdot G'_{1,0}$ für $\frac{R}{\lambda} = \infty$ *abnehmen.*

Dies ist auch in Übereinstimmung mit der semi-empirischen Formel von Martin Knudsen, da die Summe: $b \cdot \frac{1 + c_1 \cdot \bar{p}}{1 + c_2 \cdot \bar{p}}$, der entsprechenden Glieder in dieser Formel auch kein Minimum hat, und regelmässig mit wachsendem Wert von $\bar{p}$ *abnimmt,* weil $c_1 < c_2$.

Es erhellt aus diesem Verlauf der Summe $(G'_{1,p} + G'_{2,p})$ mit R/λ, bezw. aus der Summe der zwei entsprechenden Glieder der semi-empirischen Formel von Martin Knudsen, wie auch aus den späteren, exakten Berechnungen von $G'_{1,p}$ und $G'_{2,p}$,

dass die Entstehung des von MARTIN KNUDSEN gefundenen Minimums bei $R/\lambda =$ ca. 0,3 nach aller Wahrscheinlichkeit nicht von dem Verlauf der Gleitungs- und Diffusionsströmung, bezw. $(G'_{1,p} + G'_{2,p})$, herrührt, also auch nicht durch die gegenseitige Wechselwirkung der Moleküle in dem Gebiet: $R/\lambda \to 0$, wie von MARTIN KNUDSEN angedeutet, bewirkt ist. —

Die Entstehung des beobachteten Minimums, $G'_{\text{min.}} =$ ca. $0{,}93 \cdot G'_{1,0}$ bei $\left(\frac{R}{\lambda}\right)_{\text{min.}} =$ ca. 0,30, ist somit dadurch zu erklären, dass das hydrodynamische Glied: $G'_{3,p}$, welches proportional mit $\frac{R}{\lambda}$, bezw. $\bar{p}$, *wächst*, zu der Summe $(G'_{1,p} + G'_{2,p})$, welche gleichmässig mit R/λ, bezw. $\bar{p}$, *abnimmt*, hinzukommt. —

Da die aus der angenäherten theoretischen Formel berechneten Werte von $G'_{\text{min.}}$ und $\left(\frac{R}{\lambda}\right)_{\text{min.}}$ mit den beobachteten Werten gut übereinstimmen, wenn $G'_{3,p} = G'_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}$ gesetzt wird, darf also angenommen werden, dass die hydrodynamische, laminare Strömung, $G'_{3,p}$, bereits bei einem Wert von $R/\lambda =$ ca. 0,2 à 0,3 ihren vollen theoretischen Wert:

$$G'_{3,p} = G'_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda},$$

erreicht hat. —

Wir werden darum im folgenden von der Annahme ausgehen, dass in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$:

$$G'_{3,p} = G'_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}, \tag{11}$$

gesetzt werden kann und jedenfalls als eine brauchbare Annäherung in dem Gebiet $\frac{R}{\lambda} \geqq 0$ verwendet werden darf.

Auch MARTIN KNUDSEN hat in seiner semi-empirischen Formel (1) diese Lösung verwendet. —

Wir entnehmen also hieraus, dass die dreigliedrige, theoretische Formel in erster Annäherung geschrieben werden kann:

$$G' = G'_3 + G'_2 + G'_1 = \frac{2\pi}{3} m\bar{\Omega} R^3 \frac{dN}{dl} \left[\frac{3\pi}{128} \cdot \frac{2R}{\lambda} + \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}} + \frac{1}{1 + \frac{2R}{\lambda}} \right], \quad (12)$$

während die semi-empirische, allgemeine Formel von Martin Knudsen, wie früher erwähnt, folgendermassen lautet:

$$G' = \frac{2\pi}{3} m\bar{\Omega} R^3 \frac{dN}{dl} \left[\frac{3\pi}{128} \cdot \frac{2R}{\lambda} + \sqrt{\frac{\pi}{2}} \cdot \frac{\frac{2R}{\lambda}}{1 + 1{,}548 \frac{2R}{\lambda}} + \frac{1}{1 + 1{,}548 \cdot \frac{2R}{\lambda}} \right].$$

Wir ersehen hieraus, dass die dreigliedrige, theoretische Formel und die semi-empirische Formel von Martin Knudsen der Form nach vollständig übereinstimmen.

Aus der Formel von Martin Knudsen erhellt:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}323 \quad \text{und} \quad G'_{\text{min.}} = 0{,}952 \cdot G'_{1,0},$$

während die theoretische Formel ergibt:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}353 \quad \text{und} \quad G'_{\text{min.}} = 0{,}963 \cdot G'_{1,0}.$$

Experimentell wird im Mittel gefunden:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}310 \quad \text{und} \quad G'_{\text{min.}} = 0{,}930 \cdot G'_{1,0}.$$

Die Übereinstimmung kann als befriedigend angesehen werden, wenn in Betracht gezogen wird, dass die Zahlenfaktoren von R/λ in der theoretischen Formel Annäherungswerte sind und die semi-empirische, allgemeine Formel von Martin Knudsen eine mittlere Formel ist, d. h., dass die Zahlenfaktoren von R/λ in dieser Formel Mittelwerte sind.

Durch Änderung der Zahlenfaktoren von R/λ in den zwei gaskinetischen Gliedern der semi-empirischen Formel Martin Knudsens kann eine bessere Übereinstimmung mit den Beobachtungen der einzelnen Messreihen leicht erreicht werden; so

werden z. B. die Präzisionsmessungen für CO_2 mit dem Rohr No. 4, die in der Tabelle II wiedergegeben sind, befriedigend durch die folgende, semi-empirische Formel:

$$G' = G'_{1,0}\left[\frac{3\pi}{128}\cdot\frac{2R}{\lambda} + 2{,}247\cdot\frac{\frac{2R}{\lambda}}{1+2{,}767\cdot\frac{2R}{\lambda}} + \frac{1}{1+2{,}767\cdot\frac{2R}{\lambda}}\right],$$

dargestellt; aus dieser Formel folgt z. B.:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}300 \quad \text{und} \quad G'_{\text{min.}} = 0{,}927\cdot G'_{1,0},$$

und für $\frac{2R}{\lambda}\to\infty$: $G'_{\to\infty} = G'_{1,0}\left[\frac{3\pi}{128}\cdot\frac{2R}{\lambda} + 0{,}812\right]$,

während sich aus der graphischen Darstellung der Beobachtungsreihe ergab:

$$\left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}310, \quad G'_{\text{min.}} = 0{,}925\cdot G'_{1,0}$$

und $c_1/c_2 = 0{,}812$.

Für die Messungen mit CO_2 und dem Rohr No. 3 wurde für das Gebiet: $\frac{R}{\lambda} > 0{,}5$, gefunden:

$$G' = G'_{1,0}\left[\frac{3\pi}{128}\cdot\frac{2R}{\lambda} + 1{,}217\cdot\frac{\frac{2R}{\lambda}}{1+1{,}548\cdot\frac{2R}{\lambda}} + \frac{1}{1+1{,}548\cdot\frac{2R}{\lambda}}\right].$$

Aus dieser Formel folgt

$$\text{für } \frac{2R}{\lambda}\to\infty: \quad G'_{\to\infty} = G'_{1,0}\left[\frac{3\pi}{128}\cdot\frac{2R}{\lambda} + 0{,}786\right],$$

also: $c_1/c_2 = 0{,}786$. —

Für diese Beobachtungsreihe (CO_2 ~ Rohr 3) sind die Beobachtungen in dem Gebiet, $R/\lambda < 0{,}5$, in guter Übereinstimmung

mit der obengenannten Beobachtungsreihe mit CO_2 und Rohr No. 4 (Tabelle II).

§ 4. In der dreigliedrigen Formel (12) können wir die beiden gaskinetischen Glieder, nämlich 1° das Gleitungsglied, und 2° das Diffusionsglied, noch näher betrachten.

1°: *Das Gleitungsglied.* Aus der Formel ersehen wir, dass das Gleitungsglied, d. h. die der Gleitung wegen durchströmende Gasmenge, nicht konstant, sondern von R/λ abhängig ist.

Für $R/\lambda \to \infty$ ergibt sich, in Übereinstimmung mit dem Gleitungsglied, für die Gleitungsgeschwindigkeit der Moleküle, d. h. die konstante, zusätzliche Geschwindigkeit aller Moleküle im Rohre:

$$v_2 = \zeta \cdot \left(\frac{dv}{dr}\right)_{r \geq R} = k_2 \cdot \lambda \cdot \left(\frac{dv}{dr}\right)_{r \geq R} = k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl}.$$

Die Beobachtungen von Martin Knudsen ergeben für die untersuchten Gase in dem Gebiet: $R/\lambda \to \infty$, $k_2 = \text{ca. } \frac{4}{3}$, während man aus der Maxwell'schen Theorie, die aber nicht exakt ist, für $R/\lambda \to \infty$, $k_2 = 1$ erhält, unter der Voraussetzung, dass das Cosinusgesetz absolute Gültigkeit hat.

Es wäre möglich, dass das Gleitungsglied und das Diffusionsglied durch die vorliegende, semi-empirische Formel von Martin Knudsen nicht ganz einwandfrei aus dem Beobachtungsmaterial bestimmt sind, weil in dieser Formel für den Nenner beider gaskinetischen Glieder derselbe Wert $(1 + c_2 \cdot \bar{p})$ angenommen ist; auf den Wert von k_2 für: $R/\lambda \to \infty$, wird dies aber ohne Einfluss sein, da das Diffusionsglied in dem Gebiet: $R/\lambda \to \infty$, dem Gleitungsglied gegenüber nicht von wesentlicher Bedeutung ist.

Für $R/\lambda = 0$ wird, wenn das Cosinusgesetz gilt, das Gleitungsglied gleich Null, weil nur die Moleküle, welche von gegenseitigen Zusammenstössen mit anderen Molekülen im Rohr kommen, im Mittel eine Gleitungsgeschwindigkeit besitzen können.

Für $R/\lambda \geq 0$, in welchem Zustande nur vereinzelte, gegenseitige Zusammenstösse der Moleküle stattfinden, erhält man für diese Moleküle, wie später (S. 77) abgeleitet werden wird:

$$v_{2 \to 0} = \frac{\pi}{4} R\bar{\Omega} \cdot \frac{dN}{Ndl} = v_{x \to 0}.$$

und also wäre für $R/\lambda \geqq 0$: $k_2 = 2$, wenn wir die Formel: $v_2 = k_2 \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl}$, auch für diesen Zustand verwenden.

Es erhellt hieraus, dass experimentelle Bestimmungen von k_2 im Allgemeinen, und insbesondere in den Durchströmungsmethoden, nur dann Wert haben, wenn bei einer Bestimmung von k_2 auch angegeben wird, in welchem Bereich von R/λ die experimentelle Bestimmung von k_2 durchgeführt wurde, so dass beurteilt werden kann, ob der gefundene Wert von k_2 dem Grenzwert der Gleitungsgeschwindigkeit, $v_{2\to\infty}$, für $R/\lambda \to \infty$ entspricht.

Bei zukünftigen, experimentellen Bestimmungen von k_2 muss die Aufmerksamkeit hierauf gelenkt werden.

Es wäre darum von Bedeutung, in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, den Gleitungskoeffizienten und seine Abhängigkeit von R/λ nach den verschiedenen Methoden, die hierfür in Betracht kommen, experimentell zu untersuchen und die nach den verschiedenen Methoden erhaltenen Resultate für k_2 miteinander zu vergleichen.

Die verschiedenen Methoden[1] zur Bestimmung von k_2, kann man in zwei Gruppen einteilen, nämlich:

a) Die Methoden, in welchen ein Druck- bezw. Dichtheitsgradient vorhanden ist; zu dieser Gruppe gehören die hier behandelte Durchströmungsmethode von Poiseuille und Martin Knudsen und auch die Methode der Bestimmung des Luftwiderstandes, K, gegen die langsame Bewegung kleiner Kugeln (für $a/\lambda \to \infty$: $K = 6\pi\eta a \cdot U$, bezw. das Gesetz von Stokes), weil auch in diesem Falle eine laminare, hydrodynamische Strömung in dem die Kugel umgebenden Gase vorhanden ist.

Der Druckgradient in den Lamellen dieser laminaren, hydrodynamischen Strömung wird auch in diesen eine Selbstdiffusionsströmung verursachen und aufrechterhalten. —

In dieser Gruppe muss auf Grund der Beobachtungen eine Sonderung zwischen der Gleitungsströmung und der Selbstdiffusionsströmung vorgenommen werden, so dass k_2 aus der Gleitungsströmung bestimmt werden kann.

b) Die Methoden, in welchen kein Druckgradient von Be-

1) Vgl. an Literatur u. a.: W. P. J. Lignac: Dissertatie, Leiden, 1949.

deutung anwesend ist; hierzu gehören u. a. die Methode der in sich selbst schwingenden, kreisförmigen Scheibe (die Methode von MAXWELL) und die Methode der koaxialen, kreisförmigen Zylinder, von welchen der Aussenzylinder mit einer konstanten Geschwindigkeit um die gemeinschaftliche Achse rotiert; in dieser Methode wird durch die Rotation des Aussenzylinders der innere Zylinder abgelenkt, und wenn dieser an einem Torsionsfaden aufgehängt ist, entsteht eine stationäre Ablenkungsmethode, die sehr genaue Resultate ergeben und durch welche die Gleitung bestimmt werden kann. Sind die Radien der Zylinder R_1 und R_2 und ist: $\frac{R_2 - R_1}{\frac{1}{2}(R_2 + R_1)} \ll 1$, kann die Abhängigkeit zwischen k_2 und $\frac{R_2 - R_1}{\lambda}$ leicht experimentell bestimmt werden. —

Es wird wahrscheinlich schwierig sein, eine brauchbare Theorie für die theoretische Abhängigkeit zwischen ζ, bezw. v_2, und R/λ in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, aufzubauen, weil der Zustand des Gases in dem Gebiet $R/\lambda \geqq \infty$ hydrodynamisch ist, während der Zustand des Gases in dem Gebiet $R/\lambda \geqq 0$ rein gaskinetisch ist. LIGNAC[1] hat mit Hilfe der Methode der in sich selbst schwingenden Scheibe eine hübsche, experimentelle Untersuchung mit mehreren Gasen (He, H_2, N_2, O_2 und C_2H_4) durchgeführt. Der Verlauf der Dämpfung in dem ganzen Gebiet: $0 \leqq \frac{d}{\lambda} \leqq \infty$, wurde gemessen, und die Resultate für die Gebiete $d/\lambda \to 0$ und $d/\lambda \to \infty$ mit den vorliegenden Theorien verglichen. Die Übereinstimmung zwischen Theorie und Beobachtung in diesen Gebieten ist sehr befriedigend; das Zwischengebiet muss aber theoretisch noch näher geklärt werden. —

Eine vollständige, experimentelle Bestimmung der Variation von ζ, in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, liegt aber durch die direkten (MARTIN KNUDSEN und SOPHUS WEBER) und die indirekten (R. A. MILLIKAN, J. MATTAUCH u. A.) Untersuchungen über den Luftwiderstand, K, gegen die langsame Bewegung kleiner Kugeln (Gesetz von STOKES), vor. MARTIN KNUDSEN und

1) W. P. J. LIGNAC: loc. cit.

Sophus Weber[1] haben durch Beobachtung der Dämpfung von schwingenden Glaskugeln gefunden:

$$K = 6\pi\eta a \cdot U\left[1+\frac{\lambda}{a}\left(A+B\cdot e^{-c\cdot\frac{a}{\lambda}}\right)\right]^{-1}.$$

a ist der Radius und U die konstante Geschwindigkeit der Kugel.

Wird die Chapman'sche freie Weglänge, λ, verwendet, ergibt sich:

$$A = 1.10, \quad B = 0{,}57 \quad \text{und} \quad C = 1{,}15$$

oder annäherungsweise:

$$A = 1{,}10, \quad \frac{B}{A} = \text{ca.}\,\frac{1}{2} \quad \text{und} \quad C = 1{,}15.$$

Der Form nach ist diese Formel später von Millikan[2] und auch von Mattauch[3] völlig bestätigt worden, obwohl die Werte von Millikan und Mattauch für A, B und C etwas von den von Knudsen und Weber gefundenen Werten abweichen.

Aus dem Ausdruck für K wird leicht abgeleitet:

Für $a/\lambda \doteq 0$: $K_0 = \dfrac{3}{A+B}\cdot\varrho\overline{\Omega}\cdot\pi a^2\cdot U$, wo $A+B = 1{,}67$, während Millikan: $A+B = 1{,}65$, und Mattauch: $A+B = 1{,}73$, gefunden haben.

Für $a/\lambda \to \infty$ (Gesetz von Stokes mit Gleitungskorrektion) wird aus der Formel von K abgeleitet: $K \underset{\to\infty}{=} \dfrac{6\pi\eta a\cdot U}{1+A\cdot\lambda/a}$, wo $A = k_2$. — P. Epstein[4] hat für $\dfrac{a}{\lambda} = 0$, unter Berücksichtigung des zweiten Hauptsatzes der Wärmetheorie, und unter Voraussetzung der Gültigkeit des Cosinusgesetzes, den folgenden, theoretischen Ausdruck abgeleitet:

1) Martin Knudsen und Sophus Weber: Ann. d. Phys. (4) 36, S. 981, 1911.
2) R. A. Millikan: Phys. Rev. (2) 22, S. 1, 1923.
3) J. Mattauch: Z. Phys. (6) 32, S. 439, 1925.
4) P. Epstein: Phys. Rev., Aug. 1923, vgl. auch Martin Knudsen: Ann. d. Phys. 4, 46, S. 641, 1915.

$$\text{Für } \frac{a}{\lambda} = 0\colon \quad K_{0,\,\text{ber.}} = \left(\frac{4}{3} + \frac{\pi}{6}\right) \cdot \varrho\bar{\Omega} \cdot \pi a^2 \cdot U$$

$$= 1{,}857 \cdot \varrho\bar{\Omega} \cdot \pi a^2 \cdot U,$$

während nach dem obenstehenden:

$$K_{0,\,\text{beob.}} = \frac{3}{A + B} \cdot \varrho\bar{\Omega} \cdot \pi a^2 \cdot U$$

$$= 1{,}82 \cdot \varrho\bar{\Omega} \cdot \pi a^2 \cdot U.$$

Es scheint demnach, als ob der von KNUDSEN und WEBER eingeführte Akkommodationskoeffizient für die translatorische Energie, a_1, gleich Eins sei — jedenfalls für atmosph. Luft.

Schreiben wir, ausgehend von dem hydrodynamischen Zustande, $\frac{a}{\lambda} \to \infty$:

$$K \underset{\to \infty}{=} \frac{6\pi\eta a}{1 + k_2 \cdot \frac{\lambda}{a}} U,$$

ergibt sich also aus den Beobachtungen, dass:

$$k_2 = A + B \cdot e^{-c \cdot \frac{a}{\lambda}} = A\left(1 + \frac{B}{A} \cdot e^{-c \cdot \frac{a}{\lambda}}\right),$$

wo A, B und C Konstanten sind.

Wir erhalten hieraus:

$$\text{für } \frac{a}{\lambda} = 0\colon \quad k_{2,0} = A + B,$$

und

$$\text{für } \frac{a}{\lambda} \to \infty\colon \quad k_{2,\infty} = A,$$

woraus $\frac{k_{2,0}}{k_{2,\infty}} = \frac{A+B}{A} = \frac{3}{2}$, weil annäherungsweise $\frac{B}{A} = \frac{1}{2}$.

Es wäre in vielen Beziehungen von Bedeutung, die Messungen von KNUDSEN und WEBER mit anderen Gasen zu wiederholen, insbesondere mit H_2 und He, und auch die verwendeten Kugeln durch andere Körperformen, z. B. plane, kreisförmige Scheiben, zu ersetzen. — Auch eine Präzisionsuntersuchung mit Hilfe der

Ablenkungsmethode des rotierenden Aussenzylinders für verschiedene Gase wäre in dieser Verbindung von grossem Interesse. —

Obwohl in dieser Untersuchung über den Luftwiderstand, K, gegen die langsame Bewegung kleiner Kugeln dem Einfluss der Selbstdiffusionsströmung in den Lamellen der laminaren, hydrodynamischen Strömung des umgebenden Gases nicht Rechnung getragen worden ist, wird wohl kein grosser Fehler gemacht werden, wenn wir annehmen, dass für die Variation von k_2 mit $\frac{a}{\lambda}$ in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, die Formel:

$$k_2 = k_{2,\infty} \cdot \left(1 + \frac{1}{2} e^{-c_2 \cdot \frac{a}{\lambda}}\right),$$

verwendet werden darf. Wir werden später hierauf näher zurückkommen (S. 102).

2° *Das Diffusionsglied:* Dies neue Glied in der Formel von Poiseuille — herrührend von der Druckdiffusion, bezw. für reine Gase, von der Selbstdiffusion in den im Kapillarrohr strömenden Lamellen — ist annäherungsweise bestimmt durch das Glied:

$$G_1' = \frac{2\pi}{3} m \bar{\Omega} R^3 \cdot \frac{dN}{dl} \cdot \frac{1}{1 + \alpha_1 \cdot \frac{2R}{\lambda}} = \frac{1}{3} \bar{\Omega} \lambda \cdot m \frac{dN}{dl} \cdot \frac{\frac{2R}{\lambda}}{1 + \alpha_1 \cdot \frac{2R}{\lambda}} \cdot \pi R^2.$$

Für $R/\lambda = 0$ erhält man hieraus: $G_{1,0}' = \frac{2\pi}{3} m \bar{\Omega} R^3 \cdot \frac{dN}{dl}$, d. h. den Ausdruck für die reine Molekularströmung.

Dieser Ausdruck für $R/\lambda = 0$ kann auch folgendermassen geschrieben werden:

$$G_{1,0}' = \frac{2}{3} m \bar{\Omega} R \cdot \frac{dN}{dl} \cdot \pi R^2 = D_{11,0} \cdot m \frac{dN}{dl} \cdot \pi R^2,$$

wenn:

$$D_{11,0} = \frac{2}{3} \bar{\Omega} R.$$

Für $R/\lambda \to \infty$ wird für $\alpha_1 = 1$:

$$G'_{1,\infty} = \frac{1}{3}\bar{\Omega}\lambda m \frac{dN}{dl} \cdot \pi R^2 = D_{11,\infty} \cdot m \frac{dN}{dl} \cdot \pi R^2.$$

und für

$$\alpha_1 = S_1: \quad G'_{1,\infty} = \frac{1}{3}\bar{\Omega}\lambda \cdot \frac{1}{S_1} m \frac{dN}{dl} \cdot \pi R^2 = D'_{11,\infty} \cdot m \frac{dN}{dl} \cdot \pi R^2.$$

Der Wert, $D_{11,\infty} = \frac{1}{3}\bar{\Omega}\lambda$, stimmt mit dem Ausdruck für den Selbstdiffusionskoeffizienten, abgeleitet nach der einfachen kinetischen Theorie der Gase, überein, während $D'_{11,\infty} = \frac{1}{3}\bar{\Omega}\lambda \cdot 1/S_1 =$ ca. $0{,}60 \cdot \bar{\Omega}\lambda$, mit $S_1 = 0{,}56$, der exakten Theorie für harte, elastische Moleküle im MAXWELL'schen Zustande der Gase entspricht.

MARTIN KNUDSEN hat in seiner semi-empirischen Formel einen Mittelwert von $\alpha_1 = 1{,}548$ angegeben; dieser Wert ist aber wahrscheinlich zu klein für das Gebiet: $\frac{R}{\lambda} \geqq 0$, und zu gross für das Gebiet: $\frac{R}{\lambda} \to \infty$.

In den nachstehenden, angenäherten Berechnungen werden wir, in Übereinstimmung mit den vorhergehenden, angenäherten Berechnungen, den Mittelwert $\alpha_1 = 1$ verwenden, weil es sich im folgenden nur um eine orientierende Berechnung handelt.

Ob es überhaupt möglich sein wird, durch Analyse der beobachteten Kurve: $G' = G'_{1,0} \cdot f\left(\frac{R}{\lambda}\right)$, eine befriedigende Scheidung der beiden gaskinetischen Glieder zu erreichen, und einen befriedigend konstanten Wert für α_1, geltend für das ganze Gebiet $0 \leqq R/\lambda \leqq \infty$, festzustellen, kann erst näher untersucht werden, wenn neue Präzisionsmessungen für das ganze Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, vorliegen.

Im Allgemeinen kann aber angenommen werden, dass die Variation von $D_{11,p}$ mit R/λ sehr kompliziert ist, auch in Verbindung mit dem Übergang von dem molekularen in den MAXWELL'schen Zustand. Da ausserdem der Faktor C_1 in der exakten

Formel: $D_{11,\infty} = C_1 \bar{\Omega} \lambda$, in dem MAXWELL'schen Zustande der Gase von den intermolekularen Kräften abhängig ist, wird die Funktion $f\left(\frac{R}{\lambda}\right)$ in der Formel: $G' = G'_{1,0} f\left(\frac{R}{\lambda}\right)$, nur in erster Annäherung eine eindeutige Funktion von R/λ, geltend für alle Gase, sein; im Praxis aber sind, wie auch von MARTIN KNUDSEN experimentell gefunden, die Abweichungen zwischen den verschiedenen Gasen sehr gering, auch weil bei grösseren Werten von R/λ das Diffusionsglied, dem Gleitungsglied gegenüber, beinahe ohne zahlenmässige Bedeutung ist.

Es ist aber nicht ohne Interesse, die Aufmerksamkeit insbesondere auf dies Diffusionsglied, G'_1, in der laminaren Strömung reiner Gase zu lenken, weil man früher die Auffassung hatte, dass die Selbstdiffusion, bezw. der Selbstdiffusionskoeffizient, nur als eine mathematische Grösse angesehen werden könnte, und dass diese Grösse experimenteller Bestimmung nicht zugänglich wäre, weil es in einem reinen Gase nicht möglich ist, die diffundierenden Moleküle von den Molekülen, durch welche sie diffundieren, zu unterscheiden[1].

Man hat aber in den letzten Jahren — wahrscheinlich mit sehr grosser Annäherung — den Selbstdiffusionskoeffizienten in dem MAXWELL'schen Zustande der Gase mit Hilfe der gegenseitigen Diffusion von Isotopen desselben Gases, wovon einer der Isotopen radioaktiv ist, bestimmt; ausserdem hat man die gegenseitige Diffusion von Molekülen mit demselben Molekulargewicht, die in mehreren Zustandsformen auftreten können, wie z. B. Ortho- und Para-Wasserstoff[2], gemessen.

Man hat auch, um einen annähernden Wert des Selbstdiffusionskoeffizienten zu erhalten, die gegenseitige Diffusion komplexer Moleküle mit derselben Masse, wie z. B. N_2 und CO oder N_2O und CO_2[3], untersucht. Diese Messungen haben befriedigende Resultate und befriedigende Übereinstimmung mit der exakten Theorie der Selbstdiffusion gegeben, obwohl sie —

1) S. CHAPMAN und T. G. COWLING: The mathematical Theory of non-uniform Gases, Cambridge, S. 249, 14, 5, 1939, oder EARLE H. KENNARD: loc. cit. Seite 194, Sec. 110, wo Kennard schreibt: »accordingly, strict selfdiffusion has in reality become a notion devoid of physical meaning ...« usw.

2) P. HARTECK und H. W. SCHMIDT: Z. f. Phys. Chem. 21, S. 447, 1933.

3) L. E. BOARDMAN and N. E. WILD: Proc. Roy. Soc. A 162, S. 511, 1937.

streng theoretisch gesprochen — nur als sehr grosse Annäherungen angesehen werden können.

Das neue, dritte Glied in der Formel von POISEUILLE, bezw. G_1', beweist aber, dass der Selbstdiffusionskoeffizient der Gase eine physikalische Grösse ist, die gemessen werden kann, selbst wenn kein Unterschied zwischen den diffundierenden Molekülen und den Molekülen, wodurch die erstgenannten diffundieren, besteht; in einem laminar strömenden Gase ist die Selbstdiffusion direkter Messung zugänglich, weil die durch ein Kapillarrohr strömende Menge des reinen Gases durch die Selbstdiffusion vergrössert wird.

Ob es möglich sein wird, durch neue Präzisionsmessungen der laminaren, stationären Strömung reiner Gase durch Kapillarrohre, enge Spalten oder andere enge Kanäle eine befriedigende, experimentelle Bestimmung des Selbstdiffusionskoeffizienten und seiner Variation in dem ganzen Gebiet, $0 \leqq R/\lambda \leqq \infty$, zu erreichen, ist eine experimentelle Frage, die näher untersucht werden muss; in dieser Verbindung wären auch neue Untersuchungen über die Variation des Selbstdiffusionskoeffizienten, $D_{11,p}$, mit dem Druck, $\bar{p}$, bezw. mit R/λ, z. B. mit Hilfe der Isotopenmethode oder durch die gegenseitige Diffusion von Ortho- und Para-Wasserstoff, von grossem Interesse. Hierdurch würde sich u. a. herausstellen, welchen Wert α_1 in dem molekularen und in dem MAXWELL'schen Gebiet hat, bezw. wie der experimentelle Übergang von dem molekularen in das MAXWELL'sche Gebiet vonstatten geht.[1]

Anschliessend an das Vorhergehende können wir die Messungen mit Wasserstoff von MARTIN KNUDSEN in der Tabelle III betrachten. Wir ersehen hieraus, dass mit einem Kapillarrohr: Länge $L = 29{,}81$ cm und Radius $9{,}729.10^{-3}$ cm, bei der Temperatur, $t = 26°25$ C, gefunden wurde, dass $T_0 = 0.01148$ für $\frac{R}{\lambda} = 0$, und $T = 0{,}01123$ bei dem Druck: $\bar{p} = 970{,}2$ Bar., bezw. $R/\lambda = 0{,}75$, sind.

1) Vgl.: I. O. HIRSCHFELDER, R. BYRON BIRD & ELLEN L. SPOTZ: Chemical Reviews, 44, S. 205, 1949.

Hieraus erhält man mit Hilfe der Formel von MARTIN KNUDSEN das Diffusionsglied:

$$G'_{1,p} = T_0 \cdot {}_1\varrho \cdot \frac{1}{1 + \alpha_1 \cdot \frac{2R}{\lambda}} \cdot$$

Für die durchströmende Gasmenge, herrührend von der Selbstdiffusion, ergibt sich hieraus:

$$T_{0,\text{obs.}} \cdot {}_1\varrho \cdot \frac{1}{1 + \alpha_1 \cdot \frac{2R}{\lambda}} (p_1 - p_2) = D_{11,p}\, m \frac{dN}{dl} \cdot \pi R^2$$

oder:

$$D_{11,p} = \frac{L}{\pi R^2} T_{0,\text{obs.}} \cdot \frac{1}{1 + \alpha_1 \cdot \frac{2R}{\lambda}}, \quad \text{weil} \quad m \cdot \frac{dN}{dl} = {}_1\varrho \cdot \frac{p_1 - p_2}{L} \cdot$$

Verwenden wir, in Übereinstimmung mit den angenäherten Berechnungen, $\alpha_1 = 1$, erhalten wir hieraus — annäherungsweise — für Wasserstoff bei dem Druck $p = 970{,}2$ Bar., bezw. $\frac{R}{\lambda} = 0{,}75$, und $t = 26°25$ C:

$$\underline{D_{11,p} = 463 \text{ cm}^2/\text{sek.}}$$

Gehen wir andererseits aus von dem Wert des Selbstdiffusionskoeffizienten von Wasserstoff im MAXWELL'schen Zustande der Gase, bezw. dem gemessenen Wert der gegenseitigen Diffusion von Ortho- und Para-Wasserstoff:

$$D'_{11,p_0} = 1{,}28 \text{ cm}^2/\text{sek}, \text{ bei } t = 0°\text{C} \text{ und } p_0 = 760 \text{ mm } Hg,$$

erhalten wir bei 26°25 C und 760 mm Druck $= p_0$:

$$D'_{11,p_0} = 1{,}28 \left(\frac{T}{T_0}\right)^{1{,}69} = 1{,}495 \text{ cm}^2/\text{sek.}, \text{ bei } t = 26°{,}25 \text{ C.}$$

Von diesem Wert für D'_{11,p_0} ausgehend, können wir für dasselbe Rohr den Wert für $D_{11,p}$ bei dem Druck $p = 970{,}2$ Bar., bezw. $R/\lambda = 0{,}75$, schätzungsweise bestimmen.

Nehmen wir an, dass der Diffusionszustand des Gases im Rohr bei $R/\lambda = 0{,}75$ dem molekularen Zustand des Gases näher

steht als dem MAXWELL'schen Zustand, können wir schreiben:

$$D_{11,p} = D'_{11,760\,\mathrm{mm}} \cdot \frac{1{,}0108 \cdot 10^6}{970{,}2} \cdot \frac{1}{1 + \alpha_2 \cdot \frac{\lambda}{2R}} \cdot S_1,$$

wo $S_1 = 0{,}56$. Verwenden wir auch in diesem Ausdruck die Annäherung: $\alpha_2 = 1$, erhält man damit für $\frac{R}{\lambda} = 0{,}75$ und $t = 26°25$ C:

$$\underline{D_{11,p} = 525 \text{ cm}^2/\text{sek.}}$$

Es erhellt daraus, obwohl es sich hier nur um eine Schätzung handelt, weil die verwendeten Werte von α_1 und α_2 wahrscheinlich nur grobe Annäherungen sind, dass bezüglich der Grössenordnung der zwei Werte von $D_{11,p}$ Übereinstimmung vorliegt.

In den vorhergehenden Betrachtungen bin ich im Allgemeinen von dem Zustande des Gases, charakterisiert durch: $R/\lambda \to \infty$, also von dem MAXWELL'schen Zustande des Gases, ausgegangen.

Es ist aber möglich, von dem Zustande: $\frac{R}{\lambda} \geqq 0$, auszugehen, und es zeigt sich dann, dass es möglich ist, die zwei gaskinetischen Glieder in der Formel von POISEUILLE — jedenfalls für den Zustand: $\frac{R}{\lambda} \to 0$, theoretisch zu berechnen; es wird sich aber auch durch die folgenden Berechnungen zeigen, dass die Variation von $D_{11,p}$ mit R/λ, bezw. mit dem Druck $\bar{p}$, für das Gebiet: $R/\lambda \to 0$, nicht ganz so einfach ist, wie es hier der besseren Übersicht wegen angenommen worden ist.

§ 5. Wir werden nun — auch als Einleitung zu den folgenden Berechnungen — die Theorie der reinen Molekularströmung betrachten. —

Für ein kreisförmiges Rohr mit Radius, R, und Länge, L, wird der rein molekulare Zustand durch: $\lambda \to \infty$, bezw. $\frac{R}{\lambda} = 0$, charakterisiert, d. h. im Inneren des Rohres kommen keine gegenseitigen Zusammenstösse der Moleküle vor. Ist ausserdem die Bedingung: $R/L \ll 1$, erfüllt, kommen alle Moleküle im

Rohr — praktisch gesprochen — von Stössen gegen die Rohrwand her, weil der Einfluss der Endfläche des Rohres dann vernachlässigt werden kann.

Die Voraussetzungen, von welchen wir im folgenden ausgehen, sind:

1° Die gegen eine feste Wand stossenden Moleküle werden nach dem Cosinusgesetz zurückgeworfen, d. h. jedes gegen die Wand anprallende Molekül hat dieselbe Wahrscheinlichkeit, in jedem beliebigen Azimuth zurückgeworfen zu werden; die Wahrscheinlichkeit eines gegebenen Ausfallswinkels ist durch das Cosinusgesetz gegeben. —

2° Für alle von einem Oberflächenelement der Wand, dS', zurückgeworfenen Moleküle, jedes mit der Masse, m, ist die Geschwindigkeitsverteilung durch das MAXWELL'sche Geschwindigkeitsverteilungsgesetz gegeben.

Aus 1° und 2° folgt, dass die gesamte Bewegungsgrösse der Wand entlang, herrührend von den von dem Element, dS', *zurückgeworfenen* Molekülen, immer Null wird, auch im Falle, dass die auf das Element, dS', *einfallenden* Moleküle eine gemeinschaftliche, konstante Massengeschwindigkeit, w, in der Richtung der Oberfläche, besitzen, oder nicht mit der Wand in Temperaturgleichgewicht sind[1]. —

3° Ist das Gas als Gesamtheit in Ruhe und in Temperaturgleichgewicht mit der Wand, wird auch die von den *einfallenden* Molekülen an die Wand übertragene Bewegungsgrösse, in der Richtung der Oberfläche, gleich Null werden. —

Ist das Gas in Strömung mit einer konstanten Strömungsgeschwindigkeit, parallel mit der Wand, bleibt ein Komposant der von den *einfallenden* Molekülen an die Wand übertragenen Bewegungsgrösse in der Richtung der Oberfläche, bezw. in der Richtung des Stromes, als Rest übrig. — Besitzen die *einfallenden* Moleküle eine gemeinschaftliche, konstante Massengeschwindigkeit (zusätzliche Geschwindigkeit), w, in der Richtung des Stromes, wird der Komposant der Bewegungsgrösse in der Richtung der Wand: $\frac{1}{4} N \bar{\Omega} \cdot w dS'$.

1) MARTIN KNUDSEN: Ann. d. Phys., 48, S. 1113, 1915.

4° In dem stationären Strömungszustand ist der Druckgradient in dem zylindrischen, kreisförmigen Rohr konstant und also: $dp/dl = (p_1 - p_2)/L$, jedenfalls solange dp/dl klein ist; hieraus folgt, dass auch dN/dl konstant ist, wenn die Temperatur der Rohrwand überall dieselbe ist. —

Für das eigentliche Problem der Molekularströmung, d. h. die Bestimmung der durchströmenden Gasmenge in dem stationären Zustand, ist es nicht notwendig, die Geschwindigkeitsverteilung der zurückgeworfenen Moleküle zu kennen; dies erhellt u. a. aus der abgeleiteten Formel:

$$n \cdot m = \frac{2\pi}{3} \cdot m\bar{\Omega} R^3 \cdot \frac{dN}{dl} = \frac{2\pi}{3} \cdot \bar{\Omega} R^3 \cdot \frac{d\varrho}{dl},$$

oder:

$$n \cdot m = \frac{2\pi}{3} \cdot \bar{\Omega} R^3 {}_1\varrho \cdot \frac{dp}{dl} = \frac{2\pi}{3} \cdot \bar{\Omega}_{1\varrho} R^3 \cdot \frac{p_1 - p_2}{L},$$

weil hierin nur die mittlere Geschwindigkeit der Moleküle im Rohr, $\bar{\Omega}$, vorkommt; dies wird auch bestätigt dadurch, dass es möglich ist, diese Formel durch Wahrscheinlichkeitsberechnungen abzuleiten, wie von P. Clausing[1] und R. Darbord[2] dargetan.

Wünscht man aber eine Bestimmung der durch die Molekularströmung transportierten Energie und Bewegungsgrösse, oder führt man in die kinetischen Formeln thermodynamische Grössen ein, um die beobachteten Werte mit den theoretischen Werten zu vergleichen, wird es notwendig, ein Verteilungsgesetz der Molekulargeschwindigkeiten einzuführen.

Wenn die Moleküle nicht unendlich klein sind, so dass gegenseitige Stösse vorkommen, und wird — jedenfalls annäherungsweise — angenommen, dass sich das Gas im Rohr thermisch und mechanisch im Gleichgewicht befindet, wird das Verteilungsgesetz[3] das Maxwell'sche sein müssen, woraus folgt, dass:

$$\frac{\pi}{8} \cdot \bar{\Omega}^2 = \frac{1}{3} \cdot \overline{\Omega^2}.$$

1) P. Clausing: Dissertatie, Leiden, S. 106, 1928. — Physica, Bd. 9, S. 65, bezw. 75, 1929.

2) R. Darbord: Journ. de Physique, VII, S. 345, 1932

3) Martin Knudsen: The Kinetic Theory of Gases, London, 1934.

Im folgenden, wo wir versuchen wollen, den Einfluss vereinzelter, gegenseitiger Zusammenstösse auf die stationäre Molekularströmung zu berechnen, werden wir also das MAXWELL'sche Verteilungsgesetz für die Molekulargeschwindigkeiten benutzen.

Ausgehend von den obengenannten Voraussetzungen hat MARTIN KNUDSEN[1] auf einwandfreie Weise durch Addition der Anzahl der Moleküle, welche in der Zeiteinheit durch den Querschnitt des kreisförmigen Rohres, πR^2, gehen, die folgende Formel für die durch ein kreisförmiges Rohr strömende Gasmenge, gemessen durch das Produkt von Druck und Volumen,

$$Q_t = \frac{n \cdot m}{1\varrho} = \frac{4}{3}\sqrt{2\pi} \cdot \frac{1}{\sqrt{1\varrho}} \cdot R^3 \cdot \frac{p_1 - p_2}{L},$$

abgeleitet.

Später hat M. VON SMOLUCHOWSKI[2] diese Ableitung verallgemeinert und näher beleuchtet, indem er darauf hingewiesen hat, dass die durch ein Element, dS, des Querschnittes, πR^2, durchströmende Menge, dQ_t, nicht über den Querschnitt des kreisförmigen Rohres konstant ist, sondern von der Entfernung der Rohrachse abhängt. —

Im folgenden werde ich in etwa der Darstellung von J. TER HEERDT[3] folgen:

Wir wollen ein sehr langes, zylindrisches Rohr mit beliebigem Querschnitt, für welches die Bedingungen: $\lambda \to \infty$, bezw. $a/\lambda = 0$, und $\frac{a}{L} \ll 1$, erfüllt sind, betrachten; a bezeichnet hier die mittlere oder grösste lineäre Abmessung des Querschnitts. Für ein solches Rohr darf man den Einfluss der Endfläche vernachlässigen.

In dem stationären Strömungszustand muss, der Erhaltung der Masse und Bewegungsgrösse wegen, das folgende gelten:

a. Jedes Wandelement des Rohres, dS', muss in der Zeiteinheit dieselbe Anzahl von Molekülen empfangen und ausstrahlen, d. h. dieselbe Anzahl von Molekülen, welche in der Zeiteinheit gegen das Wandelement, dS', stossen, muss auch in der Zeiteinheit von dS' ausgestrahlt werden. —

1) MARTIN KNUDSEN: loc. cit. (1), S. 105—108.
2) M. VON SMOLUCHOWSKI: Ann. d. Phys., 33, S. 1559, 1910.
3) J. TER HEERDT: Dissertatie, Utrecht, S. 44—48, 1923.

b. Die an die Rohrwand übertragene Bewegungsgrösse in der Richtung der Achse muss der Differenz in Bewegungsgrösse der durch die Endfläche ein — und ausströmenden Moleküle gleich sein. —

Unter Berücksichtigung des Cosinusgesetzes ergibt sich, dass ein Wandelement dS' (vgl. Fig. 4), für welches $l = 0$ gesetzt

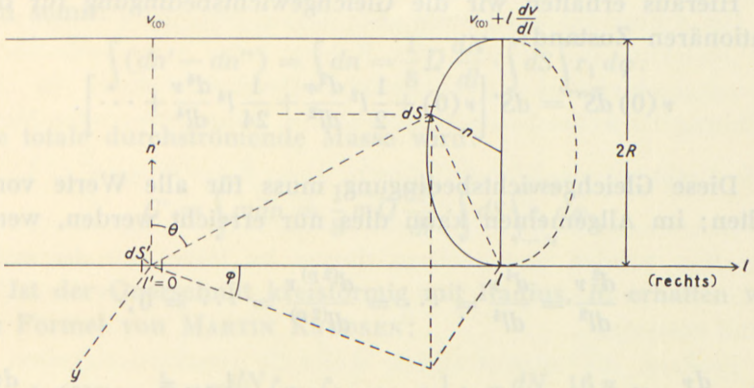


Fig. 4.

wird, in der Zeiteinheit eine Anzahl von Molekülen: $\frac{1}{4} N \bar{\Omega} dS' = \nu(0)\, dS'$, ausstrahlt; andererseits wird das Element, dS', von einer Anzahl von Molekülen, dn_1, die von der rechten Seite des Rohres kommen, getroffen, wo:

$$dn_1 = dS' \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \nu(l) \cos\theta \sin\theta \, d\theta \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\varphi \, ;$$

$\nu(l)$ ist der Wert von $\frac{1}{4} N \bar{\Omega}$ im Abstande l.

Von der linken Seite des Rohres empfängt dS' die Anzahl:

$$dn_2 = dS' \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \nu(-l) \cos\theta \sin\theta \, d\theta \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\varphi \, ,$$

und also, wenn wir für $\nu(\pm l)$ schreiben:

$$\nu(\pm l) = \nu(0) \pm \frac{l}{1!}\frac{d\nu}{dl} + \frac{l^2}{2!}\frac{d^2\nu}{dl^2} \pm \frac{l^3}{3!}\frac{d^3\nu}{dl} + \cdots,$$

$$dn_1 + dn_2 = dS'\left[\nu(0) + \frac{1}{2}l^2\frac{d^2\nu}{dl^2} + \frac{1}{24}l^4\frac{d^4\nu}{dl^4} + \cdots\right].$$

Hieraus erhalten wir die Gleichgewichtsbedingung für den stationären Zustand:

$$\nu(0)\,dS' = dS'\left[\nu(0) + \frac{1}{2}l^2\frac{d^2\nu}{dl^2} + \frac{1}{24}l^4\frac{d^4\nu}{dl^4} + \cdots\right].$$

Diese Gleichgewichtsbedingung muss für alle Werte von l gelten; im Allgemeinen kann dies nur erreicht werden, wenn:

$$\frac{d^2\nu}{dl^2} = \frac{d^4\nu}{dl^4} = \cdots = \frac{d^{(2p)}\nu}{dl^{(2p)}} = \cdots = 0,$$

oder $\frac{d\nu}{dl}$ = konstant, und also: $\nu(\pm l) = \frac{1}{4}N'\bar{\Omega} = \nu(0) \pm \frac{d\nu}{dl}l$.

Ist die Temperatur der Rohrwand konstant, wird $\frac{dN}{dl}$, bezw. $\frac{dp}{dl}$, konstant für das ganze Rohr. Dies Resultat ist in Übereinstimmung mit der Voraussetzung 4°, Seite 49.

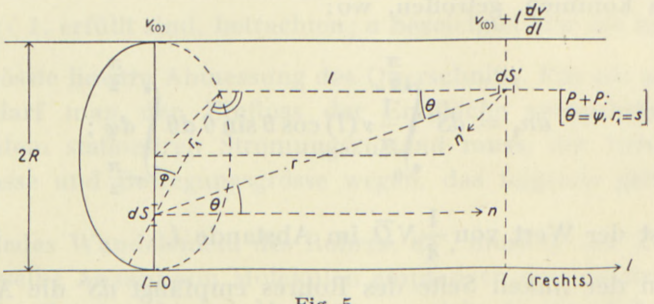


Fig. 5.

Betrachten wir (vgl. Fig. 5) ein Element, dS, in dem Querschnitt, $l = 0$, ergibt sich auf analoge Weise, weil $l = r_1 \cot\theta$, dass die Anzahl der Moleküle, die von rechts, bezw. links, kommen und das Element, dS, durchkreuzen, beträgt:

$$dn' = dS\left[\nu(0) + \frac{1}{\pi}\frac{d\nu}{dl}\int_0^{\frac{\pi}{2}} \cot\theta \cdot \sin\theta\cos\theta\, d\theta \int_{-\pi}^{+\pi} r_1\, d\varphi\right]$$

und

$$dn'' = dS\left[\nu(0) - \frac{1}{\pi}\frac{d\nu}{dl}\int_0^{\frac{\pi}{2}} \cot\theta \cdot \sin\theta\cos\theta\, d\theta \int_{-\pi}^{+\pi} r_1\, d\varphi\right],$$

und somit:

$$\int(dn' - dn'') = \int dn = \frac{1}{8}\bar{\Omega}\frac{dN}{dl}\cdot\int dS\int_{-\pi}^{+\pi} r_1\, d\varphi.$$

Die totale durchströmende Masse wird:

$$G' = \int m dn = \frac{1}{8} m\bar{\Omega}\frac{dN}{dl}\cdot\int dS\int_{-\pi}^{+\pi} r_1\, d\varphi.$$

Ist der Querschnitt kreisförmig mit Radius, R, erhalten wir die Formel von Martin Knudsen:

$$G' = \frac{1}{8} m\bar{\Omega}\frac{dN}{dl}\int dS\int r_1\, d\varphi = \frac{1}{8} m\bar{\Omega}\frac{dN}{dl}\cdot\frac{16\pi}{3}R^3,$$

oder:

$$G' = \frac{2\pi}{3} m\bar{\Omega}R^3\frac{dN}{dl}.$$

Betrachten wir ein Element, dS_1, am Rande des kreisförmigen Querschnitts, bekommen wir, weil $r_1 = 2R\cos\varphi$:

$$dG_r' = dS_1\cdot\frac{1}{2} m\bar{\Omega}R^3\frac{dN}{dl},$$

und für ein Element, dS_2, in der Mitte des kreisförmigen Querschnitts, da $r_1 = R$:

$$dG_m' = dS_2\cdot\frac{\pi}{4} m\bar{\Omega}R^3\frac{dN}{dl}.$$

Diese Variation über den Querschnitt ist zuerst von M. von Smoluchowski dargelegt worden. —

Wird für die durchströmenden Moleküle eine mittlere, hypothetische Strömungsgeschwindigkeit, v, eingeführt und wird diese definiert durch:

$$dG' = m \cdot N \cdot v \cdot dS,$$

erhält man:

$$v_{\text{Mitte}} = \frac{\pi}{4} R\bar{\Omega} \frac{dN}{Ndl},$$

$$v_{\text{Rand}} = \frac{1}{2} R\bar{\Omega} \frac{dN}{Ndl}$$

und $v_{\text{Mittel}} = \frac{2}{3} R\bar{\Omega} \frac{dN}{Ndl}$, gültig für den ganzen Querschnitt. —

Die Aufmerksamkeit muss aber darauf gerichtet werden, dass die Moleküle im Inneren des Rohres *keine* gemeinschaftliche Massengeschwindigkeit (zusätzliche Geschwindigkeit) in der Richtung der Achse besitzen, weil alle im Rohr anwesenden Moleküle direkt von der Wand kommen, wenn $R/\lambda = 0$ und $\frac{a}{L} \ll 1$; an der Rohrwand sind diese Moleküle nach dem Cosinusgesetz zurückgeworfen worden und besitzen darum im Mittel nur die molekulare Geschwindigkeit, die der konstanten Temperatur der Rohrwand entspricht.

Besitzen die ausgestrahlten Moleküle Geschwindigkeiten in Übereinstimmung mit dem MAXWELL'schen Geschwindigkeitsverteilungsgesetz, werden die Moleküle mit den grössten Geschwindigkeiten am schnellsten durch das Rohr kommen. —

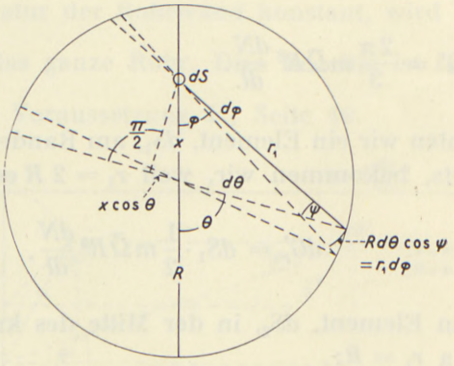


Fig. 6.

Die rein molekulare Strömung ist, auch in Verbindung mit dem konstanten Dichtheitsgradienten, dN/dl, als eine Diffusionsströmung mit einem von dem Druck unabhängigen Diffusionskoeffizienten aufzufassen; in diesem Falle, d. h. $R/\lambda = 0$, wird der Widerstand *nicht* durch ein anderes Gas, sondern durch die Wand geleistet. —

Für einen kreisförmigen Querschnitt des zylindrischen Strömungsrohres ist es nicht ohne Interesse zu untersuchen, auf welche Weise sich v und $\frac{dv}{dx}$, wenn x die Entfernung des Elementes dS von der Achse angibt, (vgl. Fig. 6) mit dem Wert von x ändern. —

Integrieren wir den Ausdruck für dG' über konzentrische Kreisringe, erhalten wir leicht, weil:

$$r_1 \, d\varphi = R \cdot d\theta \cdot \cos\psi,$$

$$dG' = \frac{1}{8} m \bar{\Omega} \frac{dN}{dl} dS \cdot \int_{-\pi}^{+\pi} \frac{R + x\cos\theta}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}} d\theta = dS \cdot mN \cdot v_x$$

und also:

$$v_x = \frac{1}{8} \bar{\Omega} R \frac{dN}{Ndl} \cdot \int_{-\pi}^{+\pi} \frac{R + x\cos\theta}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}} d\theta,$$

oder, wenn $\theta = 2\theta_1$, $k^2 = \frac{4Rx}{(R+x)^2}$ und $\frac{R+x}{2R} = \varepsilon$:

$$v_x = \frac{1}{2} \bar{\Omega} R \frac{dN}{Ndl} \left[\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\theta_1}} d\theta_1 + \right.$$

$$\left. + \varepsilon \left(\int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta_1} \, d\theta_1 - \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\theta_1}} d\theta_1 \right) \right],$$

und somit:

$$v_x = \frac{1}{2} \bar{\Omega} R \frac{dN}{Ndl} \left[(1 - \varepsilon) K + \varepsilon \cdot E \right] = \frac{1}{2} \bar{\Omega} R \frac{dN}{Ndl} \cdot \varepsilon \left[k' \cdot K + E \right],$$

wenn: $k'^2 = 1 - k^2$, und also: $k' = \frac{1-\varepsilon}{\varepsilon} = \frac{R-x}{R+x}$.

K und E sind die vollständigen, elliptischen Integrale 1^{ster} und 2^{ter} Gattung; der Modul, $k = \sin\alpha$, ist gegeben durch $\sin^2\alpha =$

$4R \cdot \frac{x}{(R+x)^2}$, während der komplementäre Modul, k', wird:

$k' = \frac{R-x}{R+x}$, da $k'^2 = 1 - k^2$, und also: $1 - \varepsilon = k' \cdot \varepsilon$.

Durch diesen Ausdruck für v_x ist der Wert von v in jeder Entfernung, x, von der Achse bestimmt und kann mit Hilfe der bekannten Tabellen[1] für K und E berechnet werden; in Fig. 7

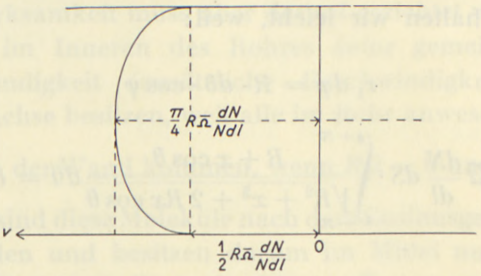


Fig. 7.

ist der Verlauf von v für einen kreisförmigen Querschnitt dargestellt.

Für die Mitte des Querschnitts ($x = 0$, $\varepsilon = \frac{1}{2}$, $k = 0$ und $k' = 1$) und für den Rand des Querschnitts ($x = R$, $\varepsilon = 1$, $k = 1$ und $k' = 0$) erhalten wir selbstverständlich dieselben Werte, wie oben auf direkte Weise abgeleitet.

Verwenden wir die bekannten Reihenentwicklungen für K und E für $x \to 0$, bezw. $k \to 0$, oder für $x \doteq R$, bezw. $k' \to 0$, erhalten wir den Verlauf von v in der Mitte und am Rande des Querschnitts. —

Wir finden z. B. an der Rohrwand, bezw. $k' \to 0$:

$$\lim v_{x \to R} = \frac{1}{4} \bar{\Omega} R \frac{dN}{Ndl} \left[1 + \frac{x}{R}\right] \left[1 + \Delta k' + \frac{1}{2}\left(\Delta - \frac{1}{2}\right) k'^2 + \frac{1}{4}(\Delta - 1) k'^3 + \cdots\right],$$

wo:

$$\Delta = ln\left(\frac{4}{k'}\right).$$

1) Siehe z. B.: Tables of Functions by Eugen Jahnke and Fritz Emde, Fourth Edition, Dover Publications, New York, p. 73, 1945.

Auf ganz analoge Weise ergibt sich:

$$\frac{dv}{dx} = -\frac{1}{4}\bar{\Omega}\frac{dN}{Ndl}\cdot\frac{1}{k^2\varepsilon}\left[(2-k^2)K - 2E\right],$$

woraus folgt:

$$\left(\frac{dv}{dx}\right)_{\text{Rand}} = \infty \quad \text{und} \quad \left(\frac{dv}{dx}\right)_{\text{Mitte}} = 0;$$

das letzte Resultat war auch aus Symmetriegründen zu erwarten.

Es ist einfach, durch die Annäherungsformeln für K und E für $x \doteq R$, bezw. $k' \to 0$, den Verlauf von $\frac{dv}{dx}$ an der Rohrwand abzuleiten; man erhält, weil:

$$k^2\cdot\varepsilon = \frac{2x}{R+x} \quad \text{und} \quad \Delta = ln\left(\frac{4}{k'}\right),$$

$$\left(\frac{dv}{dx}\right)_{x\doteq R} = -\frac{1}{4}\bar{\Omega}\frac{dN}{Ndl}\cdot\frac{R+x}{2x}\left[\Delta - 2 + \frac{1}{4}(\Delta+1)k'^2 + \frac{1}{64}\left(\Delta-\frac{1}{2}\right)k'^4\cdots\right].$$

Es erhellt aus diesen Formeln, dass der Verlauf von v und $\frac{dv}{dx}$ an der festen Wand, selbst im rein molekularen Zustand der Gase sehr kompliziert ist, so dass es nicht überraschen kann, wenn es für den Maxwell'schen Zustand der Gase bis jetzt noch nicht gelungen ist, die Geschwindigkeitsverteilung für ein längs einer festen Wand strömendes Gas abzuleiten. —

In § 2 ist erwähnt worden, dass der kinetische Zustand des Gases an der festen Wand — auch in dem Maxwell'schen Zustand der Gase — innerhalb eines Abstandes von der Wand, entsprechend der freien Weglänge, λ, oder einem Bruchteil hiervon, analog und vergleichbar ist mit dem Zustand an der Wand in der reinen Molekularströmung. Besteht also in dem Gase ein Konzentrationsgradient, $\frac{dN}{dl}$, parallel mit der Wand, und gilt für die an der Wand zurückgeworfenen Moleküle das Cosinusgesetz, können wir auf Grund dieser Analogie die folgende Schätzung für k_2 vornehmen:

$$v_{x=R} = \frac{1}{2}\bar{\Omega}R \cdot \frac{dN}{Ndl} = k_2 \cdot \frac{\pi}{8}\bar{\Omega}R \cdot \frac{dN}{Ndl},$$

oder:

$$k_2 = \frac{4}{\pi} = \text{ca. } 1{,}27.$$

Die durch ein Element im Querschnitt, dS, strömenden Moleküle transportieren in der Richtung der Achse eine Bewegungsgrösse, dM, und ausserdem eine Energie, dE.

Die von rechts kommenden Moleküle transportieren durch, dS, eine Bewegungsgrösse, dM_r, wo:

$$dM_r = dS\left[\frac{1}{6}Nm\overline{\Omega^2} + \frac{1}{6\pi} \cdot m\overline{\Omega^2}\frac{dN}{dl}\int_{-\pi}^{+\pi} r_1\, d\varphi\right],$$

und die von links kommenden Moleküle eine Bewegungsgrösse, dM_l, wo:

$$dM_l = dS\left[\frac{1}{6}Nm\overline{\Omega^2} - \frac{1}{6\pi}m\overline{\Omega^2}\frac{dN}{dl}\int_{-\pi}^{+\pi} r_1\, d\varphi\right].$$

Aus der Summe:

$$dM_r + dM_l = \frac{1}{3}Nm\overline{\Omega^2}\, dS = p\, dS,$$

geht hervor, dass der Druck p überall im Querschnitt derselbe ist und also auch der Wert von N, wenn die Temperatur konstant ist.

Aus der Differenz: $dM_r - dM_l$, erhellt, dass die totale Bewegungsgrösse, die durch den ganzen Querschnitt, $\int dS$, strömt:

$$\int dM = \int d(M_r - M_l) = \frac{1}{3\pi}m\overline{\Omega^2}\frac{dN}{dl}\int dS\int r_1\, d\varphi,$$

konstant ist, weil $\dfrac{dN}{dl}$ konstant ist, und also unabhängig von l, bezw. unabhängig von der Lage des Querschnitts im Rohr.

Die Bewegungsgrösse pro Molekül wird für die durchströmenden Moleküle:

$$\frac{\frac{1}{3\pi} m \overline{\Omega^2} \frac{dN}{dl} \cdot dS \int r_1 \, d\varphi}{\frac{1}{8} \overline{\Omega} \frac{dN}{dl} \cdot dS \int r_1 \, d\varphi} = \frac{8}{3\pi} m \frac{\overline{\Omega^2}}{\overline{\Omega}}.$$

Dieser Ausdruck wird, wenn das MAXWELL'sche Verteilungsgesetz gilt:

$$\frac{8}{3\pi} \cdot m \frac{\overline{\Omega^2}}{\overline{\Omega}} = \frac{8}{3\pi} m \cdot \frac{3\pi}{8} \frac{\overline{\Omega}^2}{\overline{\Omega}} = m \, \overline{\Omega}.$$

Hieraus erhellt, dass in der Molekularströmung die durch ein Element des Querschnitts strömenden Moleküle im Mittel eine Geschwindigkeit in der Richtung der Achse, $\overline{\Omega}$, besitzen.

Für die mittlere Energie der durchströmenden Moleküle erhalten wir in derselben Weise:

$$dE = \frac{1}{4\pi} m \overline{\Omega^3} \frac{dN}{dl} \, dS \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \int r_1 \, d\varphi,$$

woraus:

$$dE = \frac{1}{16} m \overline{\Omega^3} \frac{dN}{dl} \cdot dS \int r_1 \, d\varphi,$$

oder pro Molekül:

$$E_1 = \frac{1}{2} m \frac{\overline{\Omega^3}}{\overline{\Omega}}.$$

Wenn das MAXWELL'sche Verteilungsgesetz gilt, ist aber $\overline{\Omega^3} = \frac{4}{3} \overline{\Omega} \cdot \overline{\Omega^2}$, und demnach:

$$E_1 = \frac{1}{2} m \frac{\overline{\Omega^3}}{\overline{\Omega}} = \frac{1}{2} m \cdot \frac{4}{3} \overline{\Omega^2} = \frac{4}{3} \cdot \frac{3}{2} kT = 2 \, kT,$$

ein Resultat, das ich bereits früher abgeleitet habe[1].

Wir werden nun durch eine direkte Berechnung die Bewegungsgrösse, welche ein Element der Rohrwand, dS', von den strömenden Molekülen empfängt, ableiten. —

1) SOPHUS WEBER: Z. f. Phys., Bd. 24, S. 267, 1924.

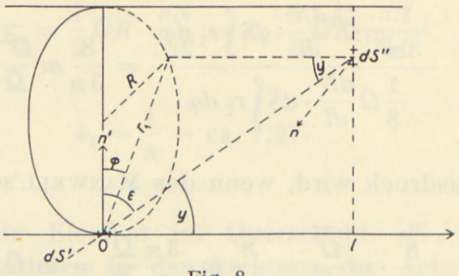


Fig. 8.

Das Wandelement, dS', (vgl. Fig. 8) wird von einer Anzahl von Molekülen, von rechts, bezw. links kommend, getroffen:

Die Anzahl von rechts wird:

$$dq_r \cdot dS' = \frac{1}{4\pi}\left(N_0\,\overline{\Omega} - \frac{d(N\Omega)}{dl}\cdot l\right)\cos\varepsilon \sin y\, dy\, d\varphi \cdot dS',$$

und von links:

$$dq_l \cdot dS' = \frac{1}{4\pi}\left(N_0\,\overline{\Omega} - \frac{d(N\Omega)}{dl}\, l\right)\cos\varepsilon \sin y\, dy\, d\varphi \cdot dS'.$$

Aus der Summe:

$$dS' \int d\left(q_r + q_l\right) = \frac{1}{2\pi} N_0\,\overline{\Omega} \cdot dS' \int \cos\varepsilon \sin y\, dy\, d\varphi,$$

ergibt sich, weil:

$$\cos\varepsilon = \sin y \cos\varphi,$$

$$\left(q_r + q_l\right) dS' = \frac{1}{2\pi} N_0\,\overline{\Omega}\, dS' \int_0^{\frac{\pi}{2}} \sin^2 y\, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos\varphi\, d\varphi$$

$$= \frac{1}{4} N_0 \overline{\Omega}\, dS' = \nu(0)\, dS',$$

in Übereinstimmung damit, dass das Element, dS', dieselbe Anzahl von Molekülen empfängt und ausstrahlt (vgl. Bedingung *a*, Seite 50).

Diese Moleküle geben dem Element dS' in der Richtung der Normale von dS' die Bewegungsgrösse:

$$dM_n = dS' \int m\Omega \cdot \frac{1}{2\pi} N_0 \Omega \cdot \cos^2 \varepsilon \sin y \, dy \, d\varphi$$

$$= dS' \cdot \frac{1}{2\pi} m N_0 \overline{\Omega^2} \int_0^{\frac{\pi}{2}} \sin^3 y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2 \varphi \, d\varphi$$

$$= \frac{1}{6} m N_0 \overline{\Omega^2} \, dS' = \frac{1}{2} p \, dS',$$

wozu eine weitere Bewegungsgrösse, $\frac{1}{2} p \, dS'$, nach Zurückwerfung der Moleküle kommt, wie auch zu erwarten war.

In der Richtung der Achse erhält das Wandelement dS' die Bewegungsgrösse:

$$dM_A = dS' \cdot \frac{1}{2\pi} m \overline{\Omega^2} \cdot \frac{dN}{dl} \cdot \int 2 R \cos^2 y \, dy \cos \varepsilon \cos \varphi \, d\varphi,$$

$$\text{weil:} \quad l = 2 R \cos \varphi \cot y.$$

Da: $\cos \varepsilon = \sin y \cdot \cos \varphi$, erhält man:

$$dM_A = dS' \cdot \frac{1}{2\pi} m \overline{\Omega^2} \cdot \frac{dN}{dl} \int_0^{\frac{\pi}{2}} 2 R \cos^2 y \sin y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2 \varphi \, d\varphi,$$

oder:

$$dM_A = B \, dS' = \frac{1}{6} m \overline{\Omega^2} \cdot \frac{dN}{dl} \cdot R \, dS',$$

und also für ein Ringelement der Wand:

$$dS' = 2\pi R \cdot dl,$$

$$dM_A = B \, dS' = \frac{1}{6} m \overline{\Omega^2} R \cdot \frac{dN}{dl} \cdot 2\pi R \, dl = \pi R^2 \cdot \frac{dp}{dl} dl,$$

in Übereinstimmung mit der Bedingung *b*, Seite 51. —

Diese Bedingung *b* folgt auch daraus, dass die durch einen Querschnitt des Rohres strömende Bewegungsgrösse konstant ist, und also unabhängig von der Lage des Querschnitts. Um dies

zu sehen, betrachten wir ein Raumelement, bestehend aus zwei Querschnitten, 1 bezw. 2, und einem Ringelement: $2\,\pi R\,dl = dS'$ (vgl. Fig. 9). Nennen wir die Drucke in den Querschnitten 1 und 2, p_1 und p_2, muss für dies Raumelement in dem stationären Zustande wegen der Erhaltung der Bewegungsgrösse gelten:

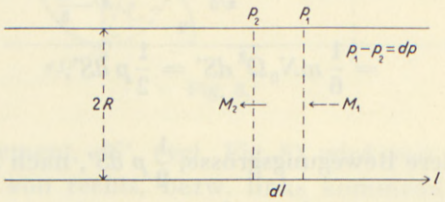


Fig. 9.

$$M_1 \cdot \pi R^2 + B\,dS' - M_2 \cdot \pi R^2 = \pi R^2 \cdot \frac{dp}{dl}\,dl,$$

wo $dS' = 2\,\pi\,R\,dl$. Da nachgewiesen ist, dass:

$$M_1 \cdot \pi R^2 = M_2 \cdot \pi R^2 = \frac{1}{3\,\pi}\,m\overline{\Omega^2}\int dS \int r_1\,d\varphi,$$

erhält man:

$$B\,dS' = \pi R^2 \frac{dp}{dl}\,dl = \frac{1}{6}\,m\overline{\Omega^2} \cdot \frac{dN}{dl} \cdot R\,dS',$$

wie oben auf gaskinetische Weise abgeleitet worden ist.

Wir ersehen also auch hieraus, dass in der Molekularströmung die durch den Druckfall: $\left(\frac{dp}{dl}\,dl\right)$, hervorgebrachte Bewegungsgrösse ganz und gar auf das Rohrstück, $2\,\pi R\,dl = dS'$, übergeht.

Aus dem Ausdruck: $B\,dS' = \frac{1}{6}\,m\overline{\Omega^2} \cdot \frac{dN}{dl} \cdot R\,dS'$, erhellt, dass den strömenden Molekülen in der reinen Molekularströmung keine konstante Massengeschwindigkeit (zusätzliche Geschwindigkeit), $v = \frac{2}{3}\,\overline{\Omega} R \frac{dN}{N dl}$, in der Richtung der Achse beigemessen werden kann.

Wäre dies der Fall, würde man finden:

$$B'\,dS' = \frac{1}{4} N\bar{\Omega} \cdot m \cdot \frac{2}{3}\bar{\Omega} R \frac{dN}{Ndl} \cdot dS' = \frac{1}{6} m\bar{\Omega}^2 \cdot \frac{dN}{dl} \cdot R\,dS'$$

und somit, weil:

$$\bar{\Omega}^2 = \frac{8}{3\pi} \cdot \overline{\Omega^2};$$

$$B'\,dS' = \frac{8}{3\pi} \cdot \frac{1}{6} m\overline{\Omega^2} \cdot \frac{dN}{dl} \cdot R\,dS'.$$

Dieser Ausdruck ist aber $\frac{8}{3\pi}$ mal zu klein, weil $B\,dS' = \frac{1}{6} m\bar{\Omega}^2 \cdot \frac{dN}{dl} \cdot R\,dS'$ sein muss.

Wünscht man also die Formeln der Molekularströmung nicht durch Addition der einzelnen Moleküle, sondern durch hydrodynamische Betrachtungen, abzuleiten, und führt man zu diesem Zweck eine Strömungsgeschwindigkeit, $v = \frac{1}{\varrho} \frac{dG'}{dS}$, ein, ersehen wir aus dem Ausdruck für $B\,dS'$, dass jedem Molekül eine hypothetische, zusätzliche Geschwindigkeit, proportional mit der eigenen Molekulargeschwindigkeit, Ω, beigemessen werden muss, wie Martin Knudsen dies in seiner zweiten Ableitung getan hat; in diesem Falle muss also, wenn die mittlere Geschwindigkeit der strömenden Moleküle, $v = \frac{2}{3}\bar{\Omega} R \frac{dN}{Ndl}$, gesetzt wird, die von Martin Knudsen angegebene Formel:

$$B\,dS' = \frac{3\pi}{32} \cdot Nm\bar{\Omega} v \cdot dS',$$

verwendet werden, weil sich nur dann der richtige Wert von $B = \frac{1}{6} m\overline{\Omega^2} \cdot R \frac{dN}{dl}$ herausstellt.

Die Formel $B\,dS' = \frac{3\pi}{32} \cdot Nm\bar{\Omega} v \cdot dS'$ ist später von Martin Knudsen[1] näher erläutert worden; dies ist aber m. E. von verschiedenen späteren Forschern[2] nicht hinreichend verstanden worden.

1) Martin Knudsen: Ann d. Phys., Bd. 34, S. 823, 1911.

2) J. A. H ter Heerdt: loc. cit. S. 46; Vgl. auch Earle H. Kennard: loc. cit. S. 305.

Da die Moleküle in der reinen Molekularströmung, für welche $R/\lambda = 0$, keine gemeinschaftliche, konstante Massengeschwindigkeit (zusätzliche Geschwindigkeit) besitzen, ist die Molekularströmung eine Diffusionserscheinung, für welche die treibende Kraft der Druckfall, bezw. der Dichtheitsgradient, ist, während der Diffusionswiderstand durch die Rohrwand geleistet wird.

Die Diffusionsgeschwindigkeit der Moleküle in der Richtung der Kraft ist durch die totale in der Zeiteinheit durch 1 cm² des Querschnitts diffundierende Masse bestimmt.

Für ein kreisförmiges Rohr mit Radius, R, erhalten wir für die mittlere Diffusionsgeschwindigkeit, v_d, wenn $\frac{R}{\lambda} = 0$, aus:

$$G' = \frac{2\pi}{3} m \bar{\Omega} R^3 \cdot \frac{dN}{dl} = \pi R^2 \cdot mN \cdot v_d$$

$$= m \cdot D_{11,0} \cdot \frac{dN}{dl} \cdot \pi R^2,$$

den Wert: $v_d = \frac{2}{3} \bar{\Omega} R \frac{dN}{N dl}$ und somit: $D_{11,0} = \frac{2}{3} \bar{\Omega} R.$

Für $\frac{R}{\lambda} \to \infty$ haben wir für den Selbstdiffusionskoeffizienten — annäherungsweise — früher abgeleitet:

$$D_{11,p} = \frac{1}{3} \bar{\Omega} \lambda \frac{1}{1 + \frac{\lambda}{2R}}, \quad \text{woraus:} \quad D_{11,0} = \frac{2}{3} \bar{\Omega} R,$$

in Übereinstimmung mit dem obengenannten Resultat aus der Molekularströmung. —

Wir ersehen also auch hieraus, dass die Molekularströmung einen Grenzfall der Selbstdiffusionsströmung darstellt.

§ 6. Wir werden nun zu bestimmen versuchen, wie die stationäre Molekularströmung beeinflusst und geändert wird, wenn gegenseitige Zusammenstösse der Moleküle im Inneren des Rohres einsetzen, d. h., wenn der Zustand durch $R/\lambda \to 0$ charakterisiert ist. Ausserdem muss in diesem Falle auch die Bedingung, $R/L \to 0$, erfüllt sein.

Aus den Beobachtungen von MARTIN KNUDSEN geht hervor, dass auch in diesem Gebiet die durchströmende Menge, Q_l, proportional ist mit $\frac{dp}{dl}$, bezw. $\frac{dN}{dl}$, wenn die Temperatur konstant ist; dies gilt jedenfalls, wenn $\frac{dp}{dl}$ klein ist.

Wir werden also einfachheitshalber auch für dies Gebiet annehmen, dass dp/dl, bezw. dN/dl, unter den gegebenen Umständen konstant ist; weiter wollen wir annehmen, dass die Moleküle

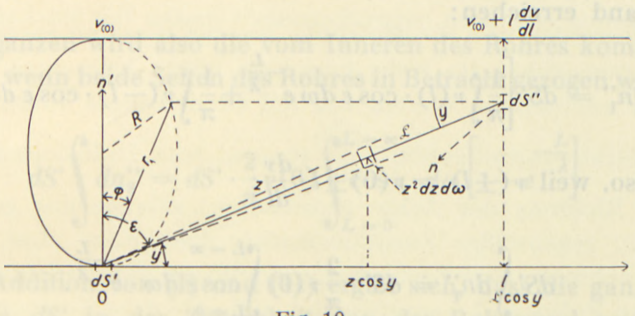


Fig. 10.

als harte elastische Kugeln aufgefasst werden können, woraus folgt, dass die Moleküle, welche von gegenseitigen Zusammenstössen kommen, gleichmässig in allen Richtungen des Raumes ausgestrahlt werden, und dass das Cosinusgesetz für die an der Rohrwand zurückgeworfenen Moleküle gilt.

Ausgehend von diesen Voraussetzungen werden wir wieder ein Wandelement dS' der Rohrwand betrachten und nachweisen, dass auch in diesem Falle die Bedingung a (Seite 50), d. h. dass dieselbe Anzahl von Molekülen, welche in der Zeiteinheit gegen dS' stossen, auch in der Zeiteinheit von dS' ausgestrahlt wird, erfüllt ist.

Ein Wandelement dS'' im Abstande L von dS', für welches $l = 0$ gesetzt wird, strahlt in der Zeiteinheit dn_1 Moleküle aus, die gegen dS' gerichtet sind, wo:

$$dn_1 \cdot dS' = dS' \cdot \frac{1}{\pi} \nu(l) \cdot \cos\varepsilon \, d\omega,$$

wenn:

$$d\omega = \frac{dS' \cos\varepsilon}{L^2} \quad \text{und} \quad \nu(l) = \frac{1}{4} N\bar{\Omega}.$$

Sind keine gegenseitigen Zusammenstösse im Rohr vorhanden, werden diese dn_1 Moleküle dS' erreichen; sind aber gegenseitige Zusammenstösse vorhanden, wird nur die Anzahl dn_1', wo:

$$dn_1' \cdot dS' = dS' \cdot \frac{1}{\pi} \nu(l) \cos\varepsilon \, d\omega \cdot e^{-\frac{L}{\lambda}},$$

dS' erreichen.

Wenn beide Seiten des Rohres in Betracht gezogen werden, wird die Gesamtzahl der Moleküle, welche dS' von der ganzen Rohrwand erreichen:

$$dS' \int dn_1'' = dS' \left[\frac{1}{\pi} \int \nu(l) \cdot \cos\varepsilon \, d\omega \, e^{-\frac{L}{\lambda}} + \frac{1}{\pi} \int \nu(-l) \cdot \cos\varepsilon \, d\omega \, e^{-\frac{L}{\lambda}} \right]$$

und also, weil $\nu(\pm l) = \nu(0) \pm l \cdot \frac{d\nu}{dl}$:

$$\text{(I)} \qquad dS' \int dn_1'' = dS' \cdot \frac{2}{\pi} \nu(0) \int_{L=0}^{L=\infty} \cos\varepsilon \, d\omega \cdot e^{-\frac{L}{\lambda}}.$$

Ausserdem empfängt dS', wenn gegenseitige Zusammenstösse vorhanden sind, eine Anzahl von Molekülen, dn_2, von dem Raumelement $z^2 \, dz \, d\omega$ in dem Abstande z von dS' (vgl. Fig. 10).

Für dn_2 können wir in der folgenden Weise einen Ausdruck ableiten: Gibt es in der Volumeneinheit N Moleküle mit der mittleren Geschwindigkeit $\bar{\Omega}$, und ist λ die mittlere freie Weglänge im freien Gase mit N Molekülen in der Volumeneinheit, werden diese Moleküle $\frac{N\bar{\Omega}}{\lambda}$ mal in der Sekunde gestossen werden.

Von den Molekülen des Raumelementes, dessen Abmessungen klein sind λ gegenüber, gehen also in der Zeiteinheit $\frac{N\bar{\Omega}}{\lambda} \cdot z^2 \, dz \, d\omega$ aus, die direkt von Zusammenstössen kommen. Von diesen ist, wenn sie gleichmässig in alle Richtungen ausgestrahlt werden, ein Teil: $\frac{1}{4\pi} \cdot \frac{\cos\varepsilon}{z^2} dS'$, gegen das Wandelement dS' gerichtet, also im ganzen eine Anzahl von:

$$\frac{1}{4\pi} \cdot dS' \cdot \frac{\cos\varepsilon}{z^2} \cdot \frac{N\bar{\Omega}}{\lambda} \cdot z^2 \, dz \, d\omega,$$

wovon nur der Bruchteil $e^{-\frac{z}{\lambda}}$ das Wandelement dS' erreichen wird. dS' empfängt also von dem ganzen Raumwinkel $d\omega$ die Anzahl:

$$dn_2' \cdot dS' = dS' \int_{z=0}^{z=L} \frac{1}{\pi}\left(\nu(0) + \frac{d\nu}{dl} \cdot l\right) \frac{d\omega}{\lambda} \cdot \cos\varepsilon \cdot e^{-\frac{z}{\lambda}}\, dz,$$

wo:

$$\nu(0) = \frac{1}{4} N_0 \bar{\Omega} \quad \text{und} \quad l = z \cos y.$$

Im ganzen wird also die vom Inneren des Rohres kommende Anzahl, wenn beide Seiten des Rohres in Betracht gezogen werden:

$$\text{(II)} \qquad dS' \int dn_2'' = dS' \cdot \frac{2}{\pi} \nu(0) \int_{L=0}^{L=\infty} \cos\varepsilon \cdot d\omega \left[1 - e^{-\frac{L}{\lambda}}\right].$$

Durch Addition von (I) und (II) ergibt sich, dass die ganze Anzahl, die dS' in der Zeiteinheit von der Rohrwand und vom Inneren des Rohres empfängt, beträgt:

$$dS' \int \left(dn_1'' + dn_2''\right) = dS' \cdot \frac{2}{\pi} \nu(0) \int \cos\varepsilon\, d\omega = \nu(0)\, dS',$$

also dieselbe Anzahl, die in der Zeiteinheit von dS' ausgestrahlt wird. —

Hieraus erhellt, dass auch in dem Falle, wo gegenseitige Zusammenstösse vorhanden sind, die Formel:

$$\nu(\pm l) = \nu(0) \pm \frac{d\nu}{dl} l,$$

in welcher $\frac{d\nu}{dl}$, bezw. $\frac{dN}{dl}$, eine Konstante darstellt, in Übereinstimmung mit der Bedingung a, Seite 50, ist. —

Die Anzahl der gegenseitigen Zusammenstösse der Moleküle im Inneren des Rohres ist $n = \frac{N\bar{\Omega}}{\lambda}$ in der Raumeinheit und also an jeder Stelle proportional mit dem dortigen N; da $\frac{dN}{dl}$ kon-

stant ist, kann also angenommen werden, dass $\frac{dn}{dl}$ auch unabhängig von l wird, wenn wir die kleine Variation von λ vernachlässigen; dies ist erlaubt, wenn $\frac{dN}{dl}$ klein ist. Wir haben ausserdem für die reine Molekularströmung gefunden, dass der Druck p, bezw. N, überall in demselben Querschnitt als eine Konstante angesehen werden kann, so dass dies auch mit n der Fall sein wird. —

Wir werden nun die Moleküle, die von direkten gegenseitigen Zusammenstössen im Rohr kommen, näher betrachten.

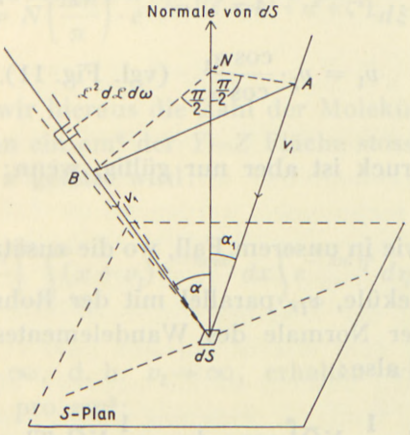
Mit wachsenden Werten von R/λ werden diese Moleküle, des Konzentrationsgradienten, $\frac{dn}{dl}$, wegen, eine Diffusionsströmung in der Richtung der Achse veranlassen; solange nur ganz vereinzelte, gegenseitige Zusammenstösse vorhanden sind, dürfen wir annehmen, dass beide zusammenstossenden Moleküle direkt von der Rohrwand kommen und also vor dem Zusammenstoss nach dem Cosinusgesetz im Mittel die molekulare Geschwindigkeit $\bar{\Omega}$, entsprechend der Temperatur der Rohrwand, besitzen; man könnte darum geneigt sein anzunehmen, wie es POLLARD und PRESENT[1] stillschweigend getan haben, dass diese Moleküle nach dem Zusammenstoss auch im Mittel die Molekular-Geschwindigkeit $\bar{\Omega}$ besitzen und also keine zusätzliche Geschwindigkeit in der Richtung der Achse erhalten haben. Bedenken wir aber, dass wir es in diesem Falle nicht mit gegenseitigen Stössen in einem stillstehenden Gase zu tun haben, sondern mit Zusammenstössen in einem Gase, worin ein Dichtheitsgradient vorhanden ist, so hat z. B. ein Molekül, von der Wand kommend, eine grössere Wahrscheinlichkeit gegen ein Molekül, von rechts kommend, als gegen ein Molekül, von links kommend, zu stossen. Das Resultat hiervon wird wahrscheinlich sein, dass die von gegenseitigen Zusammenstössen im Inneren des Rohres kommenden Moleküle *im Mittel* eine zusätzliche Geschwindigkeit, v_l, in der Richtung der Achse, bezw. des Dichtheitsgradienten, und proportional mit diesem bekommen.

Wir werden darum im folgenden damit rechnen, dass dies

1) W. G. POLLARD und R. D. PRESENT: Phys. Rev., Vol. 73, S. 765, 1948.

der Fall ist bei denjenigen Molekülen, die von gegenseitigen Zusammenstössen im Inneren des Rohres kommen und untersuchen, ob sich an obenstehenden Berechnungen hierdurch etwas ändert. Weiter wollen wir untersuchen, welcher Einfluss hiervon auf die durchströmende Menge ausgehen wird, und versuchen, mit Hilfe der Bedingung *b* (Seite 51), den Wert von v_l in dem Zustand: $\frac{R}{\lambda} \gtrless 0$, zu bestimmen. —

Martin Knudsen[1] hat bereits die Grundlage dieses Problems in Verbindung mit seinen Untersuchungen über den molekularen



$ABN \neq S$-Plan

$v_1 \cdot \cos\alpha = v_l \cdot \cos\alpha_1$.

Für $L^2 dL\, d\omega$ ist die Molekülgeschwindigkeit gegen dS:

$$\overline{\Omega} + v_1 = \left(\overline{\Omega} + v_l \cdot \frac{\cos\alpha_1}{\cos\alpha}\right).$$

Fig. 11.

Gaswiderstand gegen eine sich bewegende Platte behandelt; ob das Gas ruht und die Platte sich bewegt, oder ob das Gas strömt und die Platte stillsteht, ist selbstverständlich ohne Bedeutung.

Bewegt sich ein Körper mit der Geschwindigkeit, v_l, durch ein ruhendes Gas und ist die Zahl der Moleküle in einem cm³, N, wird ein Oberflächenelement dS in jeder Sekunde von den Molekülen, welche Geschwindigkeitsrichtungen gegen das Flä-

[1] Martin Knudsen: Ann. d. Phys., Bd. 46, S. 641, 1915.

chenelement, dS, im Raumwinkel $d\omega$ besitzen, eine Anzahl Molekülstösse, dn_3, erhalten.

Bildet die Normale des Flächenelementes, dS, mit der Bewegungsrichtung des Körpers den Winkel, α_1, und bildet weiter die Normale des Flächenelementes mit dem Erzeuger des Raumwinkels $d\omega$ den Winkel, α, wird dn_3 nach KNUDSEN bestimmt durch:

$$dn_3 = dS \cdot N \frac{d\omega}{4\pi} (\bar{\Omega} + v_1) \cdot \cos\alpha$$

$$= dS \cdot N \frac{d\omega}{4\pi} (\bar{\Omega} \cos\alpha + v_l \cos\alpha_1),$$

weil:

$$v_1 = v_l \cdot \frac{\cos\alpha_1}{\cos\alpha} \quad \text{(vgl. Fig. 11).}$$

Dieser Ausdruck ist aber nur gültig, wenn: $\frac{v_l}{\bar{\Omega}} \ll 1$.

Ist $\alpha_1 = \frac{\pi}{2}$, wie in unserem Fall, wo die zusätzliche Geschwindigkeit der Moleküle, v_l, parallel mit der Rohrachse und also senkrecht zu der Normale des Wandelementes, dS', ist, wird $\cos\alpha_1 = 0$ und also:

$$dn_3' = dS' \cdot \frac{1}{4\pi} N\bar{\Omega} \int \cos\alpha \, d\omega = \frac{1}{4} N\bar{\Omega} \, dS' = \nu(0) \, dS'.$$

Es erhellt hieraus, wie bereits bekannt, dass eine zusätzliche Geschwindigkeit, v_l, in dem Falle, wo dS' ein Wandelement des Rohres darstellt und die zusätzliche Geschwindigkeit parallel mit dS', bezw. mit der Achse des Rohres, ist, keinen Einfluss auf die Anzahl der Molekülstösse gegen das Wandelement, dS', hat.

Betrachten wir aber ein Element, dS, in dem Querschnitt des Rohres, wird $\alpha_1 = 0$ und $\cos\alpha_1 = 1$; in diesem Falle wird die Anzahl der Molekülstösse, welche dS empfängt:

$$dn_3'' = \frac{1}{4} N\bar{\Omega} \cdot dS + \frac{1}{2} N v_l \cdot dS = \frac{1}{4} N\bar{\Omega} \left[1 + 2\frac{v_l}{\bar{\Omega}}\right] dS,$$

woraus erhellt, wie auch zu erwarten war, dass die durch ein

Element des Querschnitts, dS, strömende Menge durch die zusätzliche Geschwindigkeit, v_l, vergrössert wird.

Diese Resultate lassen sich auch auf andere Weise, woraus auch die Grösse der Annäherung erhellt, ableiten; wir können z. B. diese Resultate aus dem bekannten Ausdruck für die MAXWELL'sche Geschwindigkeitsverteilung eines in der Richtung der X-Achse strömenden Gases ableiten; wie bekannt gilt hierfür:

$$dn = N\left(\frac{hm}{\pi}\right)^{3/2} \cdot e^{-hm\,[(\xi - v_l)^2 + \eta^2 + \zeta^2]}\, d\xi\, d\eta\, d\zeta .$$

Berechnen wir hieraus die Zahl der Moleküle, welche in der Zeiteinheit gegen ein cm² der Y—Z Fläche stossen, erhalten wir, wenn $\xi - v_l = x$ gesetzt wird:

$$n = N\left(\frac{hm}{\pi}\right)^{3/2} \int_{-v_l}^{\infty} (x + v_l)\, e^{-hmx^2}\, dx \int_{-\infty}^{+\infty} e^{-hm\eta^2}\, d\eta \int_{-\infty}^{+\infty} e^{-hm\zeta^2}\, d\zeta .$$

Für $v_l/\bar{\Omega} \to \infty$, d. h. $v_l \to \infty$, erhalten wir, wie auch zu erwarten wäre, pro cm²:

$$n = N v_l ,$$

in Übereinstimmung damit, dass die molekulare Geschwindigkeit, $\bar{\Omega} = \frac{2}{\sqrt{\pi hm}}$, in diesem Falle ohne Bedeutung ist, bezw. dass die Moleküle bezüglich der Molekularbewegung als stillstehend angesehen werden können.

Ist aber v_l klein im Vergleich mit der mittleren Geschwindigkeit, $\bar{\Omega} = \frac{2}{\sqrt{\pi hm}}$, wird unter Verwendung der bekannten Reihenentwicklung für: $\int_0^z e^{-z^2}\, dz$, leicht gefunden:

$$n = \frac{1}{4} N\bar{\Omega}\left[1 + 2\frac{v_l}{\bar{\Omega}} + \frac{4}{\pi}\left(\frac{v_l}{\bar{\Omega}}\right)^2 - \frac{8}{3\pi^2}\left(\frac{v_l}{\bar{\Omega}}\right)^4 - \cdots\right].$$

In erster Annäherung erhalten wir also, wenn die höheren Potenzen von $v_l/\bar{\Omega}$ vernachlässigt werden, den von MARTIN KNUDSEN angegebenen Ausdruck:

$$dn = \frac{1}{4} N\bar{\Omega}\left[1 + 2\frac{v_l}{\bar{\Omega}}\right] dS.$$

Betrachten wir beide Seiten des Elementes, dS, erhält man für die durchströmende Anzahl:

$$dn_1 - dn_2 = dS\left[\frac{1}{4} N\bar{\Omega}\left(1 + \frac{2\,v_l}{\bar{\Omega}}\right) - \frac{1}{4} N\bar{\Omega}\left(1 - \frac{2\,v_l}{\bar{\Omega}}\right)\right]$$

oder: $dn_1 - dn_2 = dS \cdot N \cdot v_l$, wie auch zu erwarten war. —

§ 7. Ausgehend von dem Zustand: $R/\lambda \geqq 0$, worin vereinzelte, gegenseitige Zusammenstösse der Moleküle in dem stationär strömenden Gase vorkommen, werden wir die Zahl der Moleküle, welche durch ein Element, dS, des Querschnitts gehen, berechnen; werden beide Seiten von dS in Betracht gezogen, erhalten wir die totale, in der Zeiteinheit durchströmende Anzahl von Molekülen. Diese Zahl besteht nur aus zwei Teilen, I und II, weil wir für diesen Zustand davon ausgehen dürfen, dass die hydrodynamische, laminare Strömung noch nicht eingesetzt hat.

Teil I: Für die durch das Element, dS, strömende Nettoanzahl von Molekülen, dq_1', welche von der Rohrwand kommen

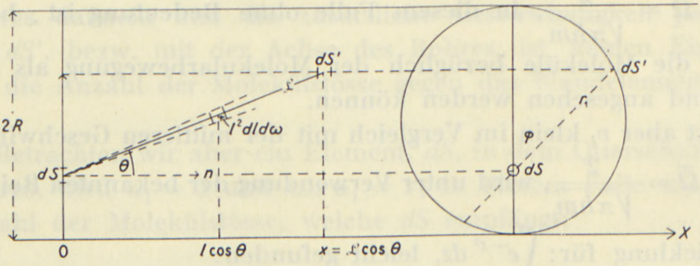


Fig. 12.

und, ohne gegen andere Moleküle zu stossen, dS erreichen, finden wir leicht (vgl. Fig. 12):

$$dq_1' = dS \cdot \frac{2}{\pi} \frac{dv}{dx} \cdot \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta \int_{-\pi}^{+\pi} r_1 \, e^{-\frac{L}{\lambda}} \, d\varphi,$$

wo $v = \frac{1}{4} N \bar{\Omega}$, $x = L \cos\theta$ und $L = \frac{r_1}{\sin\theta}$.

Hieraus ergibt sich:

$$\text{(I)} \qquad dq_1' = dS \cdot \frac{2}{\pi} \frac{dv}{dx} \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \int_{-\pi}^{+\pi} L \, e^{-\frac{L}{\lambda}} \, d\varphi.$$

Für $\frac{R}{\lambda} = 0$ und $\frac{L}{\lambda} \geqq 0$ erhält man hieraus den bekannten Ausdruck für die reine, molekulare Diffusionsströmung.

Teil II: Für die durch das Element, dS, strömende Nettoanzahl von Molekülen, welche direkt von gegenseitigen Zusammenstössen im Inneren des Rohres kommen und das Element dS erreichen, gilt nach dem vorhergehenden Paragraphen, da für diese Moleküle eine mittlere zusätzliche Geschwindigkeit (mittlere Massengeschwindigkeit), v_x, in der Richtung der Achse vorausgesetzt wird, der folgende Ausdruck:

$$dq_2' = dS \int \frac{1}{\pi} \left[v(0) + \frac{dv}{dx} l \cos\theta \right] \frac{dl}{\lambda} e^{-\frac{l}{\lambda}} \left[\cos\theta + \frac{v_x}{\bar{\Omega}} \right] d\omega$$

$$- dS \int \frac{1}{\pi} \left[v(0) - \frac{dv}{dx} l \cos\theta \right] \frac{dl}{\lambda} e^{-\frac{l}{\lambda}} \left[\cos\theta - \frac{v_x}{\bar{\Omega}} \right] d\omega,$$

weil in diesem Falle: $\cos\alpha_1 = 1$.

Hieraus erhält man leicht, da:

$$\int_0^L \frac{l}{\lambda} e^{-\frac{l}{\lambda}} dl = \lambda \int_0^L \frac{l}{\lambda} e^{-\frac{l}{\lambda}} d\left(\frac{l}{\lambda}\right) = \lambda \left[1 - e^{-\frac{L}{\lambda}} \left(1 + \frac{L}{\lambda} \right) \right],$$

$$
(\mathrm{II}) \quad \left\{ \begin{aligned} dq_2' &= dq_{2,1}' + dq_{2,2}' = \\ &= dS \cdot \frac{1}{2\pi} \bar{\Omega} \frac{dN}{dx} \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \int_{-\pi}^{+\pi} \lambda \left[1 - e^{-\frac{L}{\lambda}} \left(1 + \frac{L}{\lambda} \right) \right] d\varphi \\ &\quad + dS \cdot \frac{1}{2\pi} N v_x \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_{-\pi}^{+\pi} \left[1 - e^{-\frac{L}{\lambda}} \right] d\varphi . \end{aligned} \right.
$$

Wir ersehen hieraus, dass der Ausdruck für Teil (II) zweigliedrig ist; das erste Glied: $dq_{2,1}'$, ist bestimmt durch den Konzentrationsgradienten dN/dx, bezw. dn/dx, wo $n = \frac{N\bar{\Omega}}{\lambda}$, der im Inneren des Rohres zusammenstossenden Moleküle; dies Glied repräsentiert also die Selbstdiffusionsströmung der Moleküle, welche von gegenseitigen Zusammenstössen im Inneren des Rohres kommen; das zweite Glied von (II): $dq_{2,2}'$, rührt von der zusätzlichen Geschwindigkeit, v_x, dieser Moleküle her.

Aus dem Ausdruck (II) ersehen wir weiter, dass $q_2' = 0$ für $L/\lambda = 0$, wie auch erwartet werden müsste.

Durch Addition von (I) und (II) erhalten wir:

$$
\begin{aligned} dq = dq_1 + dq_2 &= d(q_1' + q_{2,1}') + dq_{2,2}' = \\ &= \frac{1}{2\pi} \bar{\Omega} \frac{dN}{dx} \cdot dS \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \int_{-\pi}^{+\pi} \lambda \left(1 - e^{-\frac{L}{\lambda}} \right) d\varphi \\ &\quad + \frac{1}{2\pi} v_x \cdot N dS \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_{-\pi}^{+\pi} \left(1 - e^{-\frac{L}{\lambda}} \right) d\varphi , \end{aligned}
$$

wo:

$$
L \sin\theta = r_1, \quad q_1 = q_1' + q_{2,1}' \quad \text{und} \quad q_2 = q_{2,2}' .
$$

Aus dem Ausdruck für dq erhellt, dass wir für $v_x = 0$ den von W. G. Pollard und R. D. Present[1] abgeleiteten Ausdruck erhalten, weil sie der Möglichkeit einer zusätzlichen Geschwindigkeit, v_x, der Moleküle, die direkt von gegenseitigen Zusammen-

1) Pollard und Present, loc. cit. S. 766. Formel (15).

stössen im Inneren des Rohres kommen, keine Rechnung getragen haben.

Betrachten wir die Ableitung der Ausdrücke für dq_1 und dq_2, sehen wir, dass diese Ableitung sich nicht ändert, wenn die Bedingung: $R/\lambda \geqq 0$, nicht mehr gilt; in diesem Falle muss man aber damit rechnen, dass die hydrodynamische, laminare POISEUILLE-Strömung angefangen hat, so dass noch ein drittes Glied, dq_3, der Formel für dq hinzugefügt werden muss; wir werden später auf diesen Zustand zurückkommen. —

Es lässt sich mit Hilfe der Ableitung des Ausdruckes von dq sehr einfach nachweisen, dass auch, wenn gegenseitige Zusammenstösse der Moleküle vorhanden sind, die durch einen Querschnitt des Rohres, πR^2, strömende Bewegungsgrösse konstant und unabhängig von x ist, jedenfalls wenn $R/L \ll 1$, so dass der Einfluss der Endfläche vernachlässigt werden kann; in diesem Falle geht also auch die durch den Druckfall: $\left(\frac{dp}{dx}\right) dx$, hervorgebrachte Bewegungsgrösse vollständig auf das Rohrelement: $2\pi R\, dx$, über[1].

Hierdurch wird es möglich, den Wert von v_x zu berechnen, weil N, bezw. $n = \frac{N\bar{\Omega}}{\lambda}$, über den Querschnitt als eine Konstante angenommen werden kann. —

Betrachten wir ein Wandelement dS' (vgl. Fig. 10), ist es einfach zu sehen, dass dS', solange keine hydrodynamische, laminare Strömung von POISEUILLE vorliegt, nur die folgenden Bewegungsgrössen, (a) und (b), in der Richtung der Achse empfängt:

$$\text{(a)}\quad dS' \int dM_1' = \frac{1}{2\pi} m\overline{\Omega^2} \cdot \frac{dN}{dx}\, dS' \int_0^{\frac{\pi}{2}} \cos^2 y \sin^2 y\, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} L\, e^{-\frac{L}{\lambda}} \cos\varphi\, d\varphi,$$

herrührend von den Molekülen, die von der Rohrwand kommen und, ohne im Inneren des Rohres gegen andere Moleküle zu stossen, dS', erreichen;

1) Vgl. S. 62.

(b) $dS' \int (dM'_{2,1} + dM'_{2,2})$, wo:

$$dS' \int dM'_{2,1} = \frac{1}{2\pi} m \overline{\Omega^2} \frac{dN}{dx} dS' \int_0^{+\frac{\pi}{2}} \cos^2 y \sin^2 y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - e^{-\frac{L}{\lambda}} \left(1 + \frac{L}{\lambda} \right) \right] \cos\varphi \, d\varphi,$$

und

$$dS' \int dM'_{2,2} = \frac{1}{2\pi} m \overline{\Omega} v_x \cdot N dS' \int_0^{\frac{\pi}{2}} \sin^2 y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - e^{-\frac{L}{\lambda}} \right] \cos\varphi \, d\varphi,$$

herrührend von den Molekülen, welche direkt von gegenseitigen Zusammenstössen im Inneren des Rohres kommen und dS' erreichen. —

Solange die laminare Strömung von Poiseuille noch nicht angefangen hat, d. h. solange: $R/\lambda \geqq 0$, erhalten wir also, wenn $dS' = 2\pi R dx$ gesetzt wird:

$$2\pi R \, dx \int [d(M'_1 + M'_{2,1}) + dM'_{2,2}] = \pi R^2 \cdot \frac{dp}{dx} dx,$$

oder:

$$2\pi R \, dx \int [dM'_1 + dM'_2] = \frac{1}{6} m \overline{\Omega^2} \cdot \frac{dN}{dx} \cdot R \cdot 2\pi R \, dx,$$

und also zur Bestimmung von v_x die Formel:

$$2\pi R \, dx \cdot \frac{1}{2\pi} m \overline{\Omega^2} \frac{dN}{dx} \int_0^{\frac{\pi}{2}} \cos^2 y \sin^2 y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - e^{-\frac{L}{\lambda}} \right] \cos\varphi \, d\varphi$$

$$+ 2\pi R \, dx \cdot \frac{1}{2\pi} m \overline{\Omega} v_x N \int_0^{\frac{\pi}{2}} \sin^2 y \, dy \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - e^{-\frac{L}{\lambda}} \right] \cos\varphi \, d\varphi$$

$$= 2\pi R \, dx \cdot \frac{1}{6} m \overline{\Omega^2} \frac{dN}{dx} \cdot R.$$

Für den Zustand: $R/\lambda \rightrightarrows 0$ und $\frac{L}{\lambda} \to 0$, können wir $e^{-\frac{L}{\lambda}}$ in Reihe nach L/λ entwickeln.

Wir finden hieraus, da $L = r_1/\sin y$ und $r_1 = 2R\cos\varphi$, wenn wir nur die erste Annäherung in R/λ verwenden und wenn die zusätzliche Geschwindigkeit v_x unabhängig von $\overline{\Omega}$ ist:

$$dS' \int dM_1' = \frac{1}{6} m\overline{\Omega^2} \cdot R \frac{dN}{dx}\left[1 - 4 \cdot \frac{R}{\lambda}\right] dS',$$

$$dS' \int dM_{2,1}' = \frac{1}{6} m\overline{\Omega^2} \cdot R \frac{dN}{dx} \cdot \frac{2R}{\lambda} dS'$$

und

$$dS' \int dM_{2,2}' = \frac{1}{4} Nm\overline{\Omega} \cdot v_{x\to 0} \cdot \frac{2R}{\lambda} dS',$$

woraus:

$$\frac{1}{4} Nm\overline{\Omega} \cdot v_{x\to 0} = \frac{1}{6} m\overline{\Omega^2} \cdot R \frac{dN}{dx},$$

und also:

$$v_{x\to 0} = \frac{2}{3} R \cdot \frac{\overline{\Omega^2}}{\overline{\Omega}} \cdot \frac{dN}{Ndx}.$$

Da das MAXWELL'sche Verteilungsgesetz jedenfalls für die Moleküle, welche von gegenseitigen Zusammenstössen im Inneren des Rohres kommen, gelten muss, erhalten wir:

$$v_{x\to 0} = \frac{\pi}{4} \overline{\Omega} R \frac{dN}{Ndx}.$$

Wenn wir auf analoge Weise den Wert für dq für $R/\lambda \rightrightarrows 0$, $dq_{\to 0}$, bestimmen wollten, erhellt sofort, dass auf diese Weise keine Lösung gewonnen werden kann, weil der Ausdruck für $dq_{\to 0}$ durch Reihenentwicklung von $e^{-\frac{L}{\lambda}}$ nach R/λ selbst in erster Annäherung keinen endlichen Wert für $dq_{\to 0}$ ergibt; nur für das Glied, $dq_{2,2\to 0}'$, das von v_x herrührt, erhält man auf diese Weise in erster Annäherung eine endliche Lösung.

Für $dq_{2,2\to 0}'$ findet man hierdurch für ein kreisförmiges, zylindrisches Rohr, wenn $\frac{R}{\lambda} \rightrightarrows 0$:

$$dq'_{2,2\to 0} = \frac{1}{2\pi} \cdot v_{x\to 0} \cdot N dS \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_{-\pi}^{+\pi} \frac{1}{\lambda} \cdot \frac{1}{\sin\theta} \cdot r_1 \, d\varphi ,$$

$$q'_{2,2\to 0} = \frac{1}{4} N v_{x\to 0} \cdot \frac{1}{\lambda} \int dS \int r_1 \, d\varphi = \frac{1}{4} \frac{N v_{x\to 0}}{\lambda} \cdot \frac{16}{3} \pi R^3 ,$$

oder:

$$q'_{2,2\to 0} = \frac{4}{3} N v_{x\to 0} \cdot \frac{R}{\lambda} \cdot \pi R^2 ,$$

worin:

$$v_{x\to 0} = \frac{\pi}{4} R \overline{\Omega} \frac{dN}{N dx} .$$

Diese Formel für $q'_{2,2\to 0} = \frac{1}{m} \cdot G'_{2,p\to 0}$ ist der besseren Übersicht wegen bereits in § 3, Seite 31, erwähnt und besprochen worden.

Bevor wir auf die Bestimmung von $q_{2\to 0}$ näher eingehen, wollen wir erst überlegen, welcher Zustand eintritt, wenn eine hydrodynamische, laminare Strömung in der Richtung der Achse hinzukommt.

Es ist einleuchtend, dass eine hinzukommende, laminare Strömung des Gases mit einer konstanten Massengeschwindigkeit in der Richtung der Achse keine Änderung in dem Bestehen und in der Berechnung der Diffusionsströmung, $d(q'_1 + q'_{2,1})$, und der Gleitungsströmung, $dq'_{2,2}$, verursachen wird, da diese Strömungen einander überlagern und einander nicht beeinflussen.

Zu dem Ausdruck für dq — die totale gaskinetische Strömung — wird also nur noch ein Glied, dq_3, herrührend von dieser hydrodynamischen, laminaren Strömung, hinzukommen; dasselbe wird auch der Fall sein für eine überlagerte, laminare Strömung von Poiseuille, da diese für jede ringförmige Lamelle als eine hydrodynamische Strömung mit konstanter Massengeschwindigkeit aufgefasst werden kann.

In dem Zustande: $\frac{R}{\lambda} \to \infty$, beträgt also die totale, durchströmende Anzahl:

$$dq = dq'_1 + dq'_{2,1} + dq'_{2,2} + dq_3$$

$$= dq_1 + dq_2 + dq_3 ,$$

wo:

$$q_3 = \frac{3\pi}{64} \cdot \frac{R}{\lambda} \cdot q_{1,0} \cdot -$$

Ausserdem muss für diesen Zustand: $\frac{R}{\lambda} \to \infty$, ein Glied, dM_3, herrührend von der überlagerten POISEUILLE-Strömung, zu der Formel für die an die Rohrwand übertragene Bewegungsgrösse hinzugefügt werden, so dass diese Formel lautet:

$$2\pi R\,dx \int [dM_1 + dM_2 + dM_3] = \pi R^2 \cdot \frac{dp}{dx}\,dx,$$

weil auch, jedenfalls wenn $R/L \ll 1$, für die stationäre POISEUILLE-Strömung die durch den Druckfall, $\frac{dp}{dx}\,dx$, hervorgebrachte Bewegungsgrösse vollständig auf das Wandelement: $dS' = 2\,\pi\,R\,dx$, übergeht. —

Nach der angenäherten Theorie von MAXWELL für die Gleitung wird diese Bewegungsgrösse dM_3 für ein kreisförmiges Rohr:

$$dS' \int dM_3 = \frac{1}{2}\eta \left(\frac{du}{dr}\right)_{r \geq R} \cdot dS',$$

und also, da:

$$\left(\frac{du}{dr}\right)_{r \to R} = \frac{1}{2\eta} \cdot R \cdot \frac{dp}{dx},$$

$$2\pi R\,dx \int dM_3 = \frac{1}{4} R \frac{dp}{dx} \cdot dS' = \frac{1}{4} R \frac{dp}{dx} 2\pi R\,dx = \frac{1}{2} \frac{dp}{dx} dx \cdot \pi R^2.$$

Für: $\frac{R}{\lambda} \to \infty$, erhält man somit:

$$dS' \int [d(M_1' + M_{2,1}') + dM_{2,2}'] + dS' \int dM_3 = \pi R^2 \cdot \frac{dp}{dx}\,dx,$$

oder:

$$dS' \int (dM_1 + dM_2) = \frac{1}{2}\pi R^2 \cdot \frac{dp}{dx}\,dx,$$

wo:

$$dS' = 2\pi R \cdot dx$$

Bestimmen wir hieraus: $v_{x \to \infty}$, erhält man für $\frac{R}{\lambda} \to \infty$, wenn $e^{-\frac{L}{\lambda}} = 0$ gesetzt wird:

$$\frac{1}{16} m \overline{\Omega^2} \lambda \cdot \frac{dN}{dx} + \frac{1}{4} N m \overline{\Omega} \cdot v_{x \to \infty} = \frac{1}{12} m \overline{\Omega^2} \cdot R \frac{dN}{dx},$$

und demnach, wenn das erste Glied, von der Selbstdiffusion herrührend, in dem Gebiet: $\frac{R}{\lambda} \to \infty$, vernachlässigt wird:

$$v_{x \to \infty} = \frac{\pi}{8} \overline{\Omega} R \frac{dN}{N dx}.$$

Dieser Ausdruck ist in Übereinstimmung mit dem von KUNDT und WARBURG, bezw. von MAXWELL, eingeführten Ausdruck für die Gleitungsgeschwindigkeit an der Rohrwand.

Da der MAXWELL'sche Ausdruck für M_3 nur eine Annäherung darstellt und jedenfalls zu gross ist, kann man eventuell zur Vergleichung mit den Beobachtungen: $M_3 \cdot dS' = \alpha' \cdot \frac{1}{2} \frac{dp}{dx} dx \cdot \pi R^2$, setzen und erhält dann:

$$v_{x \to \infty} = (2 - \alpha') \cdot \frac{\pi}{8} \overline{\Omega} R \frac{dN}{N dx}, \quad \text{d.h.} \quad k_{2,\infty} = 2 - \alpha'.$$

Betrachten wir die zweigliedrige Formel (I + II) für die durchströmende Anzahl der Moleküle in dem Zustande: $R/\lambda \to 0$, sehen wir, dass das erste Glied: $\int dq_1 = \int (dq_1' + dq_{2,1}')$, die Selbstdiffusionsströmung repräsentiert; für den Grenzfall: $\lambda \to \infty$, bezw. $R/\lambda = 0$ und $L/\lambda \to 0$, erhält man hieraus den Ausdruck für die KNUDSEN'sche, reine Molekularströmung, bezw. die Molekulardiffusion:

$$q_{1,0} = \frac{1}{8} \overline{\Omega} \frac{dN}{dx} \int dS \int r_1 \, d\varphi.$$

Für den Grenzfall: $\lambda \to 0$, bezw. $\frac{R}{\lambda} \to \infty$, ergibt $\int dq_1$ den Ausdruck:

$$q_{1,\infty} = \frac{1}{3} \overline{\Omega} \lambda \frac{dN}{dx} \int dS = D_{11,\infty} \cdot \frac{dN}{dx} \int dS.$$

Das zweite Glied $\int dq_2 = \int dq_{2,2}'$ ist der gaskinetische Ausdruck für die Gleitungsströmung; für $\lambda \to \infty$, bezw. $R/\lambda = 0$ und $\frac{L}{\lambda} \rightrightarrows 0$, wird $\int dq_2 = 0$, während man für $R/\lambda \to 0$ erhält:

$$q_{2\to 0} = \frac{4}{3} N v_{x\to 0} \cdot \frac{R}{\lambda} \int dS, \quad \text{wo} \quad v_{x\to 0} = \frac{\pi}{4} R \bar{\Omega} \frac{dN}{N dx}.$$

Für $\lambda \to 0$, bezw. $\frac{R}{\lambda} \to \infty$, erhält man:

$$q_{2\to\infty} = v_{x\to\infty} \cdot N \int dS, \quad \text{wo} \quad v_{\to\infty} = k_2 \cdot \frac{\pi}{8} R \bar{\Omega} \frac{dN}{N dx}.$$

Die Grenzwerte für $M_1 = M_1' + M_{2,1}'$, $M_2 = M_{2,2}'$ und M_3, für $\frac{R}{\lambda} = 0$, $R/\lambda \geqq 0$ und $R/\lambda \to \infty$, werden auf analoge Weise aus den entsprechenden Werten abgeleitet. In der folgenden Übersicht sind die verschiedenen Grenzwerte angegeben.

1°, $\frac{R}{\lambda} = 0$:

$$q_{1,0} = \frac{2}{3} \bar{\Omega} R \cdot \frac{dN}{dx} \cdot \pi R^2 = D_{11,0} \cdot \frac{dN}{dx} \cdot \pi R^2,$$

$q_{2,1}' = 0$, $q_{2,2}' = 0$ und $q_{3,0} = 0$

} Reine Molekularströmung, bezw. Molekulardiffusion.

2°, $\frac{R}{\lambda} \geqq 0$: $q_{1\to 0} = (q_{1\to 0}' + q_{2,1\to 0}')$, vorläufig unbestimmt.

$$q_{2\to 0} = q_{2,2\to 0}' = \frac{4}{3} N v_{x\to 0} \cdot \pi R^2 \cdot \frac{R}{\lambda},$$

wo: $$v_{x\to 0} = \frac{\pi}{4} R \bar{\Omega} \frac{dN}{N dx}.$$

} Gleitungsströmung.

3°, $\frac{R}{\lambda} \to \infty$:

$$q_{1\to\infty} = \frac{1}{3} \bar{\Omega} \lambda \frac{dN}{dx} \cdot \pi R^2 = D_{11,\infty} \cdot \frac{dN}{dx} \pi R^2$$

} Selbstdiffusions- bezw. Druckdiffusionsströmung.

$q_{2\to\infty} = N \cdot v_{x\to\infty} \cdot \pi R^2$ ..... Gleitungsströmung.

$q_3 = \frac{3\pi}{128} \cdot q_{1,0} \cdot \frac{2R}{\lambda}$ ..... Poiseuille-Strömung.

Die Grenzwerte für die übertragenen Bewegungsgrössen per cm^2 sind folgende:

$1°,\ \frac{R}{\lambda} = 0:\ M'_{1,0} = \frac{1}{6} m \overline{\Omega^2} \cdot R \frac{dN}{dx} = \frac{1}{2} R \frac{dp}{dx}$

$M'_{2,1} = M'_{2,2} = M_3 = 0$

} Reine Molekularströmung, bezw. Molekulardiffusion.

$2°,\ \frac{R}{\lambda} \geqq 0:\ M'_{1\to 0} = \frac{1}{6} m \overline{\Omega^2} \cdot R \frac{dN}{dx}\left[1 - 4\frac{R}{\lambda}\right]$

} Selbstdiffusionsströmung.

$M'_{2,1\to 0} = \frac{1}{6} m \overline{\Omega^2} \cdot R \frac{dN}{dx} \cdot \frac{2R}{\lambda},$

$M'_{2,2\to 0} = \frac{1}{4} N m \overline{\Omega} v_{x\to 0} \cdot \frac{2R}{\lambda},$

mit: $v_{x\to 0} = \frac{\pi}{4} R \overline{\Omega} \frac{dN}{N dx}$

} Gleitungsströmung.

$M_{3\to 0} \geqq 0$

} Poiseuille-Strömung.

$3°,\ \frac{R}{\lambda} \to \infty : M_{1\to\infty} = \frac{1}{16} m \overline{\Omega^2} \lambda \frac{dN}{dx} \cdots$

} Selbstdiffusionsströmung.

$M_{2\to\infty} = \frac{1}{4} N m \overline{\Omega} v_{\to\infty},$

mit: $v_{\to\infty} = k_2 \cdot \frac{\pi}{8} R \overline{\Omega} \frac{dN}{N dx}.$

} Gleitungsströmung.

$M_{3\to\infty} = \alpha' \cdot \frac{1}{12} m \overline{\Omega^2} R \frac{dN}{dx}$

$= \alpha' \cdot \frac{1}{4} R \frac{dp}{dx}, \quad \text{wo} \quad \alpha' = 2 - k_{2,\infty}$

} Poiseuille-Strömung.

Der genaue Verlauf von q_1 und q_2 in dem ganzen Gebiet: $0 \leqq R/\lambda \leqq \infty$, ist von grossem Interesse, auch in Verbindung mit dem Entstehen, der Grösse und der Lage des beobachteten Minimums in der Nähe von $R/\lambda = 0{,}30$.

Die Ausdrücke für q_1 und q_2, bezw.

$$dq_1 = \frac{1}{2\pi}\,\bar{\Omega}\,\frac{dN}{dx}\cdot dS\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta\,d\theta\int_{-\pi}^{+\pi}\lambda\left[1-e^{-\frac{r_1}{\lambda\sin\theta}}\right]d\varphi,$$

und:

$$dq_2 = \frac{1}{2\pi}\,v_x\,N dS\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_{-\pi}^{+\pi}\left[1-e^{-\frac{r_1}{\lambda\sin\theta}}\right]d\varphi,$$

sind aber — abgesehen von den obengenannten Grenzwerten für $R/\lambda \geqq 0$ und $R/\lambda \to \infty$ — in mathematischer Hinsicht nicht einfach, weil r_1 von φ abhängt, während v_x sich in dem Gebiet: $0 \leqq R/\lambda \leqq \infty$, von $v_{x\to 0} = \frac{\pi}{4}R\bar{\Omega}\cdot\frac{dN}{Ndx}$ bis auf $v_{x\to\infty} = k_2\cdot\frac{\pi}{8}R\bar{\Omega}\cdot\frac{dN}{Ndx}$ ändert.

Aus der obenstehenden Übersicht erhellt, dass es nicht möglich ist, durch einfache Reihenentwicklung von $e^{-L/\lambda}$ nach L/λ den Wert für $q_{1\to 0}$ für $R/\lambda \geqq 0$ zu bestimmen. Um eine endliche Lösung für $q_{1\to 0}$ zu erhalten, müssen also die, in den verschiedenen Formeln für die durchströmende Anzahl von Molekülen und für die auf die Rohrwand übertragenen Bewegungsgrössen, vorkommenden Integrale entweder in konvergenten Reihenentwicklungen vorliegen, oder sie müssen durch andere brauchbare, endliche Ausdrücke ersetzt werden. Es wäre z. B. möglich, dass man in erster Annäherung eine brauchbare und übersichtliche Lösung durch Mittelwertbildung von $e^{-L/\lambda}$, bezw. L/λ, erhalten könnte; hiervon ausgehend ist es z. B. leicht nachzuweisen, dass der mittlere Wert von $L = r_1/\sin\theta$ für die durch den Querschnitt, πR^2, strömenden Moleküle, q, in dem Zustande $R/\lambda \geqq 0$ annäherungsweise: $L = 2R$, wird, so dass in diesem Zustande, in erster Annäherung, der Ausdruck:

$$\left[1-e^{-\frac{r_1}{\lambda\sin\theta}}\right] \quad \text{durch:} \quad \left[1-e^{-\frac{2R}{\lambda}}\right] \simeq \frac{\frac{2R}{\lambda}}{1+\frac{2R}{\lambda}} = \frac{1}{1+\frac{\lambda}{2R}},$$

ersetzt werden kann; diese Annäherung ist bereits in § 3 besprochen und verwendet worden. —

Für eine einwandfreie Erklärung des Entstehens des beobachteten Minimums in der Nähe von $R/\lambda = 0{,}3$, sowie auch für die genaue Bestimmung des Verlaufs von q_1 und q_2 in dem Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, wird es aber nötig sein, die vorkommenden, bestimmten Integrale der Gattung:

$$\int_0^{\frac{\pi}{2}} \cos^n\theta \sin^m\theta \, e^{-\frac{\alpha}{\sin\theta}} \, d\theta, \quad \text{wo} \quad \alpha = \frac{r_1}{\lambda},$$

näher zu untersuchen, um zu sehen, ob es möglich ist, diese Integrale in konvergenten Reihen nach α zu entwickeln.

Bevor wir hierzu übergehen, wollen wir aber erst die Aufmerksamkeit darauf richten, dass in den vorhergehenden Berechnungen keine Rücksicht auf die Änderung von λ mit den Abständen von x und $-x$ genommen worden ist; es kann aber durch einfache Berechnungen bewiesen werden, dass der Wert von λ, der in den abgeleiteten Formeln vorkommt, der Wert von λ für $x = 0$ ist, also derjenige Wert von λ, der zu dem betrachteten Querschnitt, $x = 0$, gehört. —

Da Pollard und Present[1] bereits den entsprechenden Beweis für q_1 geliefert haben, ist es überflüssig, diesen Beweis hier zu wiederholen; hinzugefügt kann werden, dass es einfach ist, auf analoge Weise diesen Beweis für q_2 und auch für den Ausdruck zur Bestimmung von v_x zu liefern. —

§ 8. Wir werden nun erst das Integral: $W(\alpha) = \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \, d\theta$, näher untersuchen; wir sehen, dass die Grenzwerte betragen:

$$W(0) = \frac{\pi}{2} \quad \text{für} \quad \alpha = 0 \quad \text{und} \quad W(\infty) = 0 \quad \text{für} \quad \alpha \geqq \infty .$$

1) Pollard und Present: loc. cit. S. 773, App. III.

Wird in dem Integral: $W(\alpha) = \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}}\, d\theta$, $z = \frac{\alpha}{\sin\theta}$ gesetzt, erhalten wir:

$$W(\alpha) = \frac{\pi}{2} e^{-\alpha} + \int_{\infty}^{\alpha} e^{-z} \arcsin\frac{\alpha}{z}\, dz.$$

Für $\arcsin\frac{\alpha}{z}$ gilt zwischen: $-1 \leq \frac{\alpha}{z} \leq 1$, die konvergente Reihe:

$$\arcsin\left(\frac{\alpha}{z}\right) = \frac{\alpha}{z} + \frac{1}{1\cdot 2}\cdot\frac{1}{3}\left(\frac{\alpha}{z}\right)^3 + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\left(\frac{\alpha}{z}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}\left(\frac{\alpha}{z}\right)^7 + \cdots$$

und also:

$$W(\alpha) = \frac{\pi}{2} e^{-\alpha} +$$

$$+ \alpha \int_{\infty}^{\alpha} e^{-z} \left\{\frac{1}{z} + \frac{1}{1\cdot 2}\cdot\frac{1}{3}\left(\frac{\alpha^2}{z^3}\right) + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}\left(\frac{\alpha^4}{z^5}\right) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}\left(\frac{\alpha^6}{z^7}\right) + \cdots\right\} dz.$$

Wir setzen:

$$L_n' = \alpha^{n-1} \int_{\infty}^{\alpha} e^{-z} \frac{dz}{z^n};$$

also wird:

$$L_1' = \int_{\infty}^{\alpha} \frac{e^{-z}}{z}\, dz = Ei(-\alpha) = \ln(\gamma\alpha) - \alpha + \frac{1}{2}\cdot\frac{\alpha^2}{2!} - \frac{1}{3}\cdot\frac{\alpha^3}{3!} + \frac{1}{4}\cdot\frac{\alpha^4}{4!} - \cdots,$$

wo $Ei(-\alpha)$ der Integrallogarithmus ist, während $\ln\gamma = C = 0{,}557216$ (Die Euler'sche Konstante)[1].

Verwenden wir die Rekursionsformel:

$$L_n' = -\frac{1}{n-1}\left[e^{-\alpha} + \alpha L_{n-1}'\right],$$

erhalten wir hierdurch:

1) E. Jahnke and F. Emde: Tables of functions, New York, 4th Edition, S. 1, 1945.

$$W(\alpha) = \frac{\pi}{2} e^{-\alpha} -$$

$$- \alpha e^{-\alpha} \left[\frac{1}{2} \cdot \frac{1}{3 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8} + \cdots \right]$$

$$+ \alpha^2 e^{-\alpha} \left[\frac{1}{2} \cdot \frac{1}{3 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7} + \cdots \right]$$

$$- \alpha^3 e^{-\alpha} \left[\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7 \cdot 6} + \cdots \right]$$

$$+ \alpha^4 e^{-\alpha} \left[\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5} + \cdots \right]$$

$$- \alpha^5 e^{-\alpha} \left[\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} + \cdots \right]$$

$$+ \alpha^6 e^{-\alpha} \left[\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} + \cdots \right]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$+ \alpha L_1' \left[1 + \frac{1}{1 \cdot 2} \cdot \frac{1}{3 \cdot 2} \alpha^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} \alpha^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \alpha^6 + \cdots \right]$$

oder:

$$W(\alpha) = e^{-\alpha} \left[\frac{\pi}{2} - A\alpha + B\alpha^2 - D\alpha^3 + Q\alpha^4 - R\alpha^5 + \cdots \right]$$

$$+ \alpha L_1' \left[1 + \frac{1}{12} \alpha^2 + \frac{1}{320} \alpha^4 + \frac{1}{16128} \alpha^6 + \cdots \right].$$

Werden $e^{-\alpha}$ und $Ei(-\alpha)$ in Reihen nach α entwickelt, erhalten wir, da:

$$e^{-\alpha} = 1-\alpha+\frac{1}{2}\alpha^2-\frac{1}{6}\alpha^3+\frac{1}{24}\alpha^4-\frac{1}{120}\alpha^5+\cdots$$

$$W(\alpha) = \frac{\pi}{2}-m_1\alpha+\alpha\ ln\,\alpha+m_2\alpha^2-m_3\alpha^3+\frac{1}{12}\alpha^3\ ln\,\alpha+$$

$$+m_4\alpha^4-m_5\alpha^5+\frac{1}{320}\alpha^5\ ln\,\alpha+\cdots,$$

worin:

$$m_1 = \frac{\pi}{2}+A-C = 1+ln\,2-C = 1+\varepsilon = 1.11594$$

$$m_2 = \frac{\pi}{4}-1+A+B = 0$$

$$m_3 = \frac{A}{2}+B+D+\frac{\pi}{12}-\frac{1}{4}-\frac{C}{12} = \frac{1}{12}\left(\frac{4}{3}+\varepsilon\right) = 0{,}120772$$

$$m_4 = \frac{\pi}{48}+\frac{A}{6}+\frac{B}{2}+D+Q-\frac{5}{36} = 0$$

$$m_5 = \frac{\pi}{240}+\frac{A}{24}+\frac{B}{6}+\frac{D}{2}+Q+R-\frac{1}{32}-\frac{C}{320} =$$

$$= \frac{1}{320}\left(\frac{17}{10}+\varepsilon\right) = 0{,}567478\cdot 10^{-2} \quad \text{und}$$

$$\varepsilon = ln\,2-C = 0{,}693152-0{,}577216 = 0{,}11594.$$

Die Werte für A, B, D, Q und R sind in der folgenden Weise bestimmt:

Für $-1 \leqq \frac{1}{y} \leqq 1$ haben wir die folgende Identität:

$$\arcsin\frac{1}{y}=\frac{1}{y}+\frac{1}{1\cdot2}\cdot\frac{1}{3}\cdot\frac{1}{y^3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5}\cdot\frac{1}{y^5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7}\cdot\frac{1}{y^7}+\cdots$$

Wird diese Identität von y bis ∞ integriert, erhalten wir die folgende Identität:

$$\int_y^\infty\left[\arcsin\frac{1}{y}-\frac{1}{y}\right]dy=\left[y\arcsin\frac{1}{y}+ln\left(1+\sqrt{1-\left(\frac{1}{y}\right)^2}\right)\right]_y^\infty$$

$$=1+ln\,2-y\arcsin\frac{1}{y}-ln\left(1+\sqrt{1-\frac{1}{y^2}}\right)=$$

$$=\frac{1}{1\cdot2}\cdot\frac{1}{2\cdot3}\left(\frac{1}{y}\right)^2+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot4}\left(\frac{1}{y}\right)^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot6}\left(\frac{1}{y}\right)^6+\cdots,$$

weil für $y=\infty$, $\quad y\arcsin\frac{1}{y}=1$.

Aus dieser Identität erhält man für $y=1$ den folgenden Ausdruck:

$$1+ln\,2-\frac{\pi}{2}=\frac{1}{1\cdot2}\cdot\frac{1}{2\cdot3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot6}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{1}{9\cdot8}+\cdots=A.$$

Integrieren wir in analoger Weise die vorhergehende Identität noch einmal, ergibt sich:

$$\int_y^\infty\left[1+ln\,2-y\arcsin\frac{1}{y}-ln\left[1+\sqrt{1-\frac{1}{y^2}}\right]\right]dy=$$

$$=\int_y^\infty\left(\frac{1}{1\cdot2}\cdot\frac{1}{2\cdot3}\cdot\frac{1}{y^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot4}\cdot\frac{1}{y^4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot6}\cdot\frac{1}{y^6}+\cdots\right)dy.$$

Hieraus erhalten wir leicht für $y=1$ den Ausdruck:

$$\frac{\pi}{4}-ln\,2=\frac{1}{1\cdot2}\cdot\frac{1}{2\cdot3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5\cdot4\cdot3}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7\cdot6\cdot5}+\cdots=B$$

und in ganz analoger Weise:

$$\frac{7}{12} ln\, 2 - \frac{\pi}{12} - \frac{5}{36} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4} + \cdots = D$$

und:

$$\frac{\pi}{48} + \frac{1}{9} - \frac{1}{4} ln\, 2 = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} + \cdots = Q$$

und:

$$\frac{83}{960} ln\, 2 - \frac{\pi}{240} - \frac{449}{9600} =$$

$$= \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} + \cdots = R.$$

Durch diese Werte für A, B, D, Q und R sind die obenerwähnten Werte für m_1, m_2, m_3, m_4 und m_5 vollständig bestimmt.

Ausgehend von:

$$W(\alpha) = \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} d\theta = \frac{\pi}{2} - m_1 \alpha + \alpha\, ln\, \alpha + m_2 \alpha^2 - m_3 \alpha^3 +$$

$$+ \frac{1}{12} \alpha^3\, ln\, \alpha + m_4 \alpha^4 - m_5 \alpha^5 + \frac{1}{320} \cdot \alpha^5\, ln\, \alpha + \cdots,$$

worin:

$$m_2 = 0 \quad \text{und} \quad m_4 = 0,$$

kann man leicht, entweder durch Differentiation von W nach α oder durch Integration von $\int_0^{\alpha} W\, d\alpha$, neue Ausdrücke bilden.

Wir können auf diese Weise die für unsere Zwecke benötigten Ausdrücke ableiten; wir finden z. B.:

$$\int_0^{\alpha} d\alpha \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\alpha} e^{-\frac{\alpha}{\sin\theta}}\, d\left(\frac{\alpha}{\sin\theta}\right) = 1 - \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \sin\theta\, d\theta$$

$$= \int_0^{\alpha} \left[\frac{\pi}{2} - m_1\,\alpha + \alpha\, ln\,\alpha + m_2\,\alpha^2 - m_3\,\alpha^3 + \frac{1}{12}\,\alpha^3\, ln\,\alpha + \cdots\right] d\alpha,$$

woraus:

$$\int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \sin\theta\, d\theta = 1 - \frac{\pi}{2}\,\alpha + \frac{1}{2}\left(m_1 + \frac{1}{2}\right)\alpha^2 - \frac{1}{2}\,\alpha^2\, ln\,\alpha +$$

$$+ \frac{1}{4}\left(m_3 + \frac{1}{48}\right)\alpha^4 - \frac{1}{48}\,\alpha^4\, ln\,\alpha + \frac{1}{6}\left(m_5 + \frac{1}{1920}\right)\alpha^6 - \frac{1}{1920}\,\alpha^6\, ln\,\alpha + \cdots$$

und in analoger Weise:

$$\int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \sin^2\theta\, d\theta = \frac{\pi}{4} - \alpha + \frac{\pi}{4}\,\alpha^2 - \frac{1}{6}\left(m_1 + \frac{1}{2} + \frac{1}{3}\right)\alpha^3 + \frac{1}{6}\,\alpha^3\, ln\,\alpha -$$

$$- \frac{1}{20}\left(m_3 + \frac{1}{48} + \frac{1}{60}\right)\alpha^5 + \frac{1}{240}\,\alpha^5\, ln\,\alpha + 0 \cdot \alpha^6 -$$

$$- \frac{1}{42}\left[m_5 + \frac{1}{1920} + \frac{1}{2240}\right]\alpha^7 + \frac{1}{1920 \cdot 7}\,\alpha^7\, ln\,\alpha + \cdots$$

wo:

$$m_1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} + \varepsilon = D''$$

$$m_3 + \frac{1}{48} + \frac{1}{60} = \frac{1}{12}\left[\frac{107}{60} + \varepsilon\right] = \frac{1}{12}\,E''$$

$$m_5 + \frac{1}{1920} + \frac{1}{2240} = \frac{1}{320}\left[2 + \frac{1}{105} + \varepsilon\right] = H''.$$

Werden die Werte für $\int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \sin^3\theta\, d\theta$ und $\int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\theta}} \sin^4\theta\, d\theta$ in derselben Weise berechnet, erhält man u. a.:

$$\int_0^{\frac{\pi}{2}-\frac{\alpha}{\sin\theta}} [\sin^2\theta - \sin^4\theta]\, d\theta = \int_0^{\frac{\pi}{2}-\frac{\alpha}{\sin\theta}} \sin^2\theta \cos^2\theta\, d\theta =$$

$$= \frac{\pi}{16} - \frac{1}{3}\alpha + \frac{\pi}{8}\alpha^2 - \frac{A''}{6}\alpha^3 + \frac{1}{6}\alpha^3 \ln\alpha - \frac{\pi}{48}\alpha^4 + \frac{B''}{120}\alpha^5 -$$

$$- \frac{1}{240}\alpha^5 \ln\alpha + 0\cdot\alpha^6 + \frac{1}{840}F''\alpha^7 - \frac{1}{840}K''\alpha^7 \ln\alpha + \cdots$$

wo:

$$A'' = m_1 + \frac{1}{2} + \frac{1}{3} - 1 = \varepsilon + \frac{5}{6} = 6\cdot 0{,}15821$$

$$B'' = m_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 6m_3 - \frac{1}{8} - \frac{1}{10} = \frac{1}{2}\varepsilon + \frac{167}{120} =$$

$$= 120\cdot 1{,}20803\cdot 10^{-2}$$

$$F'' = m_3 + \frac{1}{48} + \frac{1}{60} + \frac{1}{72} + \frac{1}{84} - 20m_5 - \frac{1}{96} - \frac{1}{112} =$$

$$= \frac{1}{24}\left[\frac{1}{2}\varepsilon + \frac{123}{105}\right] = 840\cdot 0{,}60981\cdot 10^{-4}$$

$$K'' = \frac{1}{48} = 840\cdot 0{,}248016\cdot 10^{-4}\cdot -$$

§ 9. Wir können nun die in dem vorigen Paragraphen abgeleiteten Formeln bei der weiteren mathematischen Behandlung des früher abgeleiteten Ausdrucks:

$$d(q_1+q_2) = \frac{1}{2\pi}\bar{\Omega}\frac{dN}{dx}dS\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta\, d\theta\int_{-\pi}^{+\pi}\lambda\left[1-e^{-\frac{L}{\lambda}}\right]d\varphi$$

$$+\frac{1}{2\pi}v_x N dS\int_0^{\frac{\pi}{2}}\sin\theta\, d\theta\int_{-\pi}^{+\pi}\left[1-e^{-\frac{L}{\lambda}}\right]d\varphi,$$

wo $L\sin\theta = r_1$, in Anwendung bringen. —

Da es für einen kreisförmigen Querschnitt zweckmässig ist, die von E. KENNARD, bezw. POLLARD und PRESENT, angegebene Transformation für die Integration über den Querschnitt $\int dS$ anzuwenden, werden wir der Vergleichbarkeit und Übersicht wegen

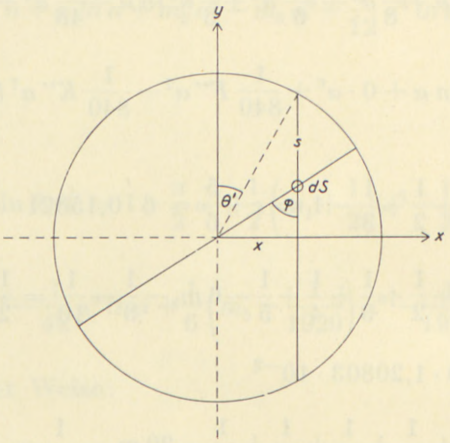


Fig. 13.

unsere Bezeichnungen in Übereinstimmung mit den von POLLARD und PRESENT benutzten Bezeichnungen bringen, indem wir setzen:

$$\theta = \varphi, \; r_1 = s \quad \text{und} \quad R = a.$$

Unter Hinweis auf die Zeichnung, Fig. 13, sehen wir, dass wir für dS schreiben können:

$$dS = dx\, dy = d\,(a \sin \theta')\, ds$$

und somit zur Kontrolle:

$$\int dS = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a \cos\theta'\, d\theta' \int_0^{2a\cos\theta'} ds = 2a^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta'\, d\theta' = \pi a^2.$$

Verwenden wir diesen Ausdruck für dS und betrachten wir s und θ' als polare Koordinaten, erhalten wir nach einer einfachen Umgestaltung:

$$\int dq_1 = \frac{1}{2\pi}\,\overline{\Omega}\,\frac{dN}{dx}\,\lambda \cdot \int_{-\pi}^{+\pi} d\varphi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a\cos\theta'\,d\theta' \int_0^{\frac{\pi}{2}} \cos^2\psi \sin\psi\,d\psi \int_0^{2a\cos\theta'} \left[1 - e^{-\frac{s}{\lambda \sin\psi}}\right] ds,$$

woraus:

$$\int dq_1 = \frac{1}{3}\,\overline{\Omega}\,\lambda\,\frac{dN}{dx} \cdot \pi a^2 \cdot \left[1 - \frac{3}{8}\left(\frac{\lambda}{\alpha}\right) + \frac{6}{\pi}\left(\frac{\lambda}{\alpha}\right) \cdot Q\left(\frac{\alpha}{\lambda}\right)\right],$$

wo:

$$Q\left(\frac{\alpha}{\lambda}\right) = \int_0^{\frac{\pi}{2}} \cos\theta'\,d\theta' \int_0^{\frac{\pi}{2}} e^{-\frac{2\alpha}{\lambda}\cdot\frac{\cos\theta'}{\sin\psi}} \sin^2\psi \cos^2\psi\,d\psi$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta' \cdot f(\theta')\,d\theta'.$$

In ganz analoger Weise erhalten wir für $\int dq_2$:

$$\int dq_2 = \frac{1}{2\pi} N v_x \int_{-\pi}^{+\pi} d\varphi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a\cos\theta'\,d\theta' \int_0^{\frac{\pi}{2}} \sin\psi\,d\psi \int_0^{2a\cos\theta'} \left[1 - e^{-\frac{s}{\lambda\sin\psi}}\right] ds$$

$$= N \cdot v_x \left[1 - \frac{1}{2}\left(\frac{\lambda}{a}\right) + \frac{2}{\pi}\left(\frac{\lambda}{a}\right) \cdot R\left(\frac{a}{\lambda}\right)\right] \cdot \pi a^2, \quad \text{wo:}$$

$$R\left(\frac{a}{\lambda}\right) = \int_0^{\frac{\pi}{2}} \cos\theta'\,d\theta' \int_0^{\frac{\pi}{2}} e^{-\frac{2a}{\lambda}\cdot\frac{\cos\theta'}{\sin\psi}} \sin^2\psi\,d\psi = \int_0^{\frac{\pi}{2}} \cos\theta' \cdot F(\theta')\,d\theta'.$$

Der Wert von v_x in dieser Formel für q_2 muss durch die Formeln für die auf die Rohrwand übertragenen Bewegungsgrössen, M_1, M_2 und M_3, bestimmt werden, wenn die Variation von v_x in dem Gebiet: $0 \leqq \frac{\alpha}{\lambda} \leqq \infty$, nicht auf andere Weise bekannt ist. —

Verwenden wir nun die in dem vorhergehenden Abschnitt abgeleiteten Reihenentwicklungen für die bestimmten Integralen:

$$f(\theta') = \int_0^{\frac{\pi}{2}} e^{-\frac{2a}{\lambda}\cdot\frac{\cos\theta'}{\sin\psi}} \sin^2\psi\cos^2\psi\, d\psi$$

und

$$F(\theta') = \int_0^{\frac{\pi}{2}} e^{-\frac{2a}{\lambda}\cdot\frac{\cos\theta'}{\sin\psi}} \sin^2\psi\, d\psi,$$

worin $\frac{2a}{\lambda}\cos\theta' = \alpha$ gesetzt wird, erhalten wir nach Integration die Reihenentwicklungen für:

$$Q\left(\frac{a}{\lambda}\right) = \int_0^{\frac{\pi}{2}} \cos\theta' \cdot f(\theta')\, d\theta' \quad \text{und} \quad R\left(\frac{a}{\lambda}\right) = \int_0^{\frac{\pi}{2}} \cos\theta' \cdot F(\theta')\, d\theta.$$

Bei diesen Integrationen werden u. a. die folgenden bestimmten Integralen verwendet:

$$\int_0^{\frac{\pi}{2}} ln\cos\theta'\, d\theta' = -\frac{\pi}{2} ln\, 2^*), \quad \int_0^{\frac{\pi}{2}} \cos^2\theta'\, ln\cos\theta'\, d\theta' = \frac{\pi}{8}[1 - 2\, ln\, 2].$$

*) Dies Integral ist bekannt und die folgenden Integralen werden mit Hilfe dieses Integrales auf einfache Weise abgeleitet. Eine einfache Ableitung von dem Integral: $\int_0^{\frac{\pi}{2}} ln\cos\theta'\, d\theta' = -\frac{\pi}{2} ln\, 2$, ist, wie folgt: $A = \int_0^{\frac{\pi}{2}} ln\cos\theta'\, d\theta'$; wird $\theta' = \frac{\pi}{2} - \alpha$ gesetzt, erhält man: $\int_{\frac{\pi}{2}}^{0} ln\sin\alpha\, d(-\alpha) = \int_0^{\frac{\pi}{2}} ln\sin\alpha\, d\alpha = A$; ausserdem ist: $\int_0^{\frac{\pi}{2}} ln\, 2\cdot d\varphi = \frac{\pi}{2} ln\, 2$, und also:

$$2A + \frac{\pi}{2} ln\, 2 = \int_0^{\frac{\pi}{2}} ln\,(2\sin\varphi\cos\varphi)\, d\varphi =$$

$$= \frac{1}{2}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}} ln\sin 2\varphi \cdot d(2\varphi) = \frac{1}{2}\int_0^{\pi} ln\sin\beta\, d\beta =$$

$$= \int_0^{\frac{\pi}{2}} ln\sin\beta\, d\beta = A, \quad \text{woraus:} \quad 2A + \frac{\pi}{2} ln\, 2 = A$$

oder:

$$A = \int_0^{\frac{\pi}{2}} ln\cos\theta'\, d\theta' = -\frac{\pi}{2} ln\, 2.-$$

$$\int_0^{\frac{\pi}{2}} \cos^4\theta' \, ln \cos\theta' \, d\theta' = \frac{7}{64}\pi - \frac{3\pi}{16} ln\, 2,$$

$$\int_0^{\frac{\pi}{2}} \cos^6\theta' \, ln \cos\theta' \, d\theta' = \frac{37}{384}\pi - \frac{15\pi}{96} ln\, 2$$

und

$$\int_0^{\frac{\pi}{2}} \cos^8\theta' \, ln \cos\theta' \, d\theta' = \frac{533}{6144}\pi - \frac{35\pi}{256} ln\, 2.$$

Wir erhalten hieraus: $G_1 = m \int dq_1 =$

$$= \frac{2\pi}{3} m \bar{\Omega} a^3 \frac{dN}{dx}\left[1 - \frac{3}{4}\left(A'' - \frac{7}{12}\right)\left(\frac{a}{\lambda}\right) + \frac{3}{4}\left(\frac{a}{\lambda}\right) ln\left(\frac{a}{\lambda}\right) - \frac{8}{15}\left(\frac{a}{\lambda}\right)^2 +\right.$$

$$+ \frac{1}{8}\left[B'' - \frac{37}{120}\right]\left(\frac{a}{\lambda}\right)^3 - \frac{1}{16}\left(\frac{a}{\lambda}\right)^3 ln\left(\frac{a}{\lambda}\right) + 0 \cdot \left(\frac{a}{\lambda}\right)^4 +$$

$$\left. + \frac{1}{840}\left[\frac{105}{2} F'' - \frac{533}{16} K''\right]\left(\frac{a}{\lambda}\right)^5 - \frac{1}{840} \cdot \frac{105}{2} K'' \left(\frac{a}{\lambda}\right)^5 ln\left(\frac{a}{\lambda}\right) + \cdots \quad \cdots\right] =$$

$$= G_1 = G_{1,0} \cdot {}_1P_1\left(\frac{a}{\lambda}\right),$$

wo:

$$\varepsilon = ln\, 2 - C = 0{,}115936; \quad \frac{3}{4}\left(A'' - \frac{7}{12}\right) = \frac{3}{4}\left(\varepsilon + \frac{1}{4}\right),$$

$$\frac{1}{8}\left(B'' - \frac{37}{120}\right) = \frac{1}{8}\left(\frac{1}{2}\varepsilon + \frac{13}{12}\right); \quad \frac{1}{840}\left[\frac{105}{2} F'' - \frac{533}{16} K''\right] = \frac{1}{768}\left(\varepsilon + \frac{41}{24}\right)$$

und:

$$\frac{1}{840} \cdot \frac{105}{2} \cdot K'' = \frac{1}{768}.$$

Setzen wir: $\frac{a}{\lambda} = x_1$ und $ln\, x_1 = 2{,}30259\, log_{10} x_1$, erhalten wir:

$${}_1P_1\left(\frac{a}{\lambda}\right) = {}_1P_1 = 1 - 0{,}274452 \cdot x_1 + 1{,}726943 \cdot x_1 \log_{10} x_1 -$$

$$- 0{,}533333 \cdot x_1^2 + 0{,}142663 \cdot x_1^3 - 0{,}143911 \cdot x_1^3 \log_{10} x_1 +$$

$$+ 0{,}00237535 \cdot x_1^5 - 0{,}00299816 \cdot x_1^5 \log_{10} x_1 + \ldots$$

oder:

$$G_1 = \frac{2\pi}{3} m \bar{\Omega} a^3 \frac{dN}{dx} \cdot {}_1P_1\left(\frac{a}{\lambda}\right) = G_{1,0} \cdot {}_1P_1 = m \cdot D_{11,0} \cdot \frac{dN}{dx} \cdot \pi a^2 \cdot {}_1P_1.$$

Hieraus ergibt sich also:

$$G_1 = m \int dq_1 = \frac{2\pi}{3} m \bar{\Omega} a^3 \frac{dN}{dx} \cdot {}_1P_1 =$$

$$= \frac{1}{3} m \bar{\Omega} \lambda \frac{dN}{dx} \cdot \pi a^2 \cdot \left[1 - \frac{3}{8}\left(\frac{\lambda}{a}\right) + \frac{6}{\pi}\left(\frac{\lambda}{a}\right) \cdot Q\left(\frac{a}{\lambda}\right)\right].$$

woraus:

$$Q\left(\frac{a}{\lambda}\right) = \frac{\pi}{16} - \frac{\pi}{6}\left(\frac{a}{\lambda}\right) + \frac{\pi}{3}\left(\frac{a}{\lambda}\right)^2 \cdot {}_1P_1\left(\frac{a}{\lambda}\right) \cdot -$$

Es ist einleuchtend, dass für $x_1 = \frac{a}{\lambda} \to \infty$ die Funktionen: $Q\left(\frac{a}{\lambda}\right)$ und $R\left(\frac{a}{\lambda}\right) = 0$ werden; es stellt sich aber heraus, dass wir in den folgenden numerischen Berechnungen bereits für: $\frac{a}{\lambda} \geqq 4$, $Q\left(\frac{a}{\lambda}\right) = 0$ und $R\left(\frac{a}{\lambda}\right) = 0$ setzen dürfen.

Wir erhalten nun die folgenden Grenzwerte für $Q\left(\frac{a}{\lambda}\right)$ und ${}_1P_1$, da $x_1^n \, ln x_1 = 0$ für $x_1 = 0$, wenn $n > 0$ ist:

Für $x_1 = \frac{a}{\lambda} = 0$: ${}_1P_1 = 1$ und $Q\left(\frac{a}{\lambda}\right)_0 = \frac{\pi}{16}$.

Für $x_1 = \frac{a}{\lambda} \geqq \infty$: $Q\left(\frac{a}{\lambda}\right)_\infty = 0$ und also: ${}_1P_1 = \frac{\lambda}{2a}\left[1 - \frac{3}{8}\frac{\lambda}{a}\right]$.

Für $x_1 = \frac{a}{\lambda} \geqq \infty$: $G_{1,\infty} = \frac{1}{3} m \bar{\Omega} \lambda \cdot \frac{dN}{dx} \cdot \pi a^2 = D_{11,\infty} \cdot m \frac{dN}{dx} \cdot \pi a^2$.

Für $x_1 = \frac{a}{\lambda} = 0$: $G_{1,0} = \frac{2\pi}{3} m \bar{\Omega} a^3 \cdot \frac{dN}{dx} = D_{11,0} \cdot m \frac{dN}{dx} \cdot \pi a^2$,

und im Allgemeinen:

$$G_{1,\frac{a}{\lambda}} = G_{1,p} = G_{1,0} \cdot {}_1P_1 = mD_{11,0} \frac{dN}{dx} \cdot \pi a^2 \cdot {}_1P_1 = mD_{11,p} \frac{dN}{dx} \cdot \pi a^2,$$

woraus: $D_{11,p} = D_{11,0} \cdot {}_1P_1 \cdot -$

Mit Hilfe dieser Formel für $G_{1,\frac{a}{\lambda}}$ können wir also auch die Brauchbarkeit der verwendeten, angenäherten Formel für die Selbstdiffusion, bezw. Druckdiffusion:

$$G_{1,\frac{a}{\lambda}} = G_{1,p} = G_{1,\infty} \cdot \frac{1}{1+\frac{\lambda}{2a}} = G_{1,0} \cdot \frac{1}{1+\frac{2a}{\lambda}}.$$

in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, untersuchen.

Betrachten wir nun den Ausdruck für: $G_2 = m \int dq_2$, erhalten wir auf ganz analoge Weise unter Verwendung der Reihenentwicklung für:

$$F(\theta') = \int_0^{\frac{\pi}{2}} e^{-\frac{2a}{\lambda} \cdot \frac{\cos\theta'}{\sin\psi}} \sin^2\psi \, d\psi = \int_0^{\frac{\pi}{2}} e^{-\frac{\alpha}{\sin\psi}} \sin^2\psi \, d\psi,$$

den Ausdruck für $\int \cos\theta' \cdot F(\theta') \, d\theta'$, und nach Durchführung der Integration hiervon den folgenden Ausdruck für die Funktion, $R\left(\frac{a}{\lambda}\right)$:

$$R\left(\frac{a}{\lambda}\right) = \frac{\pi}{4} - \frac{\pi}{2}\left(\frac{a}{\lambda}\right) \cdot {}_1N_1\left(\frac{a}{\lambda}\right),$$

wo:

$${}_1N_1\left(\frac{a}{\lambda}\right) = {}_1N_1 = 1 - \frac{4}{3} \cdot x_1 + \frac{1}{2}\left(\varepsilon + \frac{5}{4}\right) \cdot x_1^2 - \frac{1}{2} \cdot x_1^2 \, lnx_1 + 0 \cdot x_1^3 +$$

$$+ \frac{1}{24}\left(\varepsilon + \frac{7}{4}\right) \cdot x_1^4 - \frac{1}{24} \cdot x_1^4 \, lnx_1 + 0 \cdot x_1^5 + \frac{1}{384}\left(\varepsilon + 1 + \frac{3}{8}\right) \cdot x_1^6 -$$

$$- \frac{1}{384} \cdot x_1^6 \, lnx_1 + \cdots$$

mit $x_1 = \left(\frac{a}{\lambda}\right)$, $\varepsilon = 0{,}115936$ und $lnx_1 = 2{,}30259 \, log_{10} x_1$.

Hieraus erhält man:

$$
\begin{aligned}
{}_1N_1 = 1 - \frac{4}{3} \cdot x_1 + 0{,}682968 \cdot x_1^2 - 1{,}151293 \cdot x_1^2 \log_{10} x_1 + \\
+ 0{,}0534417 \cdot x_1^4 - 0{,}0959441 \cdot x_1^4 \log_{10} x_1 + \\
+ 0{,}00388264 \cdot x_1^6 - 0{,}00599631 \cdot x_1^6 \log_{10} x_1 \\
+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
$$

und somit die Grenzwerte:

$$
\text{Für } x_1 = \frac{a}{\lambda} = 0 : \quad {}_1N_1 = 1, \quad R\left(\frac{a}{\lambda}\right) = \frac{\pi}{4} \quad \text{und}
$$

$$
\text{für } x_1 = \frac{a}{\lambda} \rightarrow \infty : \quad R\left(\frac{a}{\lambda}\right)_{\rightarrow\infty} = 0, \quad {}_1N_1 \rightarrow \frac{\lambda}{2a}.
$$

Aus dem Ausdruck für $\int dq_2$ ergibt sich:

$$
m \int dq_2 = G_2 = mNv_x \cdot \pi a^2 \left[1 - \frac{1}{2}\left(\frac{\lambda}{a}\right) + \frac{2}{\pi}\left(\frac{\lambda}{a}\right) \cdot R\left(\frac{a}{\lambda}\right)\right]
$$

oder nach Einsetzung des Wertes für $R\left(\frac{a}{\lambda}\right)$:

$$
G_2 = m \int dq_2 = mN \cdot v_x \cdot \pi a^2 \left[1 - {}_1N_1\right].
$$

Um den Wert von v_x aus dem Ausdruck für die übertragenen Bewegungsgrössen zu bestimmen, können wir die bereits hierfür abgeleitete Formel schreiben, wie folgt:

$$
\pi a^2 \frac{dp}{dx} = 2\pi a \left[M_1 + M_2\right] + M_3.
$$

M_1 ist durch die Diffusionsströmung und M_2 durch die Gleitungsströmung bestimmt; M_3 rührt von der POISEUILLE-Strömung her.

Unter der Voraussetzung, dass die reine, laminare POISEUILLE-Strömung für $\frac{a}{\lambda} \geqq 0$ noch nicht eingesetzt hat oder jedenfalls nicht von Bedeutung geworden ist, erhalten wir, wenn $M_3 = 0$ gesetzt wird, den Ausdruck:

$$2\pi a\,(M_1+M_2) = \pi a^2 \frac{dp}{dx},$$

oder:

$$\frac{1}{2\pi} m\,\overline{\Omega^2}\cdot\lambda\frac{dN}{dx}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\left[1-e^{-\frac{2a}{\lambda}\cdot\frac{\cos\varphi}{\sin y}}\right]\sin^2 y\cos^2 y\,dy$$

$$+\frac{1}{2\pi} m\,\overline{\Omega}\,v_{x\to 0}\cdot N\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\left[1-e^{-\frac{2a}{\lambda}\cdot\frac{\cos\varphi}{\sin y}}\right]\sin^2 y\,dy$$

$$= \frac{1}{6} m\,\overline{\Omega^2}\cdot a\frac{dN}{dx},$$

oder:

$$\frac{1}{2\pi}\overline{\Omega^2}\cdot\lambda\frac{dN}{dx}\left[\frac{\pi}{8}-2\cdot Q\left(\frac{a}{\lambda}\right)\right]+\frac{1}{2\pi}\overline{\Omega}\,v_{x\to 0}\cdot N\left[\frac{\pi}{2}-2\cdot R\left(\frac{a}{\lambda}\right)\right]$$

$$= \frac{1}{6}\overline{\Omega^2}\cdot a\frac{dN}{dx}.$$

Hieraus erhält man:

$$v_{x\to 0} = \frac{\pi}{4}\overline{\Omega}\,a\cdot\frac{dN}{N dx}\cdot\frac{{}_1P_1}{{}_1N_1},$$

und somit, weil

$$\lim {}_1P_1 = \lim {}_1N_1 = 1 \quad \text{für} \quad \frac{a}{\lambda} = 0:$$

$$v_{x\geq 0} = \frac{\pi}{4}\overline{\Omega}\,a\cdot\frac{dN}{N dx},$$ wie früher bereits abgeleitet.

Hieraus folgt, dass für $\frac{a}{\lambda}\to 0$, d. h. solange die laminare, hydrodynamische Strömung von Poiseuille noch nicht von wesentlicher Bedeutung geworden ist, und also $M_3 = 0$ gesetzt werden kann:

$$G_{2\to 0} = \frac{2\pi}{3} m\,\overline{\Omega}\,a^3\frac{dN}{dx}\cdot\frac{3\pi}{8}\left[1-{}_1N_1\right]\frac{{}_1P_1}{{}_1N_1}$$

$$= G_{1,0}\cdot\frac{3\pi}{8}\left[1-{}_1N_1\right]\cdot\frac{{}_1P_1}{{}_1N_1}\cdot -$$

Durch die Reihenentwicklung für ${}_1N_1$ erhält man hieraus für $\frac{a}{\lambda} \to 0$:

$$G_{2\to 0} = G_{1,0} \cdot \frac{3\pi}{8}\left[1 - {}_1N_1\right]\frac{{}_1P_1}{{}_1N_1} =$$

$$= G_{1,0} \cdot \frac{\pi}{4} \cdot \frac{2a}{\lambda} \cdot \frac{{}_1P_1}{{}_1N_1}\left[1 - \frac{3}{8}\left(\varepsilon + \frac{5}{4}\right)\left(\frac{a}{\lambda}\right) + \frac{3}{8}\left(\frac{a}{\lambda}\right) ln\left(\frac{a}{\lambda}\right) \cdots\right],$$

und somit für $\frac{a}{\lambda} \gtrsim 0$:

$$G_{2\gtrsim 0} = G_{1,0} \cdot \frac{\pi}{4} \cdot \frac{2a}{\lambda}, \quad \text{weil } \lim \frac{{}_1P_1}{{}_1N_1} = 1 \text{ für } \frac{a}{\lambda} = 0.$$

Wenn, ausgehend von $a/\lambda \gtrsim 0$, die hydrodynamische Strömung von POISEUILLE allmählich von grösserer Bedeutung wird, nimmt der Wert von v_x wegen des Einflusses des Gliedes M_3 ab, so dass die nach dem Ausdruck:

$$G_{2\to 0} = G_{1,0} \cdot \frac{3\pi}{8}\left[1 - {}_1N_1\right]\frac{{}_1P_1}{{}_1N_1},$$

berechneten Werte zu gross werden; wir werden später aus den vergleichenden Tabellen ersehen, dass der Gültigkeitsbereich für diese Formel ist: $0 \leq \frac{a}{\lambda} \leq$ ca. 0,35.

Für $a/\lambda \to \infty$ folgt der Wert von $M_3 \to \infty$ aus der bekannten, laminaren Strömung von POISEUILLE, bezw. aus dem konstanten Wert der inneren Reibung, η, und dem experimentellen Wert von k_2 für $a/\lambda \to \infty$; wie früher bereits abgeleitet, erhält man hierdurch für:

$$\frac{a}{\lambda} \to \infty : v_{x\to\infty} = k_2 \cdot \frac{\pi}{8}\overline{\Omega}\, a \cdot \frac{dN}{Ndx}, \quad \text{oder}$$

$$\text{mit } k_2 = \frac{4}{3} \quad : v_{x\to\infty} = \frac{\pi}{6}\overline{\Omega}\, a \cdot \frac{dN}{Ndx}.$$

Hieraus ergibt sich:

$$G_{2\to\infty} = mNv_{x\to\infty} \cdot \pi a^2\left[1 - {}_1N_{1\to\infty}\right] = G_{1,0} \cdot \frac{\pi}{4}\left[1 - {}_1N_{1\to\infty}\right]$$

und also:

$$G_{2\to\infty} = G_{1,0} \cdot \frac{\pi}{4}\left[1 - \frac{\lambda}{2a}\right], \quad \text{wenn für}$$

$$\frac{a}{\lambda} \to \infty : {}_1N_{1\to\infty} = \frac{\lambda}{2a}, \quad \text{eingesetzt wird. —}$$

Wir können hierdurch auch die Brauchbarkeit der früher verwendeten Annäherungsformel für G_2, d. h. die Formel:

$$G_2 = G_{1,0} \cdot \frac{\pi}{4} \frac{\frac{2a}{\lambda}}{1 + \frac{2a}{\lambda}} = G_{1,0} \cdot \frac{\pi}{4} \frac{1}{1 + \frac{\lambda}{2a}} \quad \text{in dem Gebiet: } \frac{a}{\lambda} \to 0,$$

untersuchen.

Wir sehen aber leicht durch Vergleichung der Grenzwerte für $a/\lambda \doteqdot 0$ und $a/\lambda \to \infty$, dass diese einfache Formel jedenfalls brauchbar ist für das Gebiet: $\frac{a}{\lambda} \doteqdot 0$, und für das Gebiet: $\frac{a}{\lambda} \to \infty$.

§ 10. Eine vollständige Berechnung der Variation von v_x mit a/λ in dem Gebiet: $0 \leqq a/\lambda \leqq \infty$, mit Hilfe der Formel für die an die Wand übertragenen Bewegungsgrössen ist aber nicht möglich, solange wir nichts näheres über den Verlauf von M_3 in diesem Gebiet wissen.

Nach dem Vorhergehenden können wir also nur feststellen, dass:

$$v_x = k_2 \cdot \frac{\pi}{8} \overline{\Omega} a \cdot \frac{dN}{N dx},$$

wo für: $\frac{a}{\lambda} \doteqdot 0$, $\quad k_{2,0} = 2,$

für: $\frac{a}{\lambda} \to 0$, $\quad k_2 = 2 \cdot \left(\frac{{}_1P_1}{{}_1N_1}\right)_{\frac{a}{\lambda}\to 0}$

und für: $\frac{a}{\lambda} \to \infty$, $\quad k_{2,\infty} = \text{ca.}\,\frac{4}{3}$ (exp. festgestellt).

Hieraus erhalten wir also für die zirkularkapillare Durchströmungsmethode: $\frac{k_{2,0}}{k_{2,\infty}} = \frac{3}{2}$.

Wie bereits früher erwähnt, würde eine direkte Bestimmung der Gleitung nach der Ablenkungsmethode mit koaxialen Zylindern, von welchen der äusserste Zylinder in sich selbst rotiert, die Möglichkeit einer direkten Bestimmung der Abhängigkeit zwischen k_2 und a/λ eröffnen, weil in diesem Falle keine Selbstdiffusions- bezw. Druckdiffusionsströmung vorhanden sein wird.

Obwohl dies nicht ganz der Fall ist bei der Bestimmung des Gaswiderstands gegen die langsame Bewegung kleiner Kugeln (Gesetz von STOKES), gewährt diese Methode doch wohl eine angenäherte Bestimmung der Variation von k_2 mit a/λ.

Bei der diesbezüglichen experimentellen Untersuchung von MARTIN KNUDSEN und SOPHUS WEBER[1] stellte sich heraus, dass das Gesetz von STOKES für den Gaswiderstand, K, in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, wie folgt geschrieben werden kann, wenn a der Radius der Kugel ist (Vgl. Seite 41):

$$K = 6\pi\eta a \cdot U \left[1 + \frac{\zeta}{a}\right]^{-1}, \quad \text{wo} \quad \zeta = k_2 \cdot \lambda \cdot -$$

Aus den Beobachtungen ergab sich, dass:

$$k_2 = A\left[1 + \frac{B}{A} \cdot e^{-C \cdot \frac{a}{\lambda}}\right],$$

wo A, B und C Konstanten sind, während a der Radius der Kugel ist. U ist die konstante Geschwindigkeit der Kugel und λ die mittlere freie Weglänge.

Wird die CHAPMAN'sche Weglänge, λ, verwendet, sind die von KNUDSEN und WEBER gefundenen Werte:

$$A = 1.10\ , \quad \frac{B}{A} = \frac{1}{2} \quad \text{und} \quad C = 1.15.$$

Der Form nach ist diese Formel später von MILLIKAN[2] und auch von MATTAUCH[3] bestätigt worden, obwohl die experimentell gefundenen Werte etwas von einander abweichen; es kann aber

1) Loc. cit. Seite 40.
2) und 3) Loc. cit. Seite 40, vgl. auch LIGNAC, loc. cit. S. 27.

wohl kaum ein Zweifel darüber bestehen, dass diese Form der Formel festliegt und brauchbar ist. —

Wir erhalten hieraus für:

$$\frac{a}{\lambda} = 0, \quad k_{2,0} = A + B$$

und für

$$\frac{a}{\lambda} \to \infty, \quad k_{2,\infty} = A,$$

woraus wir finden:

$$\frac{k_{2,0}}{k_{2,\infty}} = \frac{A+B}{A} = \frac{3}{2},$$

in Übereinstimmung mit den obenerwähnten Resultaten der Durchströmungsmethode.

Wir dürfen also annehmen, dass in dem Ausdruck für die Gleitungsströmung, G_2, k_2 in dem Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, auch geschrieben werden kann:

$$k_2 = k_{2,\infty} \cdot \left(1 + \frac{1}{2} e^{-C_2 \cdot \frac{a}{\lambda}}\right).$$

Wir erhalten hieraus für G_2:

$$G_2 = m.N.\, k_{2,\infty} \left[1 + \frac{1}{2} e^{-C_2 \cdot \frac{a}{\lambda}}\right] \bar{\Omega} a \cdot \frac{dN}{Ndx} \cdot \pi a^2 \left[1 - {}_1N_1\left(\frac{a}{\lambda}\right)\right]$$

$$= \frac{2\pi}{3} m \bar{\Omega} a^3 \frac{dN}{Ndx} \cdot N \cdot \frac{3\pi}{16} \cdot k_{2,\infty} \cdot \left[1 + \frac{1}{2} e^{-C_2 \cdot \frac{a}{\lambda}}\right] [1 - {}_1N_1]$$

$$= G_{1,0} \cdot \frac{\pi}{4} [1 - {}_1N_1] \left[1 + \frac{1}{2} e^{-C_2 \cdot \frac{a}{\lambda}}\right], \quad \text{wenn} \quad k_{2,\infty} = \frac{4}{3},$$

woraus:

$$G_{2 \to 0} = G_{1,0} \cdot \frac{\pi}{4} \cdot \frac{2a}{\lambda} \left[1 - \frac{3}{8}\left(\varepsilon + \frac{5}{4}\right)\left(\frac{a}{\lambda}\right) \cdots\right] \left[1 - \frac{1}{3} C_2 \left(\frac{a}{\lambda}\right) \cdots\right]$$

und

$$G_{2 \to \infty} = G_{1,0} \cdot \frac{\pi}{4} [1 - {}_1N_1] = G_{1,0} \cdot \frac{\pi}{4} \left[1 - \frac{\lambda}{2a}\right].$$

Der Wert für C_2 kann am besten durch die theoretisch berechneten Werte von $G_2/G_{1,0}$ für $\frac{a}{\lambda} \to 0$, bezw. die Werte aus der Formel: $G_2/G_{1,0} = \frac{3\pi}{8}[1 - {}_1N_1] \cdot \frac{{}_1P_1}{{}_1N_1}$, wenn $\frac{a}{\lambda} <$ ca. 0,35, bestimmt werden. —

Es stellt sich, wie aus der folgenden Tabelle VII erhellt, heraus, dass $C_2 = 2$ ein angemessener Wert ist. —

§ 11. Für die Vergleichung der KNUDSEN'schen Beobachtungsreihen mit den Resultaten der entwickelten Theorie ist es zweckmässig, aus den abgeleiteten Formeln erst die benötigten Tabellen für ${}_1P_1$ und ${}_1N_1$ zu berechnen. —

In Kolonne I der folgenden Tabelle No. VI ist der Wert von a/λ angegeben. λ ist die freie mittlere Weglänge, so wie diese aus der Formel von S. CHAPMAN berechnet wird; in Kolonne II sind die nach der gefundenen Reihenentwicklung berechneten Werte für ${}_1P_1$ angegeben. Für $a/\lambda \geq 4$ ist hierbei die Annäherung: $Q\left(\frac{a}{\lambda}\right) = 0$, woraus: ${}_1P_1 = \frac{\lambda}{2a}\left[1 - \frac{3}{8}\frac{\lambda}{a}\right]$, verwendet.

In Kolonne III sind zur Kontrolle die Werte, welche für ${}_1P_1$ aus den von P. und P. angegebenen Werten für das Integral, K, berechnet werden können, angegeben. P. und P.[1] haben nämlich für $Q\left(\frac{a}{\lambda}\right)$ den folgenden Ausdruck abgeleitet:

$$Q\left(\frac{a}{\lambda}\right) = \frac{\pi}{16} - \frac{\pi}{2} \cdot K \text{ , wo:}$$

$$K = \int_0^{\frac{\pi}{2}} \left[H_1\left(i\frac{2a}{\lambda \sin\psi}\right) - iI_1\left(i\frac{2a}{\lambda \sin\psi}\right)\right] \sin^2\psi \cos^2\psi \, d\psi,$$

woraus folgt:

$${}_1P_1 = \frac{\lambda}{2a}\left[1 - 3\left(\frac{\lambda}{a}\right) \cdot K\right].$$

In diesem Ausdruck für K bezeichnen $H_1\,(iy)$ und $I_1\,(iy)$

1) W. G. POLLARD und R. D. PRESENT: loc. cit. S. 774, App. IV.

Tabelle VI.

I	II	III	IV	V
$\frac{a}{\lambda}$	${}_1P_1$, ber. $= \frac{G_1}{G_{1,0}}$, ber.	${}_1P_1$, ber. laut numerischer Integration (P & P)	$\frac{1}{1+2\frac{a}{\lambda}}$	${}_1N_1$, ber.
0·000	1·00000		1·0000	1·00000
·001	0·99454		0·9980	0·99867
·010	·96266		·9804	·98697
·015	·94852		·9709	·98063
·025	·92365		·9524	·96825
·035	·90175		·9346	·95622
·050	·8726		·9091	·9388
·10	·7948		·8333	·8850
·15	·7343		·7692	·8368
·20	·6843		·7143	·7930
·25	·6417	0·6560	·6667	·7531
·30	·6047		·6250	·7165
·35	·5700		·5882	·6828
·40	·5429		·5556	·6516
·50	·4929	0·4946	·5000	·5960
·60	·4515		·4545	·5479
·70	·4163		·4167	·5057
·80	·3863		·3846	·4687
·90	·3602		·3571	·4360
1·0	·3373	0·3401	·3333	·4070
2·0	·1994	0·2055	·2000	·2300
3·0	·1461		·1429	·1647
4·0	·1133		·1111	·1250
5·0	·0925		·0909	·1000
10·—	·0481		·0476	·0500
20·—	·02453		·02439	·0250
50·—	·00993		·00990	·0100
100·—	·00498		·00498	·0050
200·—	·00250		·00249	·0025
400·—	·001249		·001248	·00125
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
∞	0·0000		0·0000	0·00000

Für $\frac{a}{\lambda} \geqq 4$: ${}_1P_{1\to\infty} = \frac{\lambda}{2a}\left[1-\frac{3}{8}\cdot\frac{\lambda}{a}\right]$ und ${}_1N_{1\to\infty} = \frac{\lambda}{2a}$,

also: $\left(\frac{{}_1P_1}{{}_1N_1}\right)_{\to\infty} = \left[1-\frac{3}{8}\cdot\frac{\lambda}{a}\right]_{\to\infty} \geqq 1.$

die STRUVE'sche, bezw. die BESSEL'sche Zylinderfunktion, 1^{ster} Ordnung mit komplexem Argument: $iy = i \cdot \frac{2a}{\lambda} \cdot \frac{1}{\sin \psi}$.

Durch numerische Integration von dem Ausdruck für K haben P. und P. gefunden:

$$x_1 = \frac{a}{\lambda} = 0{,}25: \; K = 0{,}0560, \quad \text{woraus:} \quad {}_1P_1 = 0{,}6560$$

$$x_1 = \frac{a}{\lambda} = 0{,}50: \; K = 0{,}0834, \quad \text{woraus:} \quad {}_1P_1 = 0{,}4996$$

$$x_1 = \frac{a}{\lambda} = 1{,}00: \; K = 0{,}1066, \quad \text{woraus:} \quad {}_1P_1 = 0{,}3401$$

$$x_1 = \frac{a}{\lambda} = 2{,}00: \; K = 0{,}1186, \quad \text{woraus:} \quad {}_1P_1 = 0{,}2055$$

$$x_1 = \frac{a}{\lambda} = \infty \; : K = 0{,}1250, \quad \text{woraus:} \quad {}_1P_1 = 0{,}0000.$$

Wir ersehen hieraus, dass unsere theoretischen Zahlenwerte für ${}_1P_1$, in hinreichender Übereinstimmung sind mit den Werten, die aus den numerisch berechneten K-Werten von P. und P. abgeleitet werden können.

In Kolonne IV sind zur Vergleichung die Werte von $\frac{1}{1+\frac{2a}{\lambda}}$ angegeben; hieraus erhellt, dass der Unterschied zwischen $\frac{1}{\left(1+\frac{2a}{\lambda}\right)}$ und ${}_1P_1$ für $a/\lambda > 0{,}5$ vernachlässigt werden kann; in dem Gebiet: $0 \leq a/\lambda \leq 0{,}5$, beträgt der maximale Unterschied ca. 4 %, so dass die Diffusionsformel: $\frac{1}{D_{11,p}} = \frac{1}{D_{11,0}} + \frac{1}{D_{11,\infty}}$, jedenfalls für kreisförmige, zylindrische Rohre, als erste Annäherung brauchbar ist. —

In Kolonne V sind die in analoger Weise berechneten Werte von ${}_1N_1$ nach der gefundenen Reihenentwicklung angegeben; für $a/\lambda \geq 4$ ist bei der Berechnung die Annäherung: $R\left(\frac{a}{\lambda}\right) = 0$, ver-

wendet; hieraus folgt für das Gebiet, worin $\frac{a}{\lambda} \geqq 4$ ist, dass ${}_1N_1 = \frac{\lambda}{2a}$.

Wir werden nun die Zahlen in der Tabelle No. VII näher betrachten; in Kolonne 1 dieser Tabelle sind wieder die Werte von a/λ angegeben; in Kolonne 2 stehen die mittleren Werte von $G = \Phi\left(\frac{a}{\lambda}\right) = G_{1,2,3}/G_{1,0}$, wie diese von Dr. MELKONIAN aus dem vollständigen Beobachtungsmaterial von MARTIN KNUDSEN abgeleitet sind.

Wenn wir mit den früheren Ausdrücken:

$$\frac{{}_1\varrho \cdot Q_t}{(p_1 - p_2)} = {}_1\varrho \cdot T_1 \quad \text{und} \quad {}_1\varrho \cdot T_{1,0} = G_{1,0},$$

vergleichen, sehen wir leicht, dass:

$$\frac{G_{1,2,3}}{G_{1,0}} = \frac{G_1}{G_{1,0}} + \frac{G_2}{G_{1,0}} + \frac{G_3}{G_{1,0}} = \frac{1}{G_{1,0}} \cdot \frac{{}_1\varrho \cdot Q_t}{p_1 - p_2} = \frac{T_1}{T_{1,0}},$$

wo:

$$\frac{G_1}{G_{1,0}} = {}_1P_1 \quad \text{und} \quad \frac{G_2}{G_{1,0}} = \frac{\pi}{4}[1 - {}_1N_1] \cdot \left[1 + \frac{1}{2} e^{-\frac{2a}{\lambda}}\right],$$

(Formel I, Tabelle VII, Kolonne 9, die gültig ist in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$), bezw. $\frac{G_2}{G_{1,0}} = \frac{3\pi}{8} \cdot [1 - {}_1N_1] \cdot \frac{{}_1P_1}{{}_1N_1}$ (Formel II, Tabelle VII, Kolonne 8, die theoretisch abgeleitet ist, aber nur gültig für das Gebiet: $\frac{a}{\lambda} \to 0$) und $G_3/G_{1,0} = \frac{3\pi}{64}\left(\frac{a}{\lambda}\right)$ (Kolonne 4).

Dieser letzte Ausdruck, $G_3/G_{1,0}$, stellt die rein hydrodynamische, laminare Strömung von POISEUILLE dar.

In Kolonne 3 stehen die Werte von: $G_{1,2,3}/G_{1,0}$, welche aus der Beobachtungsreihe von MARTIN KNUDSEN mit reiner Kohlensäure und Rohr Nr. 4 hervorgegangen sind. MARTIN KNUDSEN gibt, wie bekannt, selber an, dass diese Messungen mit dem Rohr Nr. 4 (Bündel aus 24 parallel geschalteten Rohren mit $a = 0{,}003260$ cm

Tabelle

1	2	3		4	5	
$\frac{a}{\lambda}$	$\frac{G_{1,2,3}}{G_{1,0}}$ Dr. Melkonian's Mittelwerte für CO_2, H_2 und O_2	$\frac{G_{1,2,3}}{G_{1,0}}$ a) Rohrenbündel No. 4 b) Rohr No. 3 CO_2		$\frac{G_3}{G_{1,0}} = \frac{3\pi}{64}\left(\frac{a}{\lambda}\right)$ (Poiseuille)	$\frac{(G_1+G_2)}{G_{1,0}}$, beob. $= \frac{(G_{1,2,3}-G_3)}{G_{1,0}}$ (Kol. 3 — Kol. 4.)	
0·000	1·000	1·0000		0·0000	1·0000	
·001	0·987	0·9950		0·0001⁵	0·9949	
·010	0·981	0·9800		0·0015	0·9785	
·015	0·969	0·9730		0·0022	0·9708	
·025	0·958	0·9620		0·0037	0·9583	
·035	0·952	0·9560		0·0052	0·9508	
·05	0·948	0·9510		0·0074	0·9436	
·10	0·927	0·9390		0·0147	0·9243	
·15	0·925	0·9335		0·0221	0·9114	
·20	0·923	0·9300		0·0295	0·9015	
·25	0·921	0·9285		0·0368	0·8917	
·30	0·920 Min.	0·9270		0·0442	0·8828	
·35	0·921	0·9280		0·0515	0·8765	
·40	0·923	0·9295		0·0589	0·8706	
·50	0·927	0·9360		0·0736	0·8624	
		a)	b)		a)	b)
·60	0·933	0·9440	0·9492	0·0884	0·8556	0·8608
·70	0·939	0·9540	0·9566	0·1031	·8509	·8535
·80	0·948	0·9645	0·9651	0·1178	·8467	·8473
·90	0·955	0·9760	0·9749	0·1325	·8435	·8424
1·00	0·963	0·9882	0·9855	0·1473	·8409	·8382
1·50	1·017	1·0535	1·0464	0·2209	·8326	·8255
2·00	1·076	1·1220	1·1102	0·2945	·8275	·8157
3·00	1·208	1·2645	1·2485	0·4418	·8227	·8067
4·00	1·355	1·4095	1·3909	0·5890	·8205	·8019
5·0	1·494	1·5550	1·5350	0·7361	·8189	·7989
6·0	—	1·7010	—	0·8836	·8174	—
7·0	1·770	1·8475	—	1·0308	·8167	—
8·0	—	1·9945	—	1·1781	·8164	—
9·0	—	2·1413	—	1·3254	.8159	—
10·—	2·16	2·2880	2·2652	1·4726	·8154	·7926
20·—	3·66	3·7593	3·7345	2·9452	·8141	·7893
50·—	8·01	8·1760	8·1504	7·3631	·8130	·7873
100·—	15·60	15·539	15·514	14·726	·8125	·7866
200·—	30·90	30·264	30·239	29·452	·8116	·7863
400·—	59·85	59·716	59·691	58·905	·8110	·7860
∞	∞	∞	∞	∞	0.810	0.785

VII.

6	7	8	9	10
$(G_1+G_2)/G_{1,0}$, ber. (Kolonne 7 + Kolonne 9)	$G_1/G_{1,0}$, ber. = ${}_1P_1$	$G_2/G_{1,0}$, ber. = $\frac{3\pi}{8}[1-{}_1N_1]\frac{{}_1P_1}{{}_1N_1}\cdot$	$G_2/G_{1,0}$, ber. = $\frac{\pi}{4}[1-{}_1N_1]\left[1+\frac{1}{2}\cdot e^{-\frac{2a}{\lambda}}\right]$	$\frac{\pi}{4}\cdot\frac{\frac{2a}{\lambda}}{1+\frac{2a}{\lambda}}$
1·0000	1·00000	0·00000	0·00000	0·00000
0·9961	0·99454	·00156	·00157	·00157
·9779	·96266	·01498	·01525	·01540
·9711	·94852	·02208	·02259	·02288
·9605	·92365	·03569	·03680	·03740
·9522	·90175	·04863	·05040	·05138
·9424	·8726	·06703	·06980	·07140
·9221	·7948	·1217	·12730	·1309
·9101	·7343	·1687	·1758	·1812
·9014	·6843	·2104	·2171	·2244
·8944	·6417	·2478	·2527	·2618
·8885	·6047	·2819	·2838	·2945
·8810	·5700	·3120	·3110	·3234
·8780	·5429	·3420	·3351	·3490
·8686	·4929	(·3937)	·3757	·3927
·8601	.4515	(·4389)	·4086	·4284
·8527	.4163	(·4793)	·4364	·4582
·8457	.3863	—	·4594	·4833
·8398	.3602	—	·4797	·5048
·8345	.3373	—	·4972	·5236
·8179	.2510	—	·5669	·5891
·8097	.1994	—	·6103	·6283
·8030	.1461	—	·6569	·6734
·8006	.1133	—	·6873	·6981
·7994	.0925	—	·7069	·7140
—	—	—	—	—
—	—	—	—	—
—	—	—	—	—
—	—	—	—	—
·7942	0·0481	—	·7461	·7480
·7904	0·0245	—	·7658	·7662
·7875	0·0099	—	·7776	·7735
·7865	0·0050	—	·7815	·7834
·7860	0·0025	—	·7834	·7822
·7860	0·00125	—	·7848	·7853
0·7854	0.0000	$\left(\frac{3\pi}{8}=1\cdot1781\right)$	$\frac{\pi}{4}=0\cdot7854$	$\frac{\pi}{4}=0.7854$

und Länge des Bündels: L = ca. 2 cm) die bestmöglichen seiner Beobachtungsreihen darstellen.[1]

Die bei den verschiedenen Mitteldrucken: $\bar{p} = \frac{1}{2}(p_1 + p_2)$, beobachteten T_1-Werte in dem Bereich: $0{,}5 \leqq \frac{a}{\lambda} \leqq \infty$, sind unter Verwendung der Interpolationsformel:

$$T_1 = a_1\bar{p} + b_1\frac{1}{1 + c_2 \cdot \bar{p}} + b_1\frac{c_1\bar{p}}{1 + c_2 \cdot \bar{p}}, \text{ bezw.}$$

$$\frac{T_1}{b_1} = \frac{G_{1,2,3}}{G_{1,0}} = \frac{a_1}{b_1}\bar{p} + \frac{1}{1 + c_2 \cdot \bar{p}} + \frac{c_1\bar{p}}{1 + c_2 \cdot \bar{p}},$$

nach der Methode der kleinsten Quadrate ausgewertet; hierdurch sind die Konstanten dieser Formel bestimmt, und mit Hilfe dieser Konstanten sind die in der Kolonne 3 *a* angegebenen Werte berechnet. —

In derselben Weise ist das Beobachtungsmaterial von Martin Knudsen mit dem Rohr Nr. 3 (a = 0,01415 cm und L = ca. 12 cm) und reiner Kohlensäure behandelt; auch für diese Beobachtungsreihe ist das beobachtete Material für $\frac{a}{\lambda} > 0{,}5$ nach der Methode der kleinsten Quadrate behandelt und ausgewertet. Mit den hierdurch erhaltenen Konstanten sind die in der Kolonne 3 *b* angegebenen Werte für $G_{1,2,3}/G_{1,0}$ von dieser Messreihe bestimmt.

Für das Gebiet $a/\lambda > 0{,}5$ ergeben diese zwei Beobachtungsreihen einen Unterschied im Grenzwert für $a/\lambda \to \infty$, bezw. für die Konstante: c_1/c_2, die bei grossen Werten von a/λ für die Gleitung massgebend ist, da wir aus der Interpolationsformel für: $a/\lambda \to \infty$, erhalten:

$$\frac{T_1}{b_1} = \frac{G_{1,2,3}}{G_{1,0}} = \frac{a_1}{b_1}\bar{p} + \frac{c_1}{c_2} \cdot —$$

Für das Rohrenbündel Nr. 4 ergab sich $c_1/c_2 = 0{,}810$ (Kolonne 5 *a*), während für das Rohr Nr. 3, wie auch für das Rohr Nr. 1, $c_1/c_2 = 0{,}785$ gefunden wurde (Kolonne 5 *b*). Dieser letzte Wert ist auch in Übereinstimmung mit dem Wert, der annäherungsweise aus den Mittelwerten von Dr. Melkonian für das ganze Beobachtungsmaterial abgeleitet werden kann. Dieser Wert c_1/c_2 =

1) Martin Knudsen: loc. cit. S. 118, 1909.

0,785 ist in Übereinstimmung mit dem theoretischen Wert: $c_1/c_2 = \frac{\pi}{4} = 0{,}7854$, welcher sich ergab, wenn $k_2 =$ ca. $\frac{4}{3}$, in Übereinstimmung mit dem Mittelwert der Beobachtungen, gesetzt wurde.

In dem Gebiet: $0 \leqq \frac{a}{\lambda} \leqq$ ca. 0,6, sind die KNUDSEN'schen Beobachtungen für Kohlensäure mit den Rohren Nr. 4 und No. 3 graphisch ausgeglichen, weil keine angemessene Interpolationsformel in diesem Gebiet vorliegt. Die Übereinstimmung der Beobachtungen mit Rohr No. 4 und Rohr No. 3 ist in diesem Gebiet sehr befriedigend; die in der Kolonne 3 angegebenen Werte von $G_{1,2,3}/G_{1,0}$ für das Gebiet: $0 \leqq \frac{a}{\lambda} \leqq 0{,}5$, sind durch graphische Ausgleichung bestimmt. Leider waren in diesen Versuchen, wie bekannt, die anwesenden Quecksilberdämpfe nicht durch Ausfrieren entfernt.

Die Übereinstimmung zwischen den Mittelwerten der Kolonne 2 (Dr. MELKONIAN) und den Zahlen in Kolonne 3 ist auch befriedigend, wenn die Messgenauigkeit in Betracht gezogen wird. —

Es erhellt auch aus diesen Tabellen, dass das beobachtete Minimum bei: $\left(\frac{a}{\lambda}\right)_{\text{min.}} =$ ca. 0,31, liegt, während $G_{1,2,3}$ (Min.) = 0,925. $G_{1,0}$, wie schon früher erwähnt.

In der Kolonne 4 sind die berechneten Werte für $G_3/G_{1,0} = \frac{3\pi}{64} \cdot \left(\frac{a}{\lambda}\right)$ angegeben, d. h. die Werte für die hydrodynamische, laminare Strömung von POISEUILLE. Für diese Strömung haben wir als erste Annäherung angenommen, dass der Ausdruck $G_3/G_{1,0} = \frac{3\pi}{64} \cdot \left(\frac{a}{\lambda}\right)$, der jedenfalls richtig ist für grössere Werte von a/λ, auch annähernd brauchbar bleibt in dem Gebiet $a/\lambda \to 0$. Aus den Werten in Kolonne 4 erhellt, dass der Einfluss des Gliedes $G_3/G_{1,0}$ auf $G_{1,2,3}/G_{1,0}$ in dem Gebiet $a/\lambda \to 0$ jedenfalls sehr klein wird; erst bei etwas grösseren Werten von a/λ, d. h. $a/\lambda \geqq 0{,}1$ à $0{,}2$, wird der Einfluss des Gliedes $G_3/G_{1,0}$ von messbarer Bedeutung; bei dem Wert: $a/\lambda = 0{,}3$, beträgt der Einfluss ca. 5 % und bei: $a/\lambda = 0{,}6$, ca. 10 %.

In Kolonne 5 sind die Werte von:

$\left(\frac{G_1+G_2}{G_{1,0}}\right)_{\text{beob.}} = (G_{1,2,3}/G_{1,0})_{\text{beob.}} - (G_3/G_{1.0})_{\text{ber.}}$, angegeben, d. h. die Differenzen zwischen den Werten in den Kolonnen 3 und 4.

Wir ersehen hieraus, dass dieser »beobachtete« Wert von $(G_1 + G_2)/G_{1,0}$ gleichmässig von 1 (für $\frac{a}{\lambda} = 0$) bis auf 0,785 (0,810) für $\left(\frac{a}{\lambda}\right) \to \infty$ abnimmt.

Es erhellt also hieraus, dass der Wert von: $\left(\frac{G_1+G_2}{G_{1,0}}\right)_{\text{beob.}}$, mit wachsendem Wert von $\frac{a}{\lambda}$, d. h. mit wachsendem Wert von $\bar{p}$, gleichmässig *abnimmt*, so dass die natürliche Erklärung für das Entstehen des beobachteten Minimums bei $a/\lambda = 0{,}31$ gesucht werden muss in dem Eintreten der hydrodynamischen, laminaren Strömung von POISEUILLE, $G_3/G_{1,0}$, die mit wachsendem Wert von $\frac{a}{\lambda}$, bezw. $\bar{p}$, *zunimmt*, und also *nicht* in der gaskinetischen Wechselwirkung der Moleküle, bezw. in der Summe der gaskinetischen Ausdrücke für $G_1/G_{1,0}$ (herrührend von der Selbstdiffusion, bezw. Druckdiffusion) und $G_2/G_{1,0}$ (das Gleitungsglied). (Vgl. Kurve I + II, Fig. 14).

In der Kolonne 6 steht die Summe der Zahlen der Kolonne 7 (das berechnete Diffusionsglied: $G_1/G_{1,0} = {}_1P_1$) und der Kolonne 9, d. h. das berechnete Gleitungsglied:

$$\frac{G_2}{G_{1,0}} = \frac{\pi}{4}\left[1 - {}_1N_1\right]\left[1 + \frac{1}{2} \cdot e^{-\frac{2a}{\lambda}}\right].$$

Die Summe dieser beiden theoretischen Glieder (Kolonne 6) stellt also den theoretisch berechneten Wert von $\left(\frac{G_1+G_2}{G_{1,0}}\right)_{\text{ber.}}$ dar und muss somit, *wenn die Theorie brauchbar ist*, mit den observierten Zahlen in Kolonne 5, bezw. 5 *b*, übereinstimmen. Durch Vergleichung erhellt, dass dies in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, der Fall ist; die grössten Abweichungen betragen nur ca. $^3/_4$ %, aber im Mittel weniger. —

In der Kolonne 7 stehen die theoretischen Werte von $G_1/G_{1,0} =$

$= {}_1P_1$, d. h. die berechnete Selbstdiffusionsströmung, bezw. die Druckdiffusionsströmung des reinen Gases. Es kann angenommen werden, dass die theoretischen Werte von $G_1/G_{1,0} = {}_1P_1$ für grössere Werte von a/λ (bezw. im MAXWELL'schen Zustand der Gase) zu klein sind, weil unsere Formel für ${}_1P_1$ für $a/\lambda \to \infty$, $D_{11,\infty} = \frac{1}{3}\overline{\Omega}\lambda$, ergibt, während der exakte Ausdruck für den reinen MAXWELL'schen Zustand, $D_{11,\infty} = 0{,}60 \cdot \overline{\Omega}\lambda$, sein muss, jedenfalls wenn die Moleküle als harte, elastische Kugeln aufgefasst werden können. —

Der Einfluss des Gliedes, $G_1/G_{1,0} = {}_1P_1$, in der Summe $\left(\frac{G_1 + G_2}{G_{1,0}}\right)$ ist aber in dem Zustande: $a/\lambda \to \infty$, dem Gleitungsglied, $G_2/G_{1,0}$, gegenüber so gering, dass es schwierig sein wird, diesen Unterschied experimentell nachzuweisen, jedenfalls solange der theoretische Wert des Gleitungsgliedes in dem Gebiet: $\frac{a}{\lambda} \to \infty$, nicht einwandfrei theoretisch abgeleitet ist, und solange das Gleitungsglied und das Diffusionsglied in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, nicht, unabhängig von einander, mit grösster Genauigkeit experimentell untersucht worden sind.

In der Kolonne 8 stehen die nach der theoretischen Formel: $G_2/G_{1,0} = \frac{3\pi}{8}\left[1 - {}_1N_1\right]\frac{{}_1P_1}{{}_1N_1}$, berechneten Werte des Gleitungsgliedes.

Diese Formel ist aber nur gültig in dem Gebiet: $\frac{a}{\lambda} \to 0$, d. h. solange die POISEUILLE-Strömung noch nicht von Bedeutung geworden ist. Dies erhellt auch daraus, dass diese Formel für: $G_2/G_{1,0}$, ber., nicht den richtigen Grenzwert für $\frac{a}{\lambda} \to \infty$ ergibt.

In der Kolonne 9 stehen die Werte des Gleitungsgliedes, $G_2/G_{1,0}$, wenn angenommen wird, dass der Wert für v_x in analoger Weise, wie von MARTIN KNUDSEN und SOPHUS WEBER[1] in ihrer Untersuchung über den Widerstand gegen die langsame Bewegung kleiner Kugeln (Gesetz von STOKES) gefunden, mit a/λ variiert. —

Die auf diese Weise abgeleitete Formel:

1) Vgl. S. 40.

$\frac{G_2}{G_{1,0}} = \frac{\pi}{4}\left[1 - {}_1N_1\right]\left[1 + \frac{1}{2}e^{-\frac{2a}{\lambda}}\right]$, ergibt nicht allein die richtigen Werte für: $\frac{a}{\lambda} = 0$, $\frac{a}{\lambda} \to 0$ und $\frac{a}{\lambda} \to \infty$, sondern es erhellt ausserdem aus den Zahlen in den Kolonnen, 8 und 9, dass die Werte für $a/\lambda \to 0$ in befriedigender Übereinstimmung sind mit den berechneten Werten, die sich in dem Gebiet: $0 \leqq \frac{a}{\lambda} \leqq$ ca. 0,35 , aus der theoretischen Formel: $G_2/G_{1,0} = \frac{3\pi}{8}\left[1 - {}_1N_1\right] \cdot \frac{{}_1P_1}{{}_1N_1}$, erschliessen lassen.

Für grössere Werte von $\frac{a}{\lambda}$ erhält man aus dieser letzten, theoretischen Formel zu grosse Werte für $G_2/G_{1,0}$, weil in dieser Formel der Änderung von v_x bei grösseren Werten von $\left(\frac{a}{\lambda}\right)$ wegen der Poiseuille-Strömung nicht Rechnung getragen ist. Dies erhellt auch daraus, dass der Grenzwert für $a/\lambda \to \infty$: $\frac{3}{2} \cdot \frac{\pi}{4}$, wird, während dieser Grenzwert $\frac{\pi}{4}$ betragen muss.

In der Kolonne 10 sind zur Vergleichung die Werte für die Grösse: $\frac{\pi}{4} \cdot \frac{\frac{2a}{\lambda}}{1 + \frac{2a}{\lambda}}$, welche wir als Annäherung für $G_2/G_{1,0}$ verwendet haben, angegeben.

Es erhellt durch Vergleichung der Zahlen in der Kolonne 10 mit den richtigen Werten in der Kolonne 8 und Kolonne 9, dass der Ausdruck: $\frac{\pi}{4} \cdot \frac{\frac{2a}{\lambda}}{1 + \frac{2a}{\lambda}}$, eine brauchbare Annäherung für $G_2/G_{1,0}$ darstellt.

Der maximale Unterschied beträgt ca. 5 %, während die Unterschiede in den Gebieten: $a/\lambda \to 0$ und $a/\lambda \to \infty$, nur sehr gering sind. —

Es erhellt aus den vorhergehenden Betrachtungen, dass der von Martin Knudsen in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, be-

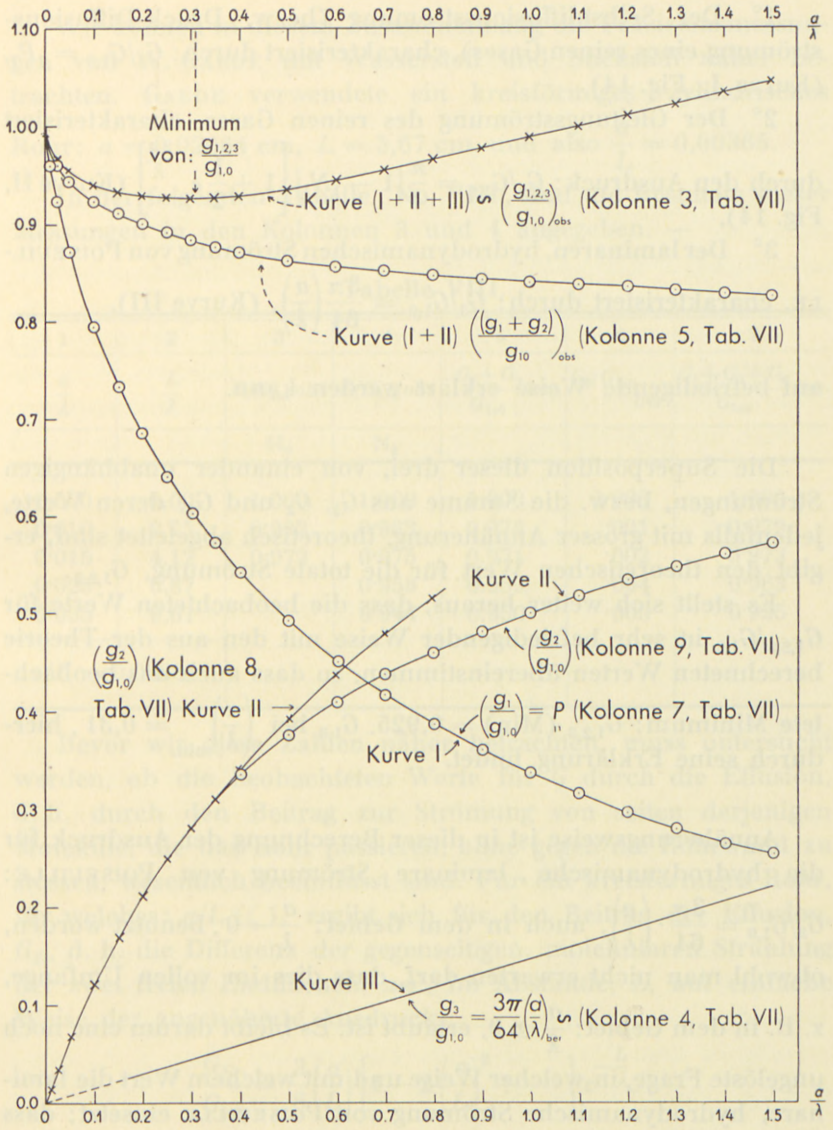


Fig. 14.

obachtete Verlauf der stationären Durchströmung eines reinen Gases durch ein kreisförmiges, zylindrisches Rohr durch die Superposition dreier von einander unabhängiger Strömungen, (vgl. Kurve: I + II + III, in Fig. 14) nämlich:

1° Der Selbstdiffusionsströmung (bezw. Druck-Diffusionsströmung eines reinen Gases), charakterisiert durch: $G_1/G_{1,0} = {}_1P_1$, (Kurve I, Fig. 14),

2° Der Gleitungsströmung des reinen Gases, charakterisiert durch den Ausdruck: $G_2/G_{1,0} = \frac{\pi}{4}[1 - {}_1N_1]\left[1 + \frac{1}{2}e^{-\frac{2a}{\lambda}}\right]$ (Kurve II, Fig. 14),

3° Der laminaren, hydrodynamischen Strömung von Poiseuille, charakterisiert durch: $G_3/G_{1,0} = \frac{3\pi}{64}\left(\frac{a}{\lambda}\right)$, (Kurve III),

auf befriedigende Weise erklärt werden kann.

Die Superposition dieser drei, von einander unabhängigen Strömungen, bezw. die Summe aus G_1, G_2 und G_3, deren Werte, jedenfalls mit grosser Annäherung, theoretisch abgeleitet sind, ergibt den theoretischen Wert für die totale Strömung, $G_{1,2,3}$.

Es stellt sich weiter heraus, dass die beobachteten Werte für $G_{1,2,3}/G_{1,0}$ in sehr befriedigender Weise mit den aus der Theorie berechneten Werten übereinstimmen, so dass auch das beobachtete Minimum: $G_{1,2,3}$ (Min) = 0,925. $G_{1,0}$ bei $\left(\frac{a}{\lambda}\right)_{\text{min.}} = 0{,}31$, hierdurch seine Erklärung findet.

Annäherungsweise ist in dieser Berechnung der Ausdruck für die hydrodynamische, laminare Strömung von Poiseuille: $G_3/G_{1,0} = \frac{3\pi}{64} \cdot \left(\frac{a}{\lambda}\right)$, auch in dem Gebiet: $\frac{a}{\lambda} \to 0$, benutzt worden, obwohl man nicht erwarten darf, dass dies im vollen Umfange, z. B. in dem Gebiet: $\frac{a}{\lambda} \geqq 0$, erlaubt ist. Es bleibt darum eine noch ungelöste Frage, in welcher Weise und mit welchem Wert die laminare, hydrodynamische Strömung von Poiseuille einsetzt; dass diese laminare, hydrodynamische Strömung bereits bei dem Wert: $\frac{a}{\lambda}$ = ca. 0,3, im vollen Umfange eingetreten ist, erhellt aber aus der Übereinstimmung zwischen dem theoretischen und beobachteten Minimum für $G_{1,2,3}$ bei $\frac{a}{\lambda} = 0{,}31$. —

Wir können in diesem Zusammenhang die Präzisionsmessungen von W. GAEDE mit Wasserstoff und Stickstoff näher betrachten. GAEDE verwendete ein kreisförmiges, zylindrisches Rohr: $a = 0{,}02064$ cm, $L = 5{,}67$ cm und also $\frac{a}{L} = 0{,}00365$.

In der folgenden Tabelle, No. VIII, sind die Resultate dieser Messungen in den Kolonnen 3 und 4 angegeben. —

Tabelle VIII.

1	2	3	4	5	6	7
$\frac{a}{\lambda}$	$\frac{L}{\lambda}$	$G/G_{1,0}$ obs	$G/G_{1,0}$ obs	$\frac{G_1+G_2}{G_{1,0}}$ ber.	$G_3/G_{1,0}$ ber.	$\frac{G_1+G_2+G_3}{G_{1,0}}$ ber.
		H_2	N_2			
0·000	0.00	1·000	1·000	1.000	0·000	1·000
0·010	2.75	0·982	0·983	0.978	·001	0·979
0·015	4.12	0·972	0·975	0.971	·002	0·973
0·025	6.87		0·958	0.959	·004	0·963
0·035	9.61		0·944	0.950	·005	0·955
$a/L = 0{,}00365$						

Bevor wir diese Zahlen näher betrachten, muss untersucht werden, ob die beobachteten Werte für G durch die Effusion, d. h. durch den Beitrag zur Strömung von seiten derjenigen Moleküle, die das Rohr passieren, ohne gegen die Rohrwand zu stossen, wesentlich beeinflusst sind. Für ein kreisförmiges Rohr, für welches: $a/L \ll 1$, ergibt sich für den Beitrag der Effusion, G_E, d. h. die Differenz der gegenseitigen, molekularen Strahlung der zwei freien Endflächen, πa^2, im Abstande, L, auf einfache Weise der angenäherte Ausdruck:

$$\frac{G_E}{G_{1,0}} = \frac{3}{8}\left(\frac{a}{L}\right)\left[1 - 2\left(\frac{a}{L}\right)^2 + \cdots\right] e^{-\frac{L}{\lambda}},$$

wenn $a/L \ll 1$. Es geht aus dieser Formel hervor, dass diese Korrektion, wenn $a/L = 0{,}00365$, ohne Bedeutung ist, selbst für den Wert für G bei $\frac{a}{\lambda} = 0{,}01$. —

In der obenstehenden Tabelle, No. VIII, stehen in Kolonne 1 die Werte für a/λ und in Kolonne 2 die Werte für L/λ. In den

Kolonnen 3 und 4 stehen die von GAEDE beobachteten und von ihm ausgewerteten Werte für $G/G_{1,0}$, wo $G_{1,0}$ den Wert für G bei $a/\lambda = 0$ darstellt. In Kolonne 5 stehen die berechneten Werte für $\frac{(G_1 + G_2)}{G_{1,0}}$, und in Kolonne 6 die berechneten Werte für $G_3/G_{1,0} = \frac{3\pi}{64}\left(\frac{a}{\lambda}\right)$. In Kolonne 7 steht die Summe der berechneten Werte für: $G_1/G_{1,0}$, $G_2/G_{1,0}$ und $G_3/G_{1,0}$.

Aus diesen GAEDE'schen Versuchsreihen, die leider nur bis $a/\lambda = 0{,}035$ durchgeführt sind, scheint hervorzugehen, dass die Übereinstimmung zwischen den beobachteten und berechneten Werten am grössten ist, wenn das POISEUILLE-Glied weggelassen wird; dies ist auch in Übereinstimmung mit den Resultaten in der Tabelle No. VII, wie durch Vergleichung der Zahlen der Kolonnen 2 und 3 mit den Zahlen der Kolonnen 5, 6 und 9 erhellt.

Es darf aber bei dieser Vergleichung nicht übersehen werden, dass bei solchen sehr niedrigen Werten von a/λ die quantitative Bedeutung des Gliedes, $G_3/G_{1,0}$, sehr klein ist gegenüber der Bedeutung des Gliedes: $\frac{(G_1 + G_2)}{G_{1,0}}$. —

Andererseits ist es in Übereinstimmung mit der molekularkinetischen Auffassung des KNUDSEN-Zustandes: $(a/\lambda) \fallingdotseq 0$, dass die Entstehung einer hydrodynamischen Strömung erst dann möglich sein wird, wenn a/λ so gross geworden ist, dass die Gasmenge in einem Raumelement des Rohres als ein Kontinuum aufgefasst werden kann und also in Analogie mit einer Flüssigkeit theoretisch behandelt werden darf; wie bekannt, ist diese Analogie vollständig in dem MAXWELL'schen Zustand der Gase, d. h. wenn $\frac{a}{\lambda} \risingdotseq \infty$.

Es wäre ohne Zweifel von Interesse und von Bedeutung, auch zur Beleuchtung dieser Analogie, wenn das vorliegende Beobachtungsmaterial, besonders in dem Gebiet: $0 \leqq \frac{a}{\lambda} \leqq$ ca. 1, durch eine Reihe von Präzisionsmessungen für Helium und Neon, sowie auch für Wasserstoff und Stickstoff nach dem Vorbilde von

W. Gaede und Martin Knudsen ergänzt werden könnte. Es wäre dann vielleicht möglich, weil nun eine quantitative Theorie für diese komplizierte Strömung in dem ganzen Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, vorliegt, etwas näheres über diese Analogie zwischen Gasen und Flüssigkeiten zu erfahren; ausserdem wäre es für die Theorie von Bedeutung , wenn die Gleitung unabhängig von der Selbstdiffusionsströmung untersucht (z. B. nach der Ablenkungsmethode) und die Abhängigkeit der Selbstdiffusion von a/λ mit Hilfe der Isotopenmethode bestimmt würde. —

§ 12. Aus der abgeleiteten Formel für: $G/G_{1,0} = G_{1,2,3}/G_{1,0}$, mit oder ohne Poiseuille-Glied, geht hervor, dass für $x_1 = \frac{a}{\lambda} \to 0$:

$$G_{1,2,3} = G_{1,0}\,[1 + \alpha_1 x_1 + \alpha_2 x_1 \ln x_1 + \alpha_3 \cdot x_1^2 + \alpha_4 \cdot x_1^2 \ln x_1 + \cdots],$$

wo:
$$G_{1,0} = \frac{4}{3}\sqrt{2\pi} \cdot \sqrt{_1\varrho}\, a^3 \cdot \frac{p_1 - p_2}{L}.$$

Hieraus erhellt, dass für $x_1 = \frac{a}{\lambda} = 0$: $G_{1,2,3} = G_{1,0}$, da $\lim x_1^n \ln x_1 = 0$ für $x_1 = 0$, wenn $n > 0$. —

Weiter erhalten wir hieraus:

$$\frac{dG_{1,2,3}}{dx_1} = G_{1,0}\,[\alpha_1 + \alpha_2 + \alpha_2 \ln x_1 + \cdots],$$

und also für $x_1 = 0$: $\frac{dG_{1,2,3}}{dx_1} = -\infty$.

Der Wert , $G'_{1,0}$, für die Molekularströmung bei $\frac{a}{\lambda} = 0$ ist für den Fall, dass die Bedingung: $\frac{a}{L} \gtrless 0$, nicht erfüllt ist, früher in mehreren Untersuchungen behandelt worden; im Allgemeinen ist man bei der Aufstellung der Formeln von der Knudsen'schen Formel[1] für die Molekulardiffusion durch ein Loch in einer dünnen Platte, ausgegangen. —

1) Martin Knudsen: Ann. d. Phys. 4, B 28, S. 999, 1909.

S. DUSHMAN[1] hat hierfür die Formel:

$$G'_{1,0} = G_{1,0} \cdot \frac{1}{1 + \frac{8}{3}\left(\frac{a}{L}\right)}\,,$$ angegeben.

Diese Formel befriedigt die Grenzwerte, sowohl für die Molekulardiffusion durch ein Loch in einer dünnen Platte $\left(\frac{a}{L} \geqq \infty\right)$, als auch für die reine Molekularströmung durch ein unendlich langes, zirkulares Rohr $\left(\frac{a}{L} = 0\right)$.

P. CLAUSING[2] hat, wenn $\frac{a}{\lambda} = 0$, dies Problem für alle Werte von $\frac{a}{L}$ eingehend theoretisch untersucht, während E. H. KENNARD[3] aus den von CLAUSING berechneten Werten für $G'_{1,0}$ die folgende Interpolationsformel:

$$G'_{1,0} = G_{1,0} \cdot \frac{1 + \frac{5}{2}\left(\frac{a}{L}\right)}{1 + \frac{19}{3}\left(\frac{a}{L}\right) + \frac{20}{3}\left(\frac{a}{L}\right)^2}\,,$$ abgeleitet hat.

In dieser Formel ist, wie auch in der Formel von DUSHMAN:

$$G_{1,0} = \frac{4\sqrt{2\pi}}{3} \cdot \sqrt{_1\varrho}\, a^3 \cdot \frac{p_1 - p_2}{L}.$$

Die Interpolationsformel von CLAUSING-KENNARD ist jedenfalls für: $\frac{a}{L} < \frac{2}{3}$, sehr befriedigend und stimmt auch mit den Grenzwerten für: $\frac{a}{L} = 0$ und $\frac{a}{L} \geqq \infty$, überein. —

POLLARD und PRESENT[4] haben für ein kreisförmiges, zylindrisches Rohr mit endlicher Länge, L, einen Ausdruck für $G'_{1 \to 0}$

1) S. DUSHMAN: Vacuum Technique, S. 101, New York, 1949.
2) P. CLAUSING: Thesis, Leiden, § 23, S. 117, 1928, und Physica, Bd. 9, S. 65, 1929.
3) E. H. KENNARD: loc. cit. § 173, S. 308.
4) POLLARD and PRESENT: loc. cit. pag. 769.

in dem Zustande, charakterisiert durch: $\lambda \gg L \gg a$, theoretisch zu berechnen versucht.

Durch eine komplizierte Annäherungsrechnung haben sie die Formel:

$$G'_{1\to 0} = G_{1,0}\left[1 - \frac{3}{4}\left(\frac{a}{L}\right) - \left(0{,}4764 + \frac{3}{4} ln \frac{L}{2a}\right)\left(\frac{a}{\lambda}\right) \cdots\right],$$

abgeleitet. —

Da P. und P. aber der zusätzlichen Geschwindigkeit, v_x, der Moleküle, die von gegenseitigen Zusammenstössen im Inneren des Rohres kommen, nicht Rechnung getragen haben, hat das Resultat von P. und P. nur auf die molekulare Diffusionsströmung, $G'_{1\to 0}$, und nicht auf die Gleitungsströmung, $G'_{2\to 0}$, Bezug, und also auch nicht auf die Totalströmung: $G_{1,2,3\to 0} = (G'_1 + G'_2)_{\to 0}$, im Zustande: $\frac{a}{\lambda} \geqq 0$.

Aus dem Ausdruck von P. und P. für $G'_{1\to 0}$:

$$G'_{1\to 0} = G_{1,0}\left[1 - \frac{3}{4}\left(\frac{a}{L}\right) - \left(0{,}4764 + \frac{3}{4} ln \frac{L}{2a}\right)\left(\frac{a}{\lambda}\right) \cdots\right],$$

ergibt sich für: $\frac{a}{\lambda} = 0$ und $\frac{a}{L} = 0$:

$$\lim_{\to 0}\left(\frac{G'_{1\to 0}}{G_{1,0}}\right) = 1 \quad \text{und} \quad \lim_{\to 0}\left[\frac{1}{G_{1,0}} \cdot \frac{dG'_{1\to 0}}{d\frac{a}{\lambda}}\right] = -\infty,$$

während für: $\frac{a}{\lambda} = 0$ und $0 < \frac{a}{L} \ll 1$, der Ausdruck: $G'_{1,0} = G_{1,0}\left[1 - \frac{3}{4}\left(\frac{a}{L}\right)\right]$, abgeleitet wird, so dass die Übereinstimmung mit den Resultaten von S. Dushman und von Clausing-Kennard nicht als befriedigend angesehen werden kann.

Wenn die von P. und P. vernachlässigte Gleitungsströmung, $G'_{2\to 0}$, keinen Beitrag von Bedeutung zu der Totalströmung: $G_{1,2\to 0} = (G'_1 + G'_2)_{\to 0}$, leisten sollte, erhält man hieraus, wenn: $\frac{a}{\lambda} = x_1 = 0$ und $0 < \frac{a}{L} \ll 1$, den Grenzwert:

$$\lim_{x_1 = 0}\left[\frac{1}{G_{1,0}} \cdot \frac{dG'_{1\to 0}}{dx_1}\right] = -\left(0{,}4764 + \frac{3}{4} ln \frac{L}{2\,a}\right),$$

so dass für ein Rohr mit endlicher Länge, im Gegensatz zu meiner Formel für ein unendlich langes Rohr, d. h. $\frac{a}{L} = 0$, ein endlicher Wert für den Differentialkoeffizienten der Kurve $\left(\frac{G'_1 + G'_2}{G_{1,0}}, \frac{a}{\lambda}\right)$ im Punkte, $\frac{a}{\lambda} = x_1 = 0$, erhalten wird.

Ob dies auch der Fall sein wird, wenn dem Einfluss des Gliedes, $\frac{G'_{2\to 0}}{G_{1,0}}$, für $\frac{a}{\lambda} \gtrsim 0$ und bei $0 < \frac{a}{L} \ll 1$, Rechnung getragen wird, muss noch näher untersucht werden. —

Es ist durch die experimentelle Untersuchung von W. Gaede[1] über die Strömung von Wasserstoff durch einen rechtangulären Spalt und insbesondere durch die schönen Untersuchungen von R. E. H. Rasmussen[2] über die Strömung der reinen Gase durch Ringspalten und andere Spalten bekannt, dass das Minimum der Kurve: $G = \Phi\left(\frac{h}{\lambda}\right)$, wo h die Höhe des Spaltes angibt, in der Nähe von: $h/\lambda =$ ca. 1 , liegt, während der Wert des Minimums: $G_{\text{min.}} =$ ca. $0{,}50 . G_{1,0}$, beträgt. —

In einer folgenden Abhandlung werde ich nachweisen, dass diese experimentellen Resultate auch mit der hier angegebenen Theorie übereinstimmen, und dass die experimentell gefundenen Werte für die Lage und Grösse des Minimums in quantitativer Übereinstimmung sind mit der Entstehung der totalen, gemessenen Strömung durch Superposition der drei erwähnten, von einander unabhängigen Strömungen, nämlich 1° der Selbstdiffusionsströmung des Gases in dem laminar strömenden Gase (bezw. der Druckdiffusionsströmung des reinen Gases) , 2° der Gleitungsströmung des reinen Gases und 3° der hydrodynamischen, laminaren Strömung von Poiseuille. —

1) W. Gaede: Ann. d. Phys, Bd. 41, S 305, 1913.
2) R. E. H. Rasmussen: loc. cit. S. 677—680.

Zusammenfassung.

In § 1 wird eine Übersicht über die grundlegenden Untersuchungen von Martin Knudsen (1909) über die stationäre Molekularströmung der reinen Gase durch zylindrische Rohre mit kreisförmigem Querschnitt (Radius R, Länge L und $R/L \ll 1$) gegeben; auch der Übergang von der Molekularströmung in die laminare, hydrodynamische Poiseuille-Strömung mit Gleitungskorrektion wird besprochen.

Ausserdem wird eine kurze Übersicht über die späteren experimentellen Arbeiten in diesem Gebiet von Martin Knudsen (1911), von W. Gaede (1913), von H. Adzumi (1939) und die theoretische Arbeit von W. G. Pollard und R. D. Present (1948) gegeben. —

Das Beobachtungsmaterial von Martin Knudsen für die reinen Gase, H_2, O_2 und CO_2, ist von Dr. Melkonian (1948) graphisch untersucht und von mir teilweise neuberechnet worden; es kann mit guter Annäherung durch die dreigliedrige, semiempirische Interpolationsformel von Martin Knudsen:

$$G' = \frac{G}{p_1 - p_2} = {}_1\varrho \cdot \frac{Q_t}{p_1 - p_2} =$$

$$= {}_1\varrho \cdot T_1 = \frac{4}{3}\sqrt{2\pi} \cdot \frac{R^3}{L}\sqrt{{}_1\varrho} \cdot \left[\frac{3\pi}{64} \cdot \frac{R}{\lambda} + \frac{\beta_1 \cdot \frac{R}{\lambda}}{1 + \alpha_1 \cdot \frac{R}{\lambda}} + \frac{1}{1 + \alpha_1 \cdot \frac{R}{\lambda}} \right], \quad (1)$$

in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, wiedergegeben werden.

G ist die in der Zeiteinheit durchströmende Gewichtsmenge des reinen Gases, wenn die Druckdifferenz zwischen den Endflächen des Rohres $(p_1 - p_2)$ beträgt, und ${}_1\varrho$ das spezifische Gewicht des Gases bei dem Druck, 1 Bar, und der Temperatur, t°C. ist.

λ ist die mittlere freie Weglänge in dem freien Gase, berechnet nach der Formel von S. Chapman, d. h. aus $p\lambda = \sqrt{\frac{\pi}{2}} \cdot \frac{\eta}{\sqrt{{}_1\varrho}}$, wo η den Koeffizienten der inneren Reibung des Gases bezeichnet.

Alle Einheiten sind absolute Einheiten. —
Für $R/\lambda = 0$ gilt die Formel von MARTIN KNUDSEN:

$$G_0 = \frac{4}{3}\sqrt{2\pi}\frac{R^3}{L}\sqrt{\varrho}\,(p_1 - p_2) = \frac{2\pi}{3} m\,\overline{\Omega} R^3 \frac{dN}{dl}.$$

Dieser Ausdruck für die reine Molekularströmung, oder vielleicht besser »die reine Molekulardiffusion«, ist von MARTIN KNUDSEN unter der Voraussetzung, dass das Cosinusgesetz absolute Gültigkeit für die Zurückwerfung der Moleküle an der festen Rohrwand hat, theoretisch abgeleitet worden. Der theoretische Ausdruck , G_0 , stimmt mit den experimentellen Resultaten der Präzisionsmessungen von MARTIN KNUDSEN und W. GAEDE in dem Gebiet: $R/\lambda \rightrightarrows 0$, überein.

In diesen letzten Messungen waren die anwesenden Dämpfe durch Ausfrieren mit flüssiger Luft entfernt. —

MARTIN KNUDSEN hat nur den Grenzfall: $R/\lambda = 0$, seiner semi-empirischen Formel theoretisch untersucht. Weiter hat er festgestellt, dass die Interpolationsformel für G, in Übereinstimmung mit den Beobachtungen, ein Minimum: $G_{\text{min.}} = \text{ca. } 0{,}95 \cdot G_0$ bei $\left(\frac{R}{\lambda}\right)_{\text{min.}} = \text{ca. } 0{,}30$, hat.

Die Frage der Entstehung und der theoretischen Berechnung dieses Minimums war bis heute noch ungeklärt geblieben.

Aus der dreigliedrigen, semi-empirischen Formel von MARTIN KNUDSEN erhellt, dass die Totalströmung G in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, aus drei Gliedern: $G = G_3 + G_2 + G_1$, zusammengesetzt werden kann; (vgl. auch § 2 und 3).

Wie bereits von MARTIN KNUDSEN angegeben, stellt das erste Glied: $G_3 = G_0 \cdot \frac{3\pi}{64} \cdot \frac{R}{\lambda}$, die laminare, hydrodynamische POISEUILLE-Strömung ohne Gleitungskorrektion dar. —

Das zweite Glied: $G_2 = G_0 \cdot \frac{\beta_1}{1 + \alpha_1 \cdot \frac{R}{\lambda}} \cdot \left(\frac{R}{\lambda}\right)$, wo $\alpha_1 = \text{ca. } 3.10$ und $\beta_1 = \text{ca. } 0.80 \cdot \alpha_1$, repräsentiert die Gleitungsströmung, die

wegen der Gleitung des Gases an der festen Rohrwand entsteht, während das dritte Glied: $G_1 = G_0 \cdot \frac{1}{1 + \alpha_1 \cdot \frac{R}{\lambda}}$, das bis jetzt stets in der Theorie der POISEUILLE-Strömung übersehen worden ist, durch die Druckdiffusion in den *strömenden Lamellen* der laminaren, hydrodynamischen POISEUILLE-Strömung entsteht.

Während diese Druckdiffusion *nicht* in einem stillstehenden Gas mit konstanter Temperatur bestehen kann, d. h. nicht stabil ist, *ist dies wohl möglich in einem laminar strömenden Gas, worin ein Druckunterschied in der Richtung der Strömung herrscht*, sowie dies in der laminaren POISEUILLE-Strömung der Fall ist. —

In einem reinen Gas ist die Druckdiffusionsströmung, G_1, bestimmt durch den Selbstdiffusionskoeffizienten des reinen Gases, $D_{11,p} = c_1 \cdot \bar{\Omega}\, \lambda$, und es erhellt also hieraus, *dass der Selbstdiffusionskoeffizient eine messbare, physikalische Grösse ist*, und *nicht*, wie bisher angenommen (S. CHAPMAN, EARLE H. KENNARD u. A.), nur eine mathematische Hilfsgrösse ohne physikalischen Sinn. —

Da diese drei Strömungen:

a) die laminare, rein hydrodynamische POISEUILLE-Strömung, (G_3),

b) die Gleitungsströmung, (G_2), und

c) die Diffusionsströmung (G_1),

als unabhängig von einander, d. h. als einander nicht beeinflussend, angesehen werden können, kann die Totalströmung, G, durch Superposition dieser drei Strömungen berechnet werden.

Aus der semi-empirischen Formel von MARTIN KNUDSEN und aus der Untersuchung des KNUDSEN'schen Beobachtungsmaterials von Dr. E. MELKONIAN erhellt, dass mit grosser Annäherung die Totalströmung, G, reiner Gase geschrieben werden kann:

$$G = G_0 \cdot f\left(\frac{R}{\lambda}\right),$$

wo $f\left(\frac{R}{\lambda}\right)$ eine eindeutige Funktion von $\frac{R}{\lambda}$ ist. —

In § 2 und 3 wird theoretisch, aber nur annäherungsweise — u. a. der besseren Übersicht wegen — die Variation der drei Glieder mit R/λ, bezw. mit dem mittleren Druck: $\bar{p} = \frac{1}{2}(p_1 + p_2)$, untersucht.

Es stellt sich heraus, dass G bei konstanter Temperatur des Rohres, $t°C$, annäherungsweise geschrieben werden kann:

$$(2) \quad \left\{ \begin{array}{c} G = G_3 + G_2 + G_1 = \\ = \frac{2\pi}{3} m \bar{\Omega} R^3 \cdot \frac{dN}{dl} \left[\frac{3\pi}{128} \cdot \frac{2R}{\lambda} + \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}} + \frac{1}{1 + \frac{2R}{\lambda}} \right], \end{array} \right.$$

wo:

$$\frac{2\pi}{3} m \bar{\Omega} R^3 \cdot \frac{dN}{dl} = G_0.$$

m ist die Masse des Moleküls und $\bar{\Omega}$ die mittlere Geschwindigkeit der Moleküle bei der Temperatur, $t°C$.

N ist die Anzahl der Moleküle per cm^3 und $\frac{dN}{dl}$ der Dichtheits- oder Konzentrationsgradient. —

Das Glied: $G_3 = \frac{2\pi}{3} m \bar{\Omega} R^3 \frac{dN}{dl} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda} = \frac{\pi}{8} \cdot \frac{1\varrho}{\eta} \cdot \frac{R^4}{L} \bar{p}(p_1 - p_2)$, stellt die rein hydrodynamische, laminare Poiseuille-Strömung dar. —

Für $R/\lambda = 0$ wird $G_3 = 0$; es muss aber theoretisch für wahrscheinlich erachtet werden, dass für $R/\lambda \doteq 0$ auch $G_3 = 0$ ist.

Für $R/\lambda \geqq$ ca. 0,2 muss aber nach den Beobachtungen angenommen werden, dass sich der Wert von G_3 mit grosser Annäherung bereits in Übereinstimmung mit dem obengenannten theoretischen Wert befindet, der, wie bekannt, gültig ist für grössere Werte von R/λ; dies erhellt u. a. aus der Lage und Grösse des beobachteten Minimums, bezw. $\left(\frac{R}{\lambda}\right)_{\text{min.}}$ und $G_{\text{min.}}$.

Das Glied: $G_2 = \frac{2\pi}{3} m \bar{\Omega} R^3 \frac{dN}{dl} \cdot \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}} = G_0 \cdot \frac{\pi}{4} \cdot \frac{\frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}}$, stellt die Gleitungsströmung dar.

Diese Strömung entsteht durch die Gleitung des Gases an der festen Wand, wenn das Cosinusgesetz für die Zurückwerfung der gegen die Wand stossenden Moleküle Gültigkeit besitzt.

Für $R/\lambda = 0$ wird $G_2 = 0$, und für $R/\lambda \geqq 0$ wird:

$$G_{2\to 0} = G_0 \cdot \frac{\pi}{4} \cdot \frac{2R}{\lambda} \cdot -$$

Für $R/\lambda \to \infty$ wird in Übereinstimmung mit den Beobachtungen: $G_2 = \frac{\pi}{4} \cdot G_0 = \text{konstant}$.

Da die theoretischen und beobachteten Werte für G_0 und auch für G_2 befriedigend übereinstimmen, kann angenommen werden, dass das Cosinusgesetz für die Zurückwerfung der Moleküle an der festen Wand Gültigkeit hat.

Das neue Glied, herrührend von der Druckdiffusion:

$$G_1 = \frac{2\pi}{3} m \overline{\Omega} R^3 \frac{dN}{dl} \cdot \frac{1}{1 + \frac{2R}{\lambda}} = G_0 \cdot \frac{1}{1 + \frac{2R}{\lambda}},$$

wird für:

$$\frac{R}{\lambda} = 0: \quad G_{1,0} = G_0 = \frac{4}{3} \sqrt{2\pi} \sqrt{\varrho} \cdot R^3 \frac{p_1 - p_2}{L}.$$

d. h. zu dem von MARTIN KNUDSEN und später von M. VON SMOLUCHOWSKI abgeleiteten, theoretischen Wert für die stationäre, reine Molekularströmung durch ein langes Rohr mit kreisförmigem Querschnitt, πR^2.

Für $\frac{R}{\lambda} \to \infty$ wird: $G_{1,\infty} = \frac{2\pi}{3} m \overline{\Omega} R^3 \frac{dN}{dl} \cdot \frac{\lambda}{2R} = \frac{1}{3} \overline{\Omega} \lambda \cdot m \cdot \frac{dN}{dl} \pi R^2 =$

$D_{11,p} \cdot m \cdot \frac{dN}{dl} \pi R^2$, wo $D_{11,\infty} = \frac{1}{3} \overline{\Omega} \lambda$, d. h. der Selbstdiffusionskoeffizient des reinen Gases in dem MAXWELL'schen Zustande des Gases, so wie dieser aus der einfachen kinetischen Theorie der Gase abgeleitet wird.

Es erhellt aus der theoretischen Annäherungsformel (2), dass diese der Form nach mit der semi-empirischen Formel von MARTIN KNUDSEN übereinstimmt.

Die numerischen Konstanten: $\alpha_1 = 3{,}097$ und $\beta_1/\alpha_1 =$ ca. 0,80, in der KNUDSEN'schen Formel sind in hinreichender Übereinstimmung mit den in der theoretischen Annäherungsformel (2) verwendeten Konstanten: $\alpha_1 = 2$ und $\beta_1/\alpha_1 = \frac{\pi}{4} = 0{,}785$.

Beide Formeln (1) und (2) ergeben ein Minimum für G: $G_{\min} =$ ca. $0{,}95 \cdot G_0$ bei $\left(\frac{R}{\lambda}\right)_{\min} =$ ca. 0,30.

Die Übereinstimmung zwischen den Formeln (1) und (2) und den Beobachtungen ist für $R/\lambda \geqq$ ca. 0,2 für beide Formeln sehr befriedigend. Beide Formeln sind aber nicht ganz befriedigend in dem Gebiet: $R/\lambda \gtrsim 0$, da die Abweichungen hier ca. 5 bis 10 % betragen.

In diesem Gebiet: $R/\lambda \gtrsim 0$, ist nur die später folgende, exakte Ableitung befriedigend. —

Es erhellt weiter aus der theoretischen Annäherungsformel — und dies ergibt sich auch aus der KNUDSEN'schen semi-empirichen Formel — dass die Summe:

$$G_1 + G_2 = G_0 \cdot \frac{1 + \frac{\pi}{4} \cdot \frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}},$$

von G_0 für $\frac{R}{\lambda} = 0$ bis auf $\frac{\pi}{4} \cdot G_0$ für $\frac{R}{\lambda} \to \infty$ gleichmässig *abnimmt*, woraus hervorgeht, dass das beobachtete Minimum seine Entstehung *nicht* dem Verlauf der Summe der zwei gaskinetischen Glieder, G_1 und G_2, zu verdanken hat, sondern dadurch zustande kommt, dass das hydrodynamische Glied, G_3, welches proportional mit $\frac{2R}{\lambda}$, bezw. $\bar{p}$, *wächst*, zu der Summe, $G_1 + G_2$, die gleichmässig mit $\frac{2R}{\lambda}$, bezw. $\bar{p}$, *abnimmt*, hinzukommt und bereits bei einem Wert von $\frac{R}{\lambda} =$ ca. 0,2 à 0,3 seinen vollen, theoretischen Wert: $G_3 = G_0 \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}$, erreicht hat.

In § 4 werden die gaskinetischen Glieder, G_1 und G_2, und deren Verlauf mit R/λ näher besprochen. Das Gleitungsglied, G_2, wird näher untersucht in Verbindung mit dem vorliegenden Be-

obachtungsmaterial aus den Strömungsuntersuchungen und unter Zugrundelegung von anderen bekannten Untersuchungen über den Gleitungskoeffizienten. —

Die bekannten Methoden zur Bestimmung des Gleitungskoeffizienten können in zwei Hauptgruppen eingeteilt werden, nämlich die Gruppe:

a) welche die Methoden, worin ein Druckgradient in dem Gase vorhanden ist, umfasst; hierzu müssen die Durchströmungsmethoden gerechnet werden, z. B. die stationäre Strömung durch Rohre und andere enge Kanäle, wie auch die Methoden, welche die Bestimmung der Druckabhängigkeit des »*Gesetz von Stokes für Gase*« als Grundlage haben, während die zweite Gruppe:

b) die Methoden, worin kein Druckgradient von Bedeutung in dem Gase vorhanden ist, umfasst, z. B. die MAXWELL'sche Methode mit der in sich selbst schwingenden, zirkularen Planscheibe und die »Ablenkungs-Torsions-Methode«, worin z. B. der äussere Zylinder von zwei koaxialen, kreisförmigen Zylindern mit konstanter Geschwindigkeit um die gemeinschaftliche Achse rotiert, wodurch der innere Zylinder in Torsionsaufhängung bei jedem Druck eine stationäre Ablenkung erfährt. —

Verwendet man die Methoden der Gruppe a) zur Bestimmung des Gleitungskoeffizienten und wird das Druckdiffusionsglied vernachlässigt, wird der hieraus berechnete Wert des Gleitungskoeffizienten zu gross, insbesondere bei niedrigen Werten von R/λ.

Da der Wert der Konstante, k_2, in der Formel für den Gleitungskoeffizienten: $\zeta = k_2 \cdot \lambda$, ausserdem etwas mit R/λ variiert, ist leider der Wert des vorliegenden Observationsmaterials zur Bestimmung des Gleitungskoeffizienten sehr zweifelhaft.

Eine systematische, experimentelle Untersuchung über den Gleitungskoeffizienten und dessen Abhängigkeit von $\frac{R}{\lambda}$ nach den verschiedenen Methoden würde von gaskinetischer Bedeutung sein.

Leider ist es aber noch nicht gelungen eine einwandfreie, theoretische Ableitung des Wertes von $\zeta = k_2 \cdot \lambda$, bezw. $k_{2,\infty}$, in dem MAXWELL'schen Gebiet der Gase zu erreichen, weil die mathematischen Schwierigkeiten zu gross sind.

Ferner wird in § 4 auch das Diffusionsglied, G_1, näher untersucht und die Grössenordnung des Selbstdiffusionskoeffizienten,

$D_{11,p}$, mit Hilfe der Präzisionsmessungen mit Wasserstoff (1911) von MARTIN KNUDSEN bestimmt.

In dieser Verbindung würden neue Messungen über die Variation des Selbstdiffusionskoeffizienten mit R/λ, bezw. $\bar{p}$, z. B. nach der Isotopen-Methode, von wesentlichem, gaskinetischem Interesse sein, weil sich diese Variation als kompliziert erweisen könnte, auch in Verbindung damit, dass der angenäherte, theoretische Ausdruck: $D_{11,\infty} = \frac{1}{3} \cdot \bar{\Omega} \lambda$, sich nicht ganz in Übereinstimmung mit dem exakten Ausdruck für den MAXWELL'schen Zustand der Gase befindet; wie bekannt, ist der exakte Ausdruck hierfür:

$$D_{11,\infty} = \frac{6}{5} \cdot \frac{\eta}{\varrho} = 0{,}60 \cdot \bar{\Omega} \lambda ,$$

wenn die gegenseitigen Zusammenstösse der Moleküle als elastische Stösse aufgefasst werden können.

In § 5 wird die exakte Theorie der reinen Molekularströmung behandelt und die Variation der Strömungsgeschwindigkeit, v, über einen kreisförmigen Querschnitt bestimmt. Es zeigt sich, dass diese Variation von v auf einfache Weise mit Hilfe der vollständigen elliptischen Integralen 1^{ster} und 2^{ter} Gattung dargestellt werden kann.

Ausserdem wird die an die Wand übertragene Bewegungsgrösse gaskinetisch berechnet; ferner wird der Transport von Energie und Bewegungsgrösse durch den Querschnitt, πR^2, des Rohres in der Molekularströmung untersucht.

Für diese letzten Berechnungen ist es aber notwendig, ein Verteilungsgesetz für die molekularen Geschwindigkeiten der Gasmoleküle einzuführen, während dies nicht nötig ist für die Bestimmung der durchströmenden Menge, G_0.

In § 6 und § 7 wird der Einfluss vereinzelter, gegenseitiger Zusammenstösse der Moleküle im Inneren des Rohres auf die Molekularströmung untersucht.

Bei der Berechnung wird vorausgesetzt, dass alle gegenseitigen Zusammenstösse gleichwertig sind, und dass die von Zusammenstössen in einem Volumenelement kommenden Moleküle

gleichmässig in alle Richtungen ausgestrahlt werden (elastische Stösse).

Es wird weiter angenommen, dass die von gegenseitigen Zusammenstössen kommenden Moleküle im Mittel eine zusätzliche Geschwindigkeit, bezw. gemeinschaftliche Massengeschwindigkeit, v_x, in der Richtung der Achse haben können. Wie bekannt, wird dies keinen Einfluss auf die Stosszahl der Moleküle gegen ein Element dS' der Rohrwand haben, weil diese, wie auch die Richtung des Druckgefälles, parallel mit der Richtung der Achse verläuft.

Auch in diesem Falle wird, wie in der exakten Theorie der reinen Molekularströmung, die durchströmende Gasmenge, $mq = G$, durch Addition der einzelnen, durch den Querschnitt des Rohres strömenden Moleküle berechnet; ferner wird mit Hilfe der an die Wand übertragenen Bewegungsgrösse, v_x, bestimmt.

Für den Zustand, $\frac{R}{\lambda} \geqq 0$, worin die laminare, hydrodynamische Strömung noch nicht von Bedeutung geworden ist, ergibt sich:

$$v_{x\to 0} = \frac{\pi}{4} \cdot R\bar{\Omega} \cdot \frac{dN}{Ndl} \cdot -$$

Für: $\frac{R}{\lambda} \geqq \infty$, ergibt sich auf ähnliche Weise:

$$v_{x\to\infty} = (2-\alpha') \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl} = k_{2,\infty} \cdot \frac{\pi}{8} R\bar{\Omega} \cdot \frac{dN}{Ndl}.$$

Durch den Wert für v_x ist die Gleitungsströmung, $mq_2 = G_2$, bestimmt. —

Die Anzahl der durchströmenden Moleküle besteht, so lange die hydrodynamische Strömung: q_3, noch nicht eingesetzt hat, aus zwei Gliedern, nämlich der Anzahl herrührend von der Selbstdiffusions- , bezw. Druckdiffusionsströmung, q_1 , und der Anzahl herrührend von der Gleitungsströmung, q_2, die durch v_x bestimmt ist.

Es ergibt sich:

$$q_1 = \int \frac{1}{2\pi}\,\bar{\Omega}\,\frac{dN}{dx}\,dS \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta\, d\theta \int_{-\pi}^{+\pi} \lambda \left[1 - e^{-\frac{r_1}{\lambda \sin\theta}}\right] d\varphi,$$

woraus,

$$\text{für} \quad \frac{R}{\lambda} = 0: \quad q_{1,0} = \frac{2}{3}\,\bar{\Omega} R \cdot \frac{dN}{dl} \cdot \pi R^2,$$

bezw.

$$\text{für} \quad \frac{R}{\lambda} \to \infty: \quad q_{1,\infty} = \frac{1}{3}\,\bar{\Omega}\lambda \cdot \frac{dN}{dl} \cdot \pi R^2;$$

und:

$$q_2 = \int \frac{1}{2\pi} N v_x dS \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_{-\pi}^{+\pi} \left[1 - e^{-\frac{r_1}{\lambda \sin\theta}}\right] d\varphi.$$

Hierzu kommt, wenn $\frac{R}{\lambda}$ grösser wird, die Anzahl, herrührend von dem Poiseuille-Glied, für welches, jedenfalls bei grösseren Werten von $\frac{R}{\lambda}$, der Ausdruck:

$$q_3 = q_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}, \text{ Gültigkeit hat.} -$$

Verwendet man in den Formeln für q_1 und q_2 die Annäherung: $\left(\frac{r_1}{\sin\theta}\right)_{\text{Mitt.}} = 2R$, welche Annäherung durch Mittelwertbildung von $\left(\frac{r_1}{\sin\theta}\right)$ in dem Gebiet: $\frac{R}{\lambda} \to 0$, abgeleitet werden kann, ergibt sich:

$$\left[1 - e^{-\frac{r_1}{\sin\theta} \cdot \frac{1}{\lambda}}\right] \simeq \left[1 - e^{-\frac{2R}{\lambda}}\right] \simeq \frac{\frac{2R}{\lambda}}{1 + \frac{2R}{\lambda}} = \frac{1}{1 + \frac{\lambda}{2R}}.$$

Durch diese Annäherung können die Werte für $mq_1 = G_1$ und $mq_2 = G_2$, die in der Annäherungsformel (2) vorkommen, welche praktisch gesprochen mit der semi-empirischen Formel (1) von Martin Knudsen übereinstimmt, auf einfache Weise abgeleitet werden.

In § 8 werden die in den Ausdrücken für q_1 und q_2 vorkommenden, bestimmten Integrale näher untersucht, da diese zu bestimmten Integralen von der Gattung:

$$\int_0^{\frac{\pi}{2}} \cos^n\theta \cdot \sin^m\theta \cdot e^{-\frac{\alpha}{\sin\theta}} d\theta,$$

reduziert werden können.

Diese bestimmten Integrale, die von Bedeutung für den Zustand: $\alpha = \frac{R}{\lambda} \to 0$, sind, können in konvergenten Reihen nach α entwickelt werden unter Verwendung des Integrallogarithmus und der bekannten Reihenentwicklung für $\arc\sin\alpha$.

Für $\alpha \to 0$ erhält man Reihen nach α, die schnell konvergieren.

Für grössere Werte von α, d. h. $\alpha > 4$, können auch brauchbare Ausdrücke für diese Integrale abgeleitet werden.

Die vorkommenden Integrale und Ausdrücke können im Allgemeinen mit Hilfe von zwei Funktionen: ${}_1P_1(\alpha)$ und ${}_1N_1(\alpha)$, ausgedrückt werden, und diese zwei Funktionen:

$${}_1P_1(\alpha) = 1 - 0{,}274452 \cdot x_1 + 1{,}726943 \cdot x_1 \log_{10} x_1 - 0{,}533333 \cdot x_1^2 + \\ + 0{,}142663 \cdot x_1^3 - 0{,}143911 \cdot x_1^3 \log_{10} x_1 + 0{,}00237535 \cdot x_1^5 - \\ - 0{,}00299816 \cdot x_1^5 \log_{10} x_1 + \ldots\ldots\ldots\ldots$$

und

$${}_1N_1(\alpha) = 1 - \frac{4}{3} \cdot x_1 + 0{,}682968 \cdot x_1^2 - 1{,}151293 \cdot x_1^2 \log_{10} x_1 + \\ + 0{,}0534417 \cdot x_1^4 - 0{,}0959441 \cdot x_1^4 \log_{10} x_1 + \\ + 0{,}00388264 \cdot x_1^6 - 0{,}00599631 \cdot x_1^6 \log_{10} x_1 + \cdots\cdots\cdots,$$

worin: $$x_1 = \frac{R}{\lambda} = \alpha,$$

können in dem ganzen Gebiet: $0 \leqq \alpha = \frac{R}{\lambda} \leqq \infty$, mit hinreichender Genauigkeit berechnet werden. —

In § 9 und 10 werden die exakten Ausdrücke für $mq_1 = G_1$ (das Selbstdiffusionsglied) und $mq_2 = G_2$ (das Gleitungsglied)

für ein langes Rohr mit Querschnitt, πR^2, berechnet; die Voraussetzung für die Gültigkeit der Formeln ist: $R/L \ll 1$. —

Für diesen Fall wird gefunden:

$$G_1 = G_{1,0} \cdot {}_1P_1\left(\frac{R}{\lambda}\right) \quad \text{und} \quad G_2 = G_{1,0} \cdot \frac{3\pi}{8} \cdot \frac{{}_1P_1\left(\frac{R}{\lambda}\right)}{{}_1N_1\left(\frac{R}{\lambda}\right)}\left[1 - {}_1N_1\left(\frac{R}{\lambda}\right)\right],$$

wo:

$$G_{1,0} = G_0 = \frac{2\pi}{3} m \overline{\Omega} \cdot R^3 \cdot \frac{dN}{dl} \cdot —$$

Es muss aber hierzu bemerkt werden, dass der Ausdruck für G_2 nur gültig ist für: $R/\lambda \to 0$, d. h. $\frac{R}{\lambda} <$ ca. 0,35, weil es noch nicht gelungen ist, die theoretische Variation von v_x mit R/λ in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, mit hinreichender Genauigkeit zu berechnen, da der Grenzwert $v_{x \to \infty}$ für $\frac{R}{\lambda} \to \infty$ noch nicht theoretisch einwandfrei abgeleitet worden ist. Es ist aber möglich nachzuweisen, dass:

$$v_{x \to \infty} = k_2 \cdot \frac{\pi}{8} \overline{\Omega} R \frac{dN}{Ndl}.$$

Da der exakte, theoretische Grenzwert von k_2 für $R/\lambda \to \infty$ leider fehlt, habe ich den experimentellen Wert: $k_{2,\infty} =$ ca. $\frac{4}{3}$ für $R/\lambda \to \infty$ verwendet, woraus:

$$v_{x \to \infty} = \frac{\pi}{6} \overline{\Omega} R \cdot \frac{dN}{Ndl}.$$

Für $v_{x \to 0}$ ist aber theoretisch abgeleitet:

$$v_{x \to 0} = \frac{\pi}{4} \overline{\Omega} R \frac{dN}{Ndl},$$

so dass die Variation von k_2 in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, ca. 50 % beträgt.

Die Variation von v_x, bezw. von k_2, mit R/λ in dem ganzen

Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, kann aber semi-empirisch festgestellt werden, da man mit Annäherung annehmen darf, dass die Druckvariation von k_2 der Form nach in Übereinstimmung ist mit der Variation, die experimentell für das »*Gesetz von Stokes für Gase*« in dem Gebiet: $0 \leqq \frac{a}{\lambda} \leqq \infty$, gefunden worden ist (Martin Knudsen und Sophus Weber, 1912).

Obwohl die Variation mit dem Druck, $\bar{p}$, des *Gesetzes von Stokes* auch nicht ganz unabhängig ist von der Druckdiffusion in dem umgebenden Gase, kann man, ohne grosse Fehler zu machen, annehmen, dass:

$$k_2 = A\left[1 + \frac{B}{A} \cdot e^{-c_2 \cdot \frac{R}{\lambda}}\right] = k_{2,\infty}\left[1 + \frac{1}{2} e^{-c_2 \cdot \frac{R}{\lambda}}\right],$$

wo $A = k_{2,\infty} = \text{ca. } \frac{4}{3}$, $\frac{B}{A} = \frac{1}{2}$, während $c_2 = 2$ in dem Gebiet: $\frac{R}{\lambda} \leqq 0{,}35$, eine befriedigende Übereinstimmung mit der theoretischen Formel:

$$G_2 = G_{1,0} \cdot \frac{3\pi}{8} \cdot \frac{{}_1P_1}{{}_1N_1}\left[1 - {}_1N_1\right],$$

die gültig ist in dem Gebiet: $0 \leqq \frac{R}{\lambda} \leqq 0{,}35$, ergibt.

In § 11 werden die theoretischen Resultate und Formeln mit dem besten vorliegenden Beobachtungsmaterial, als welches die Messungen mit CO_2 von Martin Knudsen gewählt sind, verglichen; das Beobachtungsmaterial für CO_2 ist neuberechnet und nach der Methode der kleinsten Quadrate ausgeglichen.

Es erhellt, dass die Übereinstimmung zwischen dem beobachteten Material und den Resultaten der Theorie sehr zufriedenstellend ist, so dass angenommen werden darf, dass die entwickelte Theorie befriedigend ist, und dass die totale Strömung, G, eines reinen Gases durch ein langes zylindrisches Rohr mit kreisförmigen Querschnitt, πR^2, in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, aus drei von einander unabhängigen Strömungen zusammengesetzt ist, nämlich:

1° Der gaskinetischen Selbstdiffusions- bezw. Druckdiffusionsströmung: $G_1 \cdot —$

2° Der gaskinetischen Gleitungsströmung: $G_2 \cdot —$

3° Der hydrodynamischen, laminaren POISEUILLE-Strömung: $G_3 \cdot —$

Da die theoretische Summe von G_1 und G_2 (vgl. Tabelle No. VII, Seite 108) eine gleichmässig *abnehmende* Strömungsmenge $(G_1 + G_2)$ mit wachsenden Werten von R/λ, bezw. mit wachsendem Druck, $\bar{p}$, ergibt, während die reine, laminare POISEUILLE-Strömung eine gleichmässig *wachsende* Strömungsmenge, G_3, mit wachsenden Werten von R/λ ergibt, erhellt hieraus, dass die Ursache für die Entstehung des Minimums nicht zu finden ist in dem Verlauf der Summe aus den zwei gaskinetischen Gliedern, G_1 und G_2, sondern darin, dass die hydrodynamische, laminare POISEUILLE-Strömung bereits bei kleineren Werten von R/λ, d. h. R/λ = ca. 0,2 à 0,3, voll entwickelt ist und ihren theoretischen Wert: $G_3 = G_{1,0} \cdot \frac{3\pi}{64} \cdot \frac{R}{\lambda}$, erreicht hat.

Aus den Tabellen geht hervor, dass die beobachteten und theoretischen Werte für die Grösse und Lage des Minimums *genau* übereinstimmen. Diese sind:

$$G_{\text{min.}} = 0{,}925 \cdot G_{1,0} \quad \text{und} \quad \left(\frac{R}{\lambda}\right)_{\text{min.}} = 0{,}31 .$$

Dies Resultat ist auch in Übereinstimmung mit den von Dr. MELKONIAN nach graphischer Auswertung aus dem vollständigen Beobachtungsmaterial von MARTIN KNUDSEN gefundenen Werten.

Ferner werden, in Verbindung mit den Zahlen in der Tabelle VII die beobachteten und berechneten Werte von $G = G_1 + G_2 + G_3$ in dem ganzen Gebiet: $0 \leqq \frac{R}{\lambda} \leqq \infty$, verglichen, und es stellt sich heraus, dass die Übereinstimmung, auch in dem Gebiet: $0 \leqq \frac{R}{\lambda} \leqq$ ca. 0,2, sehr befriedigend ist.

Es erhellt u. a., dass die Summe $(G_1 + G_2)/G_{1,0}$ von 1 für

$R/\lambda = 0$, auf $\frac{\pi}{4} = 0{,}785$ für $R/\lambda \to \infty$, gleichmässig abnimmt. Dieser Wert von 0,785 für $R/\lambda \to \infty$, entspricht einem Wert von $k_{2,\infty} = \text{ca.}\ \frac{4}{3}$.

Ausserdem erhellt aus den Präzisionsmessungen für H_2 von MARTIN KNUDSEN (1911) und insbesondere aus den Präzisionsmessungen von W. GAEDE (1913) in dem Gebiet: $R/\lambda \geqq 0$, dass die POISEUILLE-Strömung in diesem Gebiet, d. h. $0 \leqq \frac{R}{\lambda} \leqq$ ca. 0,04, nach aller Wahrscheinlichkeit noch nicht eingesetzt hat, wie man auch theoretisch erwarten würde, während man, wie es aus der Grösse und Lage des Minimums hervorgeht, annehmen muss, dass die reine POISEUILLE-Strömung ihren vollen theoretischen Wert, $G_3 = G_{1,0} \cdot \frac{3\pi}{128} \cdot \frac{2R}{\lambda}$, bereits bei dem Wert von $\frac{R}{\lambda} = 0{,}20$ à 0,30 erreicht hat.

In § 12 wird der experimentelle und theoretische Verlauf der Molekularströmung, G_1', in der Umgebung des Punktes: $R/\lambda = 0$, näher beleuchtet.

Zum Schluss kann noch einmal die Aufmerksamkeit darauf gelenkt werden, *dass der Selbstdiffusionskoeffizient eines reinen Gases eine messbare, physikalische Grösse ist*, obwohl es vielleicht schwierig sein wird, diese Grösse mit Hilfe der Strömung reiner Gase durch Rohre mit befriedigender Genauigkeit zu bestimmen.

Eine weitere Vergleichung der Theorie mit dem vorliegenden Beobachtungsmaterial für rechtanguläre Spalten und kreisförmige Ringspalten (R. E. H. RASMUSSEN, 1937) wird wahrscheinlich eine grössere Genauigkeit in der Bestimmung des Selbstdiffusionskoeffizienten ergeben, und vielleicht auch nähere Aufschlüsse darüber, wann und auf welche Weise die laminare, hydrodynamische Strömung von POISEUILLE einsetzt.

In dieser Verbindung kann noch mitgeteilt werden, dass die von R. E. H. RASMUSSEN gefundenen Werte für die Grösse und Lage des Minimums für Ringspalten und rechtanguläre Spalten mit Höhe, h:

$$G_{\min.} = \text{ca. } 0{,}50 \cdot G_{1{,}0} \quad \text{bei} \quad \left(\frac{h}{\lambda}\right)_{\min.} = \text{ca. } 1\,,$$

in guter Übereinstimmung mit der hier entwickelten Theorie sind, und hierdurch auch ihre einwandfreie Erklärung erhalten.

Auch bei dieser Gelegenheit möchte ich der Direktion des Dänischen Carlsbergfonds meinen aufrichtigen Dank für gewährte Stütze aussprechen. —

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ON THE PSEUDOSCALAR INTERACTION IN β-DECAY

BY

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Assuming the β-decaying nucleon to move in a simple scalar potential, the Foldy-Wouthuysen transformation is used to obtain an equivalent non-relativistic form for the pseudoscalar interaction. The effect of the recoil of the nucleus has been taken into account. The observed shapes of allowed β-spectra permit an estimate of an upper limit to the pseudoscalar coupling constant, which is difficult to reconcile with the value derived from the analysis by PETSCHEK and MARSHAK of the Ra E spectrum. This conclusion is dependent on the assumption of a simple nuclear potential.

The pseudoscalar β-interaction differs from the other four types of interactions in not possessing any non-relativistic analogy. The matrix elements therefore may depend essentially on the nuclear forces acting upon the decaying particles, and great care is needed in the derivation of equivalent non-relativistic forms which permit a comparison with other β-decay interactions. The significance of the pseudoscalar interaction for the shape of certain forbidden spectra has recently been suggested[1].

The contribution of the pseudoscalar interaction to β-transitions may in the plane wave approximation for the leptons be expanded as follows:

$$\langle f|\beta\gamma_5 L|i\rangle = \langle f|\beta\gamma_5|i\rangle L_0 + \frac{1}{6}\langle f|\beta\gamma_5 r^2|i\rangle (\Delta L)_0 \qquad (1.1)$$

$$+\langle f|\beta\gamma_5 r_i|i\rangle (\nabla_i L)_0 \qquad (1.2)$$

$$+\frac{1}{2}\langle f|\beta\gamma_5\left(r_i r_k - \frac{1}{3}\delta_{ik} r^2\right)|i\rangle (\nabla_i \nabla_k L)_0 \qquad (1.3)$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the transforming nucleus and L is the lepton covariant $\psi_e^* \beta\gamma_5 \psi_\nu$. The index 0 means that the functions will be evaluated at the position

1) A. G. PETSCHEK and R. E. MARSHAK, Phys. Rev. 85, 698 (1952).

of the nucleus. The operator Q, which transforms neutrons into protons, is omitted in the following. The first two terms (1.1) contribute to first forbidden $\Delta J = 0$ (yes) transitions. The third (1.2) contributes to allowed transitions $\Delta J = 0, \pm 1$ (no), whereas the fourth contributes to first forbidden $\Delta J = 0, \pm 1, \pm 2$ (yes) transitions.

An especially well suited method for estimating the nuclear matrix elements in (1) is provided by the Foldy-Wouthuysen transformation[2) 3)]. The Foldy-Wouthuysen transformation also proves very useful for the evaluation of other relativistic matrix elements which occur in β-theory. The approximations made by AHRENS and FEENBERG[4)] in a similar evaluation stand out clearly when this method is used.

The Hamiltonian to be transformed is

$$H = H_0 + H_{\text{int}} + H_\beta + H_0^{\beta\nu} \tag{2}$$

where H_0 is the free particle Hamiltonian for the nucleons $\sum_{i=1}^{A} (-\vec{\alpha}\cdot\vec{p} - \beta M)_i$, H_{int} is the nucleon-nucleon interaction, H_β is the β-interaction $\sum g_5 \beta\gamma_5 L$, and $H_0^{\beta\nu}$ is the free lepton Hamiltonian. The usual nucleon-nucleon interactions do not contain odd operators, i. e., operators which mix large and small components.

Treating $H_\beta + H_0^{\beta\nu}$ as a perturbation it is easy (cf. ref. 2) to construct a unitary transformation e^S which transforms $H_0 + H_{\text{int}}$ into a Schrödinger Hamiltonian:

$$S = \frac{\beta}{2M} \sum \vec{\alpha}\cdot\vec{P} + \frac{1}{4M^2}\left[\sum \vec{\alpha}\cdot\vec{P},\ H_{\text{int}}\right] - \frac{\beta}{6M^3}\left(\sum \vec{\alpha}\cdot\vec{P}\right)^3. \tag{3}$$

If we take H_{int} simply to be a scalar central potential βV we get the non-relativistic equivalent of the pseudoscalar matrix element:

2) L. L. FOLDY and S. A. WOUTHUYSEN, Phys. Rev. **78**, 29 (1950). This method was first applied to β-decay by HERBST and BUSHKOVITCH, Phys. Rev. **91**, 442 (1953).

3) Previous attempts have been made by T. AHRENS, E. FEENBERG and H. PRIMAKOFF, Phys, Rev. **87**, 663 (1953) and T. AHRENS, Phys. Rev. **90**, 974 (1953) to express the relativistic matrix element $\langle f|\beta\gamma_5 L|i\rangle$ in terms of the non-relativistic matrix elements. The results are essentially different from those obtained here, and it also appears that some of the approximations involved cannot be justified.

4) T. AHRENS and E. FEENBERG, Phys. Rev. **86**, 64 (1952).

$$\left.\begin{aligned}\langle f|\beta\gamma_5 L|i\rangle \approx \frac{i}{2M}\langle f|\vec{\sigma}\cdot\vec{\nabla} L|i\rangle + \frac{i}{4M^2}\langle f|L\vec{\sigma}\cdot\vec{r}\frac{1}{r}\frac{dV}{dr}|i\rangle \\ -\frac{i}{4M^2}\langle f|(E_f - E_i - V_f + V_i)L\vec{\sigma}\cdot\vec{\nabla}|i\rangle .\end{aligned}\right\} \quad (4)$$

Here we have omitted terms of the order $\frac{V}{M^2}$ in the first term. Also the last term having the same selection rules as the second term will be neglected in the following.

If one transforms the total Hamiltonian H into a Schrödinger Hamiltonian one will get an additional term from $H_0^{\beta\nu}$. However, this term vanishes since it is of the form[5)]

$$\langle f|[H,[\sum\vec{\alpha}\cdot\vec{p},\sum g_5\beta\gamma_5 L]]|i\rangle . \quad (5)$$

Both methods are thus identical, and this result can be used to show that one need not care about the operator Q. The formula (4) applies as well to the scalar potential βV as to the static part of the vector potential V.

In the plane wave approximation (4) becomes

$$\langle f|\beta\gamma_5 L|i\rangle \approx$$

$$\frac{i}{4M^2}\langle f|\vec{\sigma}\cdot\vec{r}\frac{1}{r}\frac{dV}{dr}|i\rangle (L)_0 + \frac{i}{6M}\langle f|\vec{\sigma}\cdot\vec{r}|i\rangle (\Delta L)_0 \quad (6.1)$$

$$+\frac{i}{2M}\langle f|\sigma_i|i\rangle (\nabla_i L)_0 + \frac{i}{4M^2}\langle f|\vec{\sigma}\cdot\vec{r}\frac{1}{r}\frac{dV}{dr}r_i|i\rangle (\nabla_i L)_0 \quad (6.2)$$

$$+\frac{i}{4M}\langle f|\sigma_i r_k + \sigma_k r_i - \frac{2}{3}\delta_{ik}\vec{\sigma}\cdot\vec{r}|i\rangle (\nabla_i\nabla_k L)_0 \quad (6.3)$$

where the terms are ordered according to selection rules, the second term in (6.1) being of the order of magnitude of the (6.3) term. In (6.3) only the major term is included.

The terms (6.1) correspond to the first terms (1.1) in (1). The terms (6.2) correspond to (1.2), and so on. This correspondence can also be seen directly by evaluating the matrix elements using single particle Dirac wave functions for the nucleons.

The relative order of magnitude of the terms (6.1) can also

5) This was pointed out to us by L. L. Foldy.

be found in this simple case. The result for a square well potential is

$$\frac{\frac{i}{4M^2}\langle f|\vec{\sigma}\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}|i\rangle}{\frac{i}{6M}\langle f|\vec{\sigma}\cdot\vec{r}|i\rangle} \approx \frac{\langle f|\beta\gamma_5|i\rangle}{\frac{i}{6M}\langle f|\vec{\sigma}\cdot\vec{r}|i\rangle} \approx E^2 \tag{7}$$

where E is the kinetic energy of the decaying nucleon. This means that for low Z the ratio of the first term to the second in (6.1) is of the order of $E^2/(\vec{p}+\vec{q})^2$. For heavy nuclei, however, this ratio will tend towards $E^2/(\alpha Z/2\varrho)^2$, that is the terms will be of nearly the same order of magnitude. By ϱ we denote the nuclear radius, $\vec{p}$ is the electron and $\vec{q}$ the neutrino momentum.

As regards the absolute magnitude of the matrix elements we find for the square well potential

$$\langle f|\beta\gamma_5|i\rangle \approx \frac{E^2}{\varrho M^2} \tag{8}$$

which for heavy elements is of the order of magnitude 10^{-4}. The smallness of this matrix element reveals the peculiarity of the pseudoscalar coupling. Furthermore, for the ratio of the two terms in (6.2) we find

$$\frac{\frac{1}{2M}\langle f|\vec{\sigma}\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}r_i|i\rangle}{\langle f|\sigma_i|i\rangle} \approx \frac{2E}{M}. \tag{9}$$

The term $\langle f|\vec{\sigma}\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}\left(r_i r_k-\frac{1}{3}\delta_{ik}r^2\right)|i\rangle(\nabla_i\nabla_k L)_0$ is omitted in (6.3), because of the smallness of this ratio.

These considerations are limited to simple potentials. RUDERMAN[6] has shown that in the case of pseudoscalar meson theory with pseudoscalar coupling $\langle f|\beta\gamma_5|i\rangle$ may be large.

For the neutron decay the series expansion in (1) and (6) is not appropriate. The β-spectrum of the neutron[7] may be calculated from either (1) or (6) by using plane waves for all particles. The inclusion of the momentum difference between

6) M. RUDERMAN, Phys. Rev. **89**, 1227 (1953).
7) L. MICHEL, Proc. Phys. Soc. **63A**, 514 (1950).

the neutron and the recoiling proton is essential to get a non vanishing matrix element.

Thus one might expect that recoil effects add essentially to the contribution to allowed transitions from the pseudoscalar interaction. This is, however, not the case as can be seen in the following way. To calculate the recoil effect the true final state, $\langle f|$, must be represented in the rest system of the daughter nucleus, where the state is represented by $\langle f'|$:

$$\langle f| \approx \langle f'| \left(1 - iM\vec{v}_R \sum_{i=1}^{A} \vec{r}^{(i)}\right). \tag{10}$$

Here the recoil momentum is $MA\vec{v}_R$ and the centre of mass coordinate is $\frac{1}{A}\sum_{i=1}^{A} \vec{r}^{(i)}$. After the substitution of (10) into (6), operators acting on two different particles have been neglected. Conservation of momentum requires that

$$\vec{v}_R = -\frac{1}{MA}\left(\vec{p} + \vec{q}\right) \tag{11}$$

and the result of operation with $\vec{v}_R$ on the lepton covariant L is

$$\vec{v}_R L = -\frac{i}{MA}\vec{\nabla} L. \tag{12}$$

Thus we obtain

$$\langle f|\beta\gamma_5 L|i\rangle \approx$$

$$\frac{i}{4M^2}\langle f'|\vec{\sigma}\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}|i\rangle (L)_0 + \frac{i}{6M}\left(1-\frac{1}{A}\right)\langle f'|\vec{\sigma}\cdot\vec{r}|i\rangle (\Delta L)_0 \tag{13.1}$$

$$+\frac{i}{2M}\langle f'|\sigma_i|i\rangle (\nabla_i L)_0 + \frac{i}{4M^2}\left(1-\frac{1}{A}\right)\langle f'|\vec{\sigma}\cdot\vec{r}\,\frac{1}{r}\frac{dV}{dr}r_i|i\rangle (\nabla_i L)_0 \tag{13.2}$$

$$+\frac{i}{4M}\left(1-\frac{1}{A}\right)\langle f'|\sigma_i r_k + \sigma_k r_i - \frac{2}{3}\delta_{ik}\vec{\sigma}\cdot\vec{r}|i\rangle (\nabla_i \nabla_k L)_0. \tag{13.3}$$

This may also be seen directly in the case of single particle wave functions for the nuclei. In the low velocity approximation for the Lorenz transformation we get

$$\left.\begin{aligned} \langle f| &\approx \langle f'| \exp\left(-\frac{\vec{v}_R}{2}\sum_{i=1}^{A}\vec{\alpha}^{(i)}\right)\exp\left(-iM\vec{v}_R\sum_{i=1}^{A}\vec{r}^{(i)}\right) \\ &\approx \langle f'|\left(1-\frac{\vec{v}_R}{2}\sum_{i=1}^{A}\vec{\alpha}^{(i)}-iM\vec{v}_R\sum_{i=1}^{A}\vec{r}^{(i)}\right). \end{aligned}\right| \qquad (14)$$

The first exponential transforms the spinors while the second transforms the space part of the phase. By the insertion of (14) into (1) and the identification of corresponding terms in (1) and (6) one is again led to (13).

For the neutron the series expansion of the last factor in (14) is not appropriate, and the only contribution will arrive from the recoil term

$$\langle f'| \exp\left(-iM\vec{v}_R\cdot\vec{r}\right)\beta\vec{\sigma}\cdot\vec{\nabla}L|i\rangle$$

since the term

$$\langle f'|\beta\gamma_5\exp\left(-iM\vec{v}_R\cdot\vec{r}\right)L|i\rangle$$

vanishes for free particles.

As specific applications of the above estimates of the pseudoscalar matrix elements we consider the relatively simple cases of allowed transitions with maximum spin change, and first forbidden $0 \to 0$ (yes) transitions. In these cases one may restrict the calculations to a mixture of tensor and pseudoscalar interactions.

The shape of allowed β-spectra can easily be found from (13) and has been evaluated for $\Delta J = \pm 1$ (no) transitions. The second term in (13.2) has been omitted. In the plane wave approximation we get

$$\left.\begin{aligned} P(W)\,dW = &\frac{1}{(2\pi)^3}q^2 p\,W dW g_3^2\,|\int\vec{\beta\sigma}|^2 \\ &\left\{1+\left[\frac{p^2+q^2}{3}-\frac{2}{9}\frac{p^2q}{W}\right]\left(\frac{g_5/g_3}{2M}\right)^2\right. \\ &\left.-\frac{2}{3}\left(\frac{p^2}{W}-q\right)\frac{g_5/g_3}{2M}\right\} \end{aligned}\right| \qquad (15)$$

where g_3 and g_5 are the tensor and pseudoscalar coupling constants respectively. It is seen that the main effect of the pseudo-

scalar interaction is proportional to $(p^2/W - q)$ and therefore large for large maximum β-energies. Thus disintegrations like the He^6 and B^{12} decays provide the most sensitive tests.

For $(g_5/g_3)/2M = \pm 1/20$ the spectrum (15) has been compared with the experimental He^6 spectrum[8] in Fig. 1. Such a

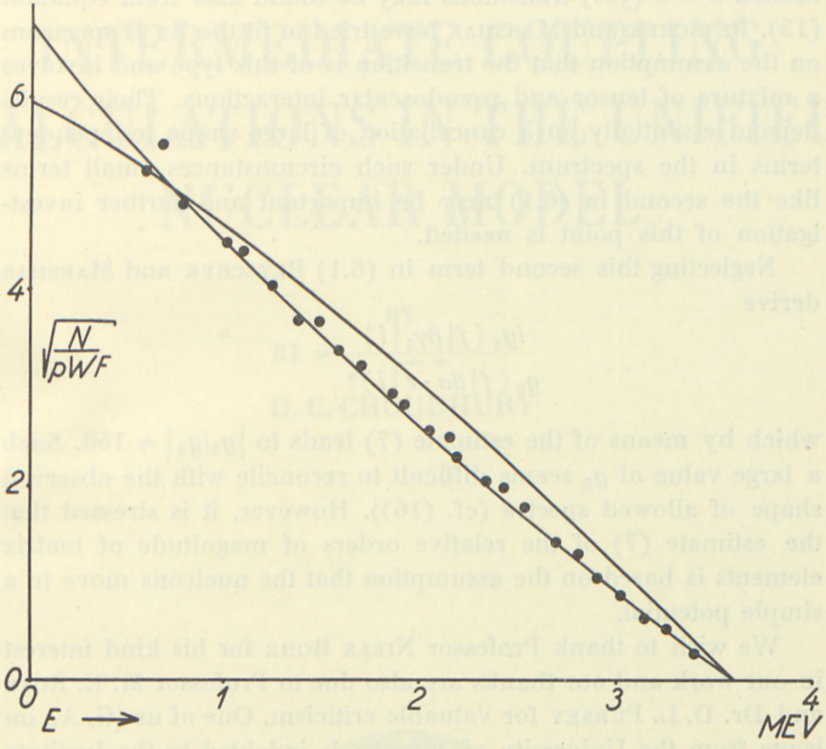

Fig. 1. Comparison of the spectrum (15) for $\frac{g_5/g_3}{2M} = \pm 1/20$ (full drawn curves) with the experimental data for He^6. The data are fitted best possibly to the curve corresponding to $(g_5/g_3)/2M = +1/20$.

value of g_5 would lead to a significant deviation from the observed spectrum and the figure shows that

$$|g_5/g_3| \lesssim 100$$

represents an upper limit to g_5.

8) C. S. Wu, B. M. Rustad, V. Perez-Mendez and L. Lidofsky, Phys. Rev. 87, 1140 (1952).

A comparison with the experimental B^{12} spectrum indicates that this limit may be lowered by a factor of two.

The Coulomb correction to the spectrum (15) is, apart from the usual factor $F(Z, W)$, small for these low Z spectra.

The influence of the pseudoscalar interaction on first forbidden $0 \rightarrow 0$ (yes) transitions may be found also from equation (13). PETSCHEK and MARSHAK have tried to fit the Ra E spectrum on the assumption that the transition is of this type and involves a mixture of tensor and pseudoscalar interactions. Their results depend essentially on a cancellation of large shape independent terms in the spectrum. Under such circumstances small terms like the second in (6.1) may be important and further investigation of this point is needed.

Neglecting this second term in (6.1) PETSCHEK and MARSHAK derive

$$\frac{ig_5 \langle f | \beta\gamma_5 | i \rangle}{g_3 \langle f | \beta\vec{\sigma} \cdot \vec{r} | i \rangle^*} \approx 13$$

which by means of the estimate (7) leads to $|g_5/g_3| \approx 150$. Such a large value of g_5 seems difficult to reconcile with the observed shape of allowed spectra (cf. (16)). However, it is stressed that the estimate (7) of the relative orders of magnitude of matrix elements is based on the assumption that the nucleons move in a simple potential.

We wish to thank Professor NIELS BOHR for his kind interest in our work and our thanks are also due to Professor M. E. ROSE and Dr. D. L. PURSEY for valuable criticism. One of us (G. A., on leave from the University of Zagreb) is indebted to the Institute Rudjer Bošković for financial support enabling him to continue his work in Copenhagen.

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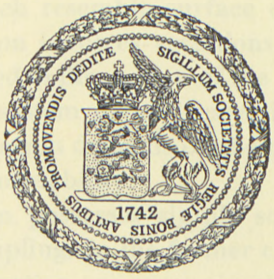
INTERMEDIATE COUPLING CALCULATIONS IN THE UNIFIED NUCLEAR MODEL

BY

D. C. CHOUDHURY



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The intermediate coupling treatment of the unified nuclear model, discussed by A. Bohr and B. Mottelson, is extended to a study of the nuclear level structure for a single $j = 5/2$ particle coupled to the nuclear surface oscillations of the quadrupole type. Magnetic moments and quadrupole moments for the nuclear ground states for $j = 5/2$ and $j = 7/2$ configurations are also considered. Wave functions are used, including all unperturbed states with up to three phonons, and the energies and moments are calculated as a function of the coupling strength. We should need the inclusion of states with still more phonons to make detailed contact with the strong coupling region. However, even for the intermediate coupling strength for which the present results are valid, various features of the strongly coupled system are beginning to develop.

I. Introduction.

In the unified description of nuclear dynamics[1,2], the nucleus is considered as a shell structure capable of performing collective oscillations. The state of the system is thus described in terms of individual-particle and collective degress of freedom. The former represent the most loosely bound particles, while the latter represent the bulk of the nucleons which cannot be individually excited at moderate nuclear excitation energies. The most important of the collective types of the motion are oscillations in the nuclear shape which resemble surface oscillations.

The collective motion involves variations in the nuclear field and is, therefore, coupled to the motion of the individual nucleons. The properties of the system depend essentially on the strength of this coupling, which again depends on the particle configuration and on the nuclear deformability.

The coupled system possesses simple solutions in the limit of weak and strong coupling. In the former case, the particle and the collective types of motion are approximately independent and the effect of the coupling can be treated as a small perturbation.

[1] A. Bohr, Dan. Mat. Fys. Medd. **26**, no. 14, 1952; in the text quoted as A.
[2] A. Bohr and B. Mottelson, Dan. Mat. Fys. Medd. **27**, no. 16, 1953; in the text quoted as B.-M.

In the latter case, the system bears analogies to molecular structures; the nucleus acquires a large deformation and the stationary states can be characterized by the motion of the particles with respect to the deformed nucleus and the vibration and rotation of the structure as a whole.

A weak coupling situation is expected in the vicinity of major closed shells, where the high stability of the spherical nuclear shape makes the coupling relatively ineffective. With the addition of particles, the coupling increases and, in regions far removed from closed shells, the strong coupling situation appears to be rather accurately realized, as evidenced in particular by the nuclear rotational spectra (cf., e.g., B.-M. § VIc).

In many nuclei, however, neither the weak nor the strong coupling solutions are adequate, and it is necessary to develop methods to analyze the properties of the system also in the intermediate coupling region. Some calculations to this effect have been performed (B.-M. § IIb.iii), using the representation of uncoupled motion similar to that used by FOLDY and MILFORD[1], and diagonalizing the Hamiltonian, including states containing up to a certain number of quanta of the surface oscillations. The method is somewhat similar to the Tamm-Dancoff[2,3] method of field theory. In the present paper, we apply this method to further studies of nuclear properties in the intermediate coupling region.

Calculations of this type become rather complex when the number of unperturbed states included in the wave function becomes too great, and we therefore restrict ourselves to the simplest type of system, that of a single particle with a constant angular momentum j coupled to the nuclear surface oscillations of lowest order (the quadrupole oscillations). In the description of nuclear states, it will often be necessary to consider many-particle configurations and also to take into account that the interaction with the surface oscillations may couple together particle states with different values of j. It is expected, however, that the simplified system considered will contain many of the characteristic features of the intermediate coupling situation and illustrate the gradual transition from weak to strong coupling.

[1] L. L. FOLDY and F. J. MILFORD, Phys. Rev. **80**, 751, 1950.
[2] I. TAMM, J. Phys. U.S.S.R. **9**, 449. 1945.
[3] S. M. DANCOFF, Phys. Rev. **78**, 382, 1950.

In Section II, we give the equation of motion and derive the general formula for the evaluation of matrix elements for the particle-surface interaction. In Sections III, IV, and V, we consider level structure, magnetic moments and quadrupole moments, respectively, for a number of configurations.

II. Equation of Motion; Method of Treatment.

We consider the dynamical system consisting of a single particle with angular momentum j coupled to the quadrupole oscillations of the nuclear surface. The total Hamiltonian of the system has the form (we follow the notation of A. and B.-M.)

$$H = H_s(\alpha_{2\mu}) + H_p(x) + H_{\text{int}}(\alpha_{2\mu}, x) \cdots, \tag{1}$$

where the surface energy H_s is equivalent to that of a system of harmonic oscillators and is given by

$$H_s(\alpha_{2\mu}) = \sum_{\mu} \left\{ \frac{1}{2} B_2 \left| \dot{\alpha}_{2\mu} \right|^2 + \frac{1}{2} C_2 \left| \alpha_{2\mu} \right|^2 \right\} \cdots \cdots \tag{2}$$

in terms of the mass parameter B_2 and nuclear deformability C_2 (defined in A. and B.-M.). The frequency of the surface oscillations is given by

$$\omega = \sqrt{\frac{C_2}{B_2}} \cdots \cdots. \tag{3}$$

The particle Hamiltonian H_p is a constant, since the particle is assumed to remain in the same orbits so that only the degree of freedom associated with the orientation of its state is taken into account.

The interaction energy H_{int} represents the coupling of the particle to the nuclear deformation and to first order in $\alpha_{2\mu}$ takes the form

$$H_{\text{int}}(\alpha_{2\mu}, x) = -k \sum_{\mu} \alpha_{2\mu} Y_{2\mu}(\theta, \varphi) \cdots \cdots, \tag{4}$$

where θ, φ are the polar angles of the particle and k is the coupling constant.

The effect of the particle-surface interaction may be conveniently characterized by the dimensionless coupling parameter given by

$$x = \sqrt{\frac{5}{4\pi}} \cdot \frac{1}{\sqrt{4j}} \cdot \frac{k}{\sqrt{\hbar\omega C}} \cdots \cdots \tag{5}$$

For $x\sqrt{j} \ll 1$, the coupling is relatively ineffective and can be treated by perturbation method; for $x \gg 1$ the strong coupling treatment again provides a simple solution of the coupled motion.

In the present intermediate coupling treatment, we shall use the representation of uncoupled motion and thus write the wave function

$$\Psi = \sum_{NR} |j; NR; IM\rangle \langle j; NR; IM| \rangle \cdots \cdots, \tag{6}$$

where the state of the surface is characterized by the number of phonons N, each having an anugular momentum of two units, and by R, the total angular momentum of the surface. I and M denote the total angular momentum of the nucleus and its Z component, respectively.

In order to evaluate the matrix elements of H_{int}, it is convenient to write the surface variable $\alpha_{2\mu}$ in terms of creation and destruction operators b^*_μ and b_μ, respectively. The expression (4) for H_{int} then becomes

$$H_{\text{int}}(\alpha_{2\mu}) = -k \sum_{\mu} \sqrt{\frac{\hbar\omega}{2C}} \cdot (b_\mu + (-1)^\mu b^*_{-\mu}) Y_{2\mu}(\theta, \varphi). \tag{7}$$

Using the decomposition

$$|j: NR; IM\rangle = \sum_{m\mu'} |jm\rangle |NR\mu'\rangle (jRm\mu' | jRIM) \cdots \cdots, \tag{8}$$

where the first two terms represent the particle state vector and the surface state vector, respectively, and where $(jRm\mu' | jRIM)$ is the Clebsch-Gordan coefficient[1] for the composition of angular momenta; we obtain for $N' > N$

$$\left.\begin{aligned} &\langle j; NR; IM | H_{\text{int}} | j; N'R'; IM\rangle \\ &= -k \sqrt{\frac{\hbar\omega}{2C}} \sum_{mm'\mu\mu'\mu''} \langle jm | Y_{2\mu} | jm'\rangle \cdot \langle NR\mu' | b_\mu | N'R'\mu''\rangle \\ &\times (jRm\mu' | jRIM) \cdot (jR'm'\mu'' | jR'IM) \cdots \cdots \end{aligned}\right\} \tag{9}$$

[1] E. U. Condon and G. H. Shortley. The Theory of Atomic Spectra, Cambridge University Press, 1935.

The general matrix elements of $\langle Y_{2\mu} \rangle$ have the form

$$\langle lsjm \,|\, Y_{2\mu} \,|\, l'sj'm' \rangle = \langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle \cdot (j'2\,m'm-m' \,|\, j'2jm)^* \cdots \cdots, \quad (10)$$

where l is the orbital angular momentum and s the spin of the particle.

The function $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ is evaluated and is given by

$$\left.\begin{aligned} &\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle \\ &= (-1)^{j+l'-\frac{1}{2}} \cdot \sqrt{\frac{5}{4\pi}} \cdot \sqrt{(2l'+1)(2j'+1)}\, W(j'l'jl \,|\, {}^1\!/_2 2) \cdot (l'200 \,|\, l'2l0) \cdots, \end{aligned}\right\} \quad (11)$$

where $W(j'l'jl \,|\, {}^1\!/_2 2)$ is a Racah coefficient, the values of which have been tabulated[1]. It can be shown that $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ depends only on j and j' and not on l and l', provided the combining states have the same parity.

Similarly, we can write out the matrix elements of $\langle b_\mu \rangle$ in the form

$$\langle NRM_R \,|\, b_\mu \,|\, N'R'M'_R \rangle = \langle NR \,\|\, b \,\|\, N'R' \rangle \cdot (R\,2\,M_R\mu \,|\, R\,2\,R'M_R+\mu) \cdots \cdots \quad (12)$$

In evaluating the function $\langle NR \,\|\, b \,\|\, N'R' \rangle$, it is convenient to write the phonon states in terms of Boson creation operators acting on the ground state of the nuclear surface. As an illustration, the two-phonon state with angular momentum R can be written as

$$|\, 2\,Rm \rangle = \frac{1}{A} \sum_{\mu\mu'} (22\,\mu\mu' \,|\, 22\,Rm) \cdot b^*_\mu b^*_{\mu'} \,|\, 0 \rangle \cdots \cdots \cdots \cdots, \quad (13)$$

where A is the normalization factor of the state vector and is obtained by computing the absolute value of $\sum_{\mu\mu'} (22\,\mu\mu' \,|\, 22\,Rm) \times b^*_\mu b^*_{\mu'} \,|\, 0 \rangle$. The values of $\langle NR \,\|\, b \,\|\, N'R' \rangle$ have been evaluated for all states involving up to three phonons and are given in Table I.

By means of (10) and (12), one obtains from (9) an expression for the matrix elements of H_{int} which involves products of four

* Note that our notation $\langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle$ is not the same as RACAH's. The relation is $\sqrt{2j+1} \cdot \langle lsj \,\|\, Y_2 \,\|\, l'sj' \rangle_{\text{our}} = (lsj \,\|\, Y_2 \,\|\, l'sj')_{\text{Racah's}}$ (Cf. Phys. Rev. **62**, 438, 1942).

[1] L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Rev. Mod. Phys. **24**, 249, 1952.

TABLE I.

The values of $\langle NR \| b \| N'R' \rangle$ *factors.*

$\langle 00 \| b \| 12 \rangle = 1$	$\langle 22 \| b \| 33 \rangle = \sqrt{\frac{15}{7}}$
$\langle 12 \| b \| 20 \rangle = \sqrt{2}$	$\langle 22 \| b \| 34 \rangle = \sqrt{\frac{11}{7}}$
$\langle 12 \| b \| 22 \rangle = \sqrt{2}$	$\langle 24 \| b \| 32 \rangle = \sqrt{\frac{36}{35}}$
$\langle 12 \| b \| 24 \rangle = \sqrt{2}$	$\langle 24 \| b \| 33 \rangle = -\sqrt{\frac{6}{7}}$
$\langle 20 \| b \| 32 \rangle = \sqrt{\frac{7}{5}}$	$\langle 24 \| b \| 34 \rangle = \sqrt{\frac{10}{7}}$
$\langle 22 \| b \| 30 \rangle = \sqrt{3}$	$\langle 24 \| b \| 36 \rangle = \sqrt{3}$
$\langle 22 \| b \| 32 \rangle = \sqrt{\frac{4}{7}}$	

Clebsch-Gordan coefficients. Summing over the magnetic quantum numbers we finally obtain

$$\left.\begin{aligned} &\langle j; NR; IM \left| H_{\text{int}} \right| j; N'R'; IM \rangle \\ &= (-1)^{R+I+1-j} \cdot k \cdot \sqrt{\frac{\hbar\omega}{2C}} \cdot \sqrt{(2j+1)(2R'+1)}\, W(R'Rjj \,|\, 2I) \\ &\quad \times \langle lsj \| Y_2 \| lsj \rangle \cdot \langle NR \| b \| N'R' \rangle \cdot \cdots \cdots \end{aligned}\right\} \quad (14)$$

We now obtain an approximate solution of the equation of motion by including in the wave function (6) only states involving up to a certain number of phonons and by diagonalizing this restricted system. Assuming first a value for the eigenvalue, one may solve the linear equations in the amplitudes $\langle j; NR; IM \rangle$ and, in turn, obtain an improved energy value. Such an iteration method was found to converge rapidly.

III. Level Structure.

We have studied the level structure for a particle with $j = 5/2$ coupled to the nuclear surface. The case $j = 5/2$ was chosen because, with increasing j, the number of phonons occurring for a given coupling strength increases. On the other hand, for $j = 1/2$, the coupling vanishes and, in case of $j = 3/2$, there is no regular strong coupling solution with which to compare.

We have calculated the energies as a function of the coupling parameter x of the two lowest $I = 5/2$ states of which the lower represents the ground state of the system; besides these,

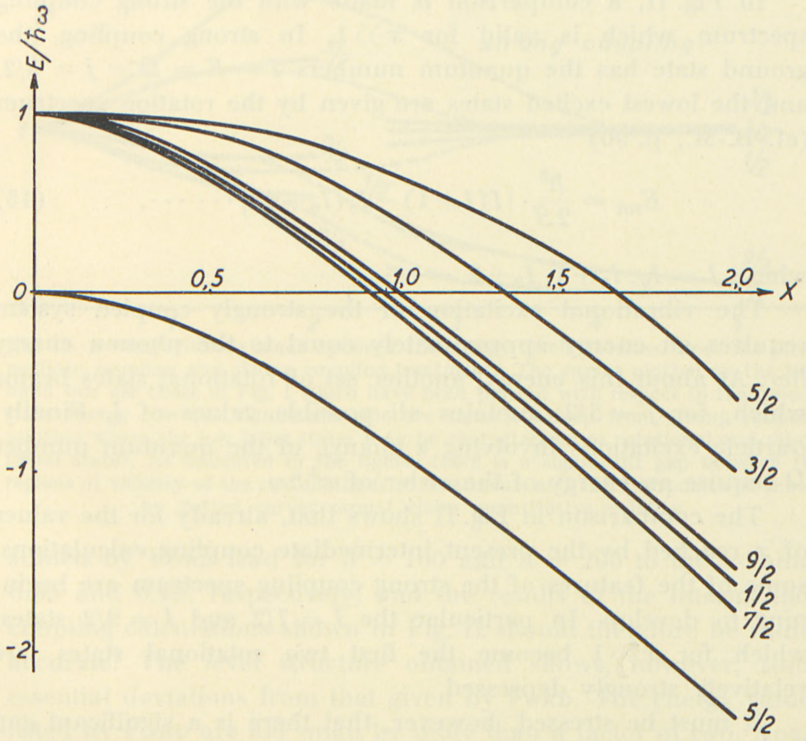

Fig. I. Level structure of a single $j = 5/2$ particle coupled to the nuclear surface oscillations. Energies of the two lowest states of $I = 5/2$ in units of $\hbar\omega$ have been plotted as a function of the coupling parameter $x = \sqrt{\frac{5}{4\pi}} \cdot \sqrt{\frac{1}{4j}} \cdot \frac{k}{\sqrt{\hbar\omega C}}$. The lower curve represents the ground state of the system. The other four curves represent the energies of the lowest states for types $I = 1/2$, $3/2$, $7/2$, and $9/2$. Wave-functions including all unperturbed states with up to three phonons have been used in the calculations. The results obtained are believed to be reliable for $x < 1 \cdot 5$.

the energies of the lowest states of types $I = 1/2$, $3/2$, $7/2$, and $9/2$ have also been evaluated. In the calculations we have included all the unperturbed states including up to three phonons. The results obtained are shown in Fig. I.

The region of validity of the present calculations can be seen from the magnitude of the amplitudes of the three phonon states and it is found that significant effects of states with a higher number of phonons are to be expected for x larger than 1.5. It may be added that a comparison with the results obtained by only including states up to two phonons shows that the latter approximation is only valid for $x < 1$.

In Fig. II, a comparison is made with the strong coupling spectrum which is valid for $x \gg 1$. In strong coupling, the ground state has the quantum numbers $I = K = \Omega = j = 5/2$, and the lowest excited states are given by the rotation spectrum (cf. B.-M., p. 96)

$$E_{\text{rot}} = \frac{\hbar^2}{2\mathcal{J}} \cdot [I(I+1) - I_0(I_0+1)] \cdots \cdots, \qquad (15)$$

where $I = I_0, I_0+1, I_0+2, \cdots \cdots$

The vibrational excitation of the strongly coupled system requires an energy approximately equal to the phonon energy $\hbar\omega$. At about this energy, another set of rotational states begins which, for $j = 5/2$, contains all possible values of I. Finally, particle excitations involving a change of the quantum number Ω require an energy of the order of $x^2\hbar\omega$.

The comparison in Fig. II shows that, already for the values of x reached by the present intermediate coupling calculations, some of the features of the strong coupling spectrum are beginning to develop. In particular the $I = 7/2$ and $I = 9/2$ states, which for $x \gg 1$ become the first two rotational states, are relatively strongly depressed.

It must be stressed, however, that there is a significant gap between the regions of validity of the two methods of treatment, and the interpolations, shown in the figure by dotted curves, cannot therefore claim quantitative validity. In order to carry the present calculations to such large values of x that a detailed contact with the strong coupling spectrum can be made, it would be necessary to include in the wave function many more phonons,

which would make calculations with the present techniques rather impracticable.

The present results can be compared with those obtained by FORD[1] who has calculated the level structure for a single particle ($j = 5/2$) configuration in the coupled system by using the strong coupling approximation. The nuclear parameters as-

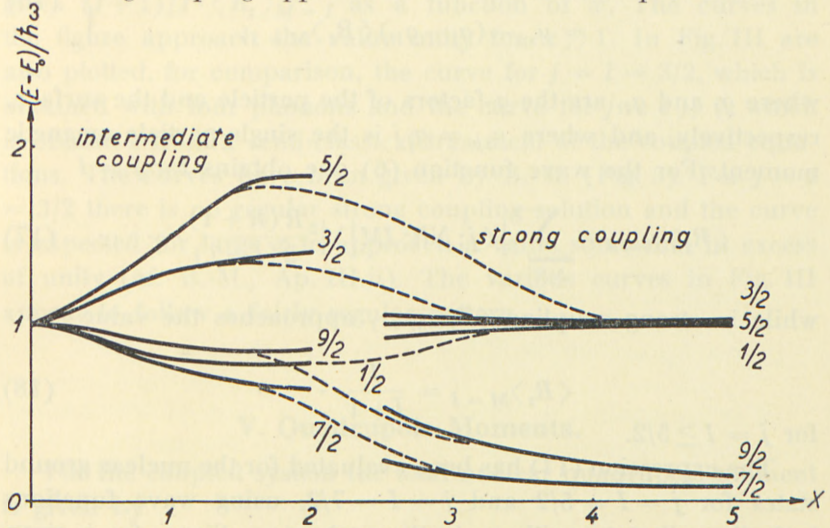

Fig. II. A comparison is made between the energy levels obtained from the intermediate coupling and strong coupling treatments. The curves plotted on the left-hand side are those in Fig. I which have been plotted with respect to the ground state energy. On the right-hand side are those obtained from strong coupling solution where the low lying states may be characterized as rotational and vibrational states. As indicated in the figure, there is a significant gap between the regions of validity of the two methods of treatment, and the interpolations shown by dotted curves cannot claim quantitative validity.

sumed by FORD lead for $A = 100$ and $A = 200$ to the x-values 0.65 and 0.90, respectively, and the results of the intermediate coupling calculations shown in Fig. II should therefore be rather accurate. The level structure obtained shows, however, some essential deviations from that given by FORD. The energy values listed by FORD are too small by more than a factor of two. Apart from the three lowest states, the strong coupling approximation also leads to a different level order from that obtained by the proper intermediate coupling calculations. The conclusion that a single particle configuration does not lead to a strong coupling situation was also drawn by FORD from other arguments.

[1] K.W. FORD, Phys. Rev. **90**, 29, 1953.

IV. Magnetic Moments.

For the coupled system consisting of a single particle and the nuclear surface oscillations, the magnetic moment is given by

$$\left.\begin{aligned}\mu &= \langle g_j j_z + g_R R_z \rangle_{M=I} \\ &= \mu_{sp} - (g_j - g_R)\langle R_z \rangle_{M=I},\end{aligned}\right\} \quad (16)$$

where g_j and g_R are the g-factors of the particle and the surface, respectively, and where $\mu_{sp} = g_j j$ is the single-particle magnetic moment. For the wave function (6) one obtains for $j = I$

$$\langle R_z \rangle_{M=I} = \sum_{NR} |\langle j; NR; IM | \rangle|^2 \frac{R(R+1)}{2(I+1)} \cdots\cdots \quad (17)$$

while in strong coupling $\langle R_z \rangle_{M=I}$ approaches the value

$$\langle R_z \rangle_{M=I} = \frac{I}{I+1} \cdots\cdots \quad (18)$$

for $j = I \geq 5/2$.

The expression (17) has been evaluated for the nuclear ground states for $j = I = 5/2$ and $j = I = 7/2$, using wave functions including all states with up to three phonons. Since, for a given value of x the number of phonons present in the coupled system

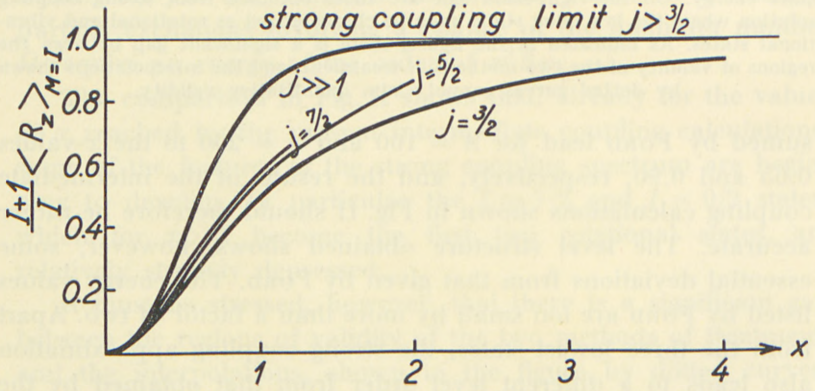

Fig. III. The figure illustrates the transfer of angular momentum from the particle to the surface oscillators as a function of the coupling strength. The curves for $j = I = 5/2$ and $j = I = 7/2$ have been obtained by using wave functions including all unperturbed states with up to three phonons. For comparison, the curves obtained by B.-M. for $j = I = 3/2$ and $j = I \rangle\rangle 1$ have also been given (cf. their Fig. 10).

increases with j, the three phonon approximation begins to break down for lower values of x in the case of $j = I = 7/2$ than for $j = I = 5/2$. In the former case, the results are considered to be reliable only for $x < 1$.

The results of the calculations are shown in Fig. III which gives $(I+1)/I \cdot \langle R_z \rangle_{M=I}$ as a function of x. The curves in the figure approach the value unity for $x \gg 1$. In Fig. III are also plotted, for comparison, the curve for $j = I = 3/2$, which is obtained with four phonons and the curve for $j = I \gg 1$, which is obtained from a semi-classical treatment of the coupled equations. The curves have been given by B.-M. (Fig. 5). For $j = I = 3/2$ there is no regular strong coupling solution and the curve is expected for large x to approach a value somewhat in excess of unity (cf. B.-M., Ap. III.ii). The various curves in Fig. III appear to follow a fairly regular pattern.

V. Quadrupole Moments.

For the coupled system the total nuclear quadrupole moment is given by

$$Q = Q_p + Q_s \cdots\cdots, \tag{19}$$

where Q_p is the contribution of the particle and Q_s is that of the surface deformation. It is expected that, in general, the latter term contributes the main part of the total Q.

The quadrupole moment Q_s for a uniformly charged nucleus is given by

$$Q_s = \frac{3\,ZR_0^2}{\sqrt{5\pi}} \cdot \langle \alpha_{20} \rangle_{M=I} \cdots\cdots, \tag{20}$$

where Z is the nuclear charge and R_0 the mean radius of the nucleus. Since the particle-surface interaction energy is a linear expression in $\alpha_{2\mu}$, the matrix element in (20) may be conveniently evaluated by using the Schrödinger wave equation*; one then obtains

$$\langle \alpha_{20} \rangle_{M=I} = -\frac{\sqrt{5\pi}}{5} \cdot \frac{8\,I}{(2I+3)} \cdot \frac{\hbar\omega}{k} \cdot \sum_{NR}{}' |\langle j; NR; IM | \rangle|^2 (N + |\varepsilon|) \cdots\cdots \tag{21}$$

* I am indebted to Dr. K. Alder for suggesting this method.

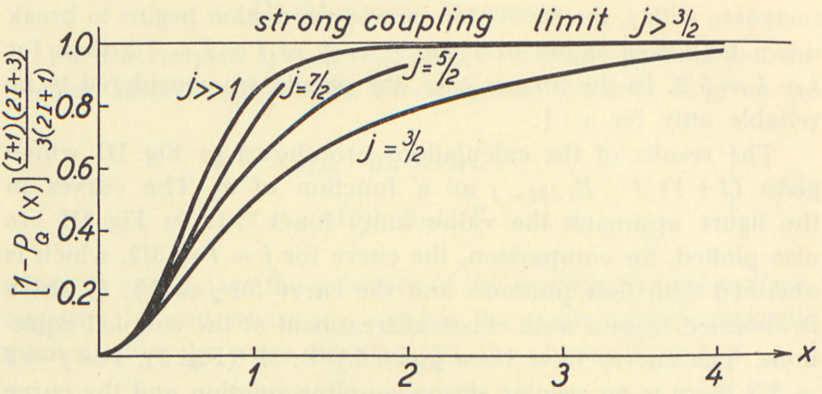

Fig. IV. The figure shows the gradual development of the projection factor for the quadrupole moment in the coupled system as a function of the coupling strength. The curves for $j = I = 5/2$ and $j = I = 7/2$ have been obtained by using wave functions including all unperturbed states with up to three phonons. For comparison, the curves obtained by B.-M. for $j = I = 3/2$ and $j = I \gg 1$ have also been given (cf. their Fig. 10).

It is convenient to write the collective part of the nuclear quadrupole moment in the form (cf. B.-M., Chapter V)

$$Q_s = Q_0 \times P_Q(x) \cdots\cdots\cdots, \tag{22}$$

where

$$Q_0 = -\frac{3\,ZR_0^2}{4\pi} \cdot \frac{k}{C} \cdot \frac{(2I-1)}{2(I+1)}, \tag{23}$$

and where $P_Q(x)$ represents a projection factor which is equal to unity for $x \ll 1$ and approaches for $x \gg 1$, the value

$$P_Q(x) = \frac{I(2I-1)}{(I+1)(2I+3)} \cdots\cdots \tag{24}$$

In strong coupling, where the nucleus performs small vibrations about a certain equilibrium deformation, the quantity Q_0 represents the instrinsic nuclear quadrupole moment measured with respect to the axis of the deformed nucleus.

The transition from weak to strong coupling may be conveniently described in terms of the gradual development of the projection factor (24). For the intermediate coupling wave function, one obtains from (20), (21), (22), and (23)

$$P_Q(x) = \frac{1}{x^2} \cdot \frac{4(I+1)}{(2I+3)(2I-1)} \cdot \sum_{NR} |\langle j; NR; IM|\rangle|^2 (N+|\varepsilon|) \cdots\cdots (25)$$

This expression has been evaluated for the states $j = I = 5/2$ and $j = I = 7/2$, for the wave function including all possible states with up to three phonons, and the results are given in Fig. IV. In this figure are also shown the intermediate coupling projection factors obtained by B.-M. (cf. their fig. 10) for the states $j = I = 3/2$ and $j = I \gg 1$. The present results tend to confirm the interpolated values of $P_Q(x)$ employed by B.-M. in their analysis of empirical quadrupole moments (cf. Table IX of this reference).

I wish to express my most sincere gratitude to Professor Niels Bohr for the privilege of working in his Institute and for his kind interest in this problem. I am also greatly indebted to Drs. K. Alder, A. Bohr, and B. Mottelson for many helpful discussions and suggestions.

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ON THE EQUIVALENCE OF INVARIANCE UNDER TIME REVERSAL AND UNDER PARTICLE-ANTIPARTICLE CONJUGATION FOR RELATIVISTIC FIELD THEORIES

BY

GERHART LÜDERS

København
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For relativistic field theories, in a sense specified in section 2, the invariance under time reversal "of the second kind" (time reversal including particle-antiparticle conjugation) is proved mathematically. Consequently, the postulate of invariance under time reversal ("of the first kind") is, for field theories of this type, completely equivalent to the postulate of invariance under particle-antiparticle conjugation.

Introduction.

It was found by several authors that the postulates of invariance of the laws of nature under time reversal or under particle-antiparticle conjugation[1] allow one to rule out some kinds of couplings which, nevertheless, are in accordance with the postulate of relativistic invariance. Two applications are hitherto known, viz.

(1) Coupling between one Bose field ("mesons") and one Dirac field ("nucleons"). Simultaneous coupling with and without derivatives is forbidden for scalar and pseudovector fields[2,3].

(2) Fermi coupling of four Dirac fields. In a sum of several covariant couplings, the phases of the coupling constants must be the same[4,5].

It is a remarkable fact that both results follow from each of the two postulates. Therefore, one might be led to assume that, quite generally, a relativistic field theory is invariant either

[1] We prefer the term "particle-antiparticle conjugation" (though more lengthy) to the more commonly used denotation "charge conjugation".

[2] As a consequence of particle-antiparticle conjugation, cf. LÜDERS, G., R. OEHME, and W. E. THIRRING, Z. Naturforschung 7 a, 213 (1952); PAIS, A., and R. JOST, Phys. Rev. 87, 871 (1952).

[3] As a consequence of time reversal, cf. LÜDERS, G., Z. Physik 133, 325 (1952).

[4] As a consequence of time reversal, cf. BIEDENHARN, L. C., and M. E. ROSE, Phys. Rev. 83, 459 (1951).

[5] As a consequence of both time reversal and particle-antiparticle conjugation, cf. TOLHOEK, H. A. and S. R. DE GROOT, Phys. Rev. 84, 151 (1951).

under both transformations or under neither of them. In the present note, the proof of this conjecture will be given. For the sake of simplicity, the considerations are restricted to local field theories constituted by the usual fields of spin 0, 1, and $^1/_2$. The coupling Hamiltonians shall contain no derivatives of the Dirac fields and no higher derivatives than the first of the Bose fields. The theories shall be relativistic in a sense specified in section 2 by the postulates I, II, I′, II′, Ia, IIa. It seems very likely that the result of the considerations, i. e., the equivalence of the two kinds of invariance postulates, holds true also under more general conditions.

In the following, two types of time reversal will appear: the time reversal "of the first kind" which, loosely speaking, consists in a reversal of the motion[1] of all particles, and the time reversal "of the second kind", a simultaneous performance of a proper time reversal and a particle-antiparticle conjugation[2]. (The only type of time reversal which is of relevance for the principle of detailed balance and, perhaps, for the foundation of thermodynamics is that of the first kind[3].) In this paper it is proved that a relativistic field theory (in the sense specified in section 2) is automatically invariant under time reversal of the second kind[4]. For the validity of the proof, one has explicitly to assume that the field theory in question is invariant under reflection in space; a formal reflection in time acts as an intermediate step in the proof.

As the time reversal of the second kind is identical with a simultaneous application of a time reversal of the first kind (i. e., time reversal in the proper sense) and a particle-antiparticle conjugation, a relativistic theory (in the restricted sense of this paper) is either invariant under both operations or under neither of them. Therefore, both of these invariance postulates

[1] In a previous paper (Z. Physik **133**, 325 (1952)), we preferred the term "Bewegungsumkehr" (reversal of motion) to "Zeitumkehr" (time reversal) for this operation.

[2] The different types of time reversal were also considered by J. Tiomno in his Princeton thesis. Further, S. Watanabe (Phys. Rev. 84, 1008 (1951)) uses two types of time reversal; his "standpoint I" corresponds to our time reversal of the second kind, and vice versa.

[3] Coester, F., Phys. Rev. 84, 1259 (1951), Lüders, G., Z. Physik **133**, 325 (1952).

[4] This conjecture was suggested to the writer in a correspondence with B. Zumino, New York.

lead to the same consequences, e. g., the exclusion of some couplings. It seems to be a matter of taste which of these two postulates is considered the more fundamental one.

We shall be concerned with various types of symmetry operations: viz., two types of time reversal, particle-antiparticle conjugation, and reflections in space and in time. These operations will be uniformly treated as substitutions; the prescriptions for these substitutions are summarized in Table 1. In this table, $\varphi(\vec{r})$ means the operator of a spin 0 field, $\dot{\varphi}(\vec{r})$ the operator corresponding to its derivative with respect to time; furthermore $\varphi_k(\vec{r})$ ($k = 1, 2, 3$) are the space components of a spin 1 field, and $\varphi_0(\vec{r})$ is the time component; finally, $\dot{\varphi}_k(\vec{r})$, $\dot{\varphi}_k(\vec{r})$ are the derivatives with respect to the time, and $\psi(\vec{r})$ is the operator of a Dirac field. All operators shall be understood in the Schrödinger representation, where one, of course, has identities of the form

$$\dot{\varphi}(\vec{r}) \equiv i\,[H, \varphi(\vec{r})]. \tag{0.1}$$

Further, one has subsidiary conditions for spin 1 fields with non-vanishing rest mass. In the table, the quantities ε_0, ε_1, $\varepsilon_{1/2}$, η_0, etc., are multiplicative c-numbers, whereas the symbols U, C, T mean four-rowed matrices acting on the spinor indices.

In section 1, the mathematical definitions of the two kinds of time reversal and of particle-antiparticle conjugation are summarized. In section 2, the proof is given for the invariance of a relativistic field theory under time reversal of the second kind. In Appendix 2, a lemma on the covariant quantities which can be constructed from Dirac spinors is proved.

1. Time reversal of the first and second kind, and particle-antiparticle conjugation.

The *time reversal of the first kind* was formulated for the first time by WIGNER[1]; the substitutions on the field operators, corresponding to this operation, were given in a previous paper[2]. These substitutions are summarized in the second column of Table 1. The most essential prescription is that, according to the

[1] WIGNER, E. P., Göttinger Nachr. math.-phys. Kl. **1932**, 546.
[2] LÜDERS, G., Z. Physik **133**, 325 (1952), in the following quoted as A.

TABLE 1

General comments are given at the end of the introduction and at the beginning of section 1.

	1	2	3	4	5	6
Spin	Field operator	Time reversal of the first kind	Particle-antiparticle conj.	Time reversal of the second kind	Reflection in space	Formal reflection in time
0	$\varphi\ (\vec{r})$ $\varphi^*(\vec{r})$ $\dot{\varphi}\ (\vec{r})$ $\dot{\varphi}^*(\vec{r})$	$\varepsilon_0 \varphi\ (\vec{r})$ $\varepsilon_0^* \varphi^*(\vec{r})$ $-\varepsilon_0 \dot{\varphi}\ (\vec{r})$ $-\varepsilon_0^* \dot{\varphi}^*(\vec{r})$	$\eta_0 \varphi^*(\vec{r})$ $\eta_0^* \varphi\ (\vec{r})$ $\eta_0 \dot{\varphi}^*(\vec{r})$ $\eta_0^* \dot{\varphi}\ (\vec{r})$	$\delta_0 \varphi^*(\vec{r})$ $\delta_0^* \varphi\ (\vec{r})$ $-\delta_0 \dot{\varphi}^*(\vec{r})$ $-\delta_0^* \dot{\varphi}\ (\vec{r})$	$\xi_0 \varphi\ (-\vec{r})$ $\xi_0^* \varphi^*(-\vec{r})$ $\xi_0 \dot{\varphi}\ (-\vec{r})$ $\xi_0 \dot{\varphi}^*(-\vec{r})$	$\xi_0 \varphi\ (\vec{r})$ $\xi_0^* \varphi^*(\vec{r})$ $-\xi_0 \dot{\varphi}\ (\vec{r})$ $-\xi_0^* \dot{\varphi}^*(\vec{r})$
1	$\varphi_k(\vec{r})$ $\varphi_k^*(\vec{r})$ $\dot{\varphi}_k(\vec{r})$ $\dot{\varphi}_k^*(\vec{r})$	$\varepsilon_1 \varphi_k(\vec{r})$ $\varepsilon_1^* \varphi_k^*(\vec{r})$ $-\varepsilon_1 \dot{\varphi}_k(\vec{r})$ $-\varepsilon_1^* \dot{\varphi}_k^*(\vec{r})$	$\eta_1 \varphi_k^*(\vec{r})$ $\eta_1^* \varphi_k(\vec{r})$ $\eta_1 \dot{\varphi}_k^*(\vec{r})$ $\eta_1^* \dot{\varphi}_k(\vec{r})$	$\delta_1 \varphi_k^*(\vec{r})$ $\delta_1^* \varphi_k(\vec{r})$ $-\delta_1 \dot{\varphi}_k^*(\vec{r})$ $-\delta_1^* \dot{\varphi}_k(\vec{r})$	$-\xi_1 \varphi_k(-\vec{r})$ $-\xi_1^* \varphi_k^*(-\vec{r})$ $-\xi_1 \dot{\varphi}_k(-\vec{r})$ $-\xi_1^* \dot{\varphi}_k^*(-\vec{r})$	$\xi_1 \varphi_k(\vec{r})$ $\xi_1^* \varphi_k^*(\vec{r})$ $-\xi_1 \dot{\varphi}_k(\vec{r})$ $-\xi_1^* \dot{\varphi}_k^*(\vec{r})$
	$\varphi_0(\vec{r})$ $\varphi_0^*(\vec{r})$ $\dot{\varphi}_0(\vec{r})$ $\dot{\varphi}_0^*(\vec{r})$	$-\varepsilon_1 \varphi_0(\vec{r})$ $-\varepsilon_1^* \varphi_0^*(\vec{r})$ $\varepsilon_1 \dot{\varphi}_0(\vec{r})$ $\varepsilon_1^* \dot{\varphi}_0^*(\vec{r})$	$\eta_1 \varphi_0^*(\vec{r})$ $\eta_1^* \varphi_0(\vec{r})$ $\eta_1 \dot{\varphi}_0^*(\vec{r})$ $\eta_1^* \dot{\varphi}_0(\vec{r})$	$-\delta_1 \varphi_0^*(\vec{r})$ $-\delta_1^* \varphi_0(\vec{r})$ $\delta_1 \dot{\varphi}_0^*(\vec{r})$ $\delta_1^* \dot{\varphi}_0(\vec{r})$	$\xi_1 \varphi_0(-\vec{r})$ $\xi_1^* \varphi_0^*(-\vec{r})$ $\xi_1 \dot{\varphi}_0(-\vec{r})$ $\xi_1^* \dot{\varphi}_0^*(-\vec{r})$	$-\xi_1 \varphi_0(\vec{r})$ $-\xi_1^* \varphi_0^*(\vec{r})$ $\xi_1 \dot{\varphi}_0(\vec{r})$ $\xi_1^* \dot{\varphi}_0^*(\vec{r})$
$^1/_2$	$\psi\ (\vec{r})$	$\varepsilon_{\frac{1}{2}} U\ \psi\ (\vec{r})$	$\eta_{\frac{1}{2}} C^* \psi^*(\vec{r}) = \eta_{\frac{1}{2}} \bar{\psi}(\vec{r})\, \beta C^*$	$\delta_{\frac{1}{2}} T^* \psi^*(\vec{r}) = \delta_{\frac{1}{2}} \bar{\psi}(\vec{r})\, \beta T$	$\xi_{\frac{1}{2}} \beta\ \psi\ (-\vec{r})$	$\xi_{\frac{1}{2}} T\ \psi\ (\vec{r})$
	$\psi^*(\vec{r})$ $\bar{\psi}(\vec{r}) = \psi^*(\vec{r})\beta$	$\varepsilon^*_{\frac{1}{2}} U^* \psi^*(\vec{r})$ $\varepsilon^*_{\frac{1}{2}} U^* \bar{\psi}\ (\vec{r})$	$\eta^*_{\frac{1}{2}} C \psi\ (\vec{r})$ $\eta^*_{\frac{1}{2}} \beta^* C \psi\ (\vec{r})$	$\delta^*_{\frac{1}{2}} T \psi\ (\vec{r})$ $\delta^*_{\frac{1}{2}} \beta T \psi\ (\vec{r})$	$\xi^*_{\frac{1}{2}} \beta^* \psi^*(-\vec{r})$ $\xi_{\frac{1}{2}} \bar{\psi}(-\vec{r})\, \beta$	$\xi^*_{\frac{1}{2}} T^* \psi^*(\vec{r})$ $-\xi^*_{\frac{1}{2}} \bar{\psi}(\vec{r})\, T$
	Real Bose & Majorana fields	$\varepsilon_{0,1} = \pm 1,$ $\varepsilon_{\frac{1}{2}} = \pm i$	$\eta_{0,1,\frac{1}{2}} = \pm 1$	$\delta_{0,1} = \pm 1, \delta_{\frac{1}{2}} = \pm i$	$\xi_{0,1} = \pm 1,$ $\xi_{\frac{1}{2}} = \pm i$	$\xi_{0,1} = \pm 1,$ $\xi_{\frac{1}{2}} = \pm i$
	c-number	(c-number)*	c-number	(c-number)*	c-number	c-number
	Order of factors	unchanged	unchanged	unchanged	unchanged	reversed

second line from below, each c-number is to be replaced by its complex conjugated value. Because of the appearance of i in the Schrödinger equation, this leads to the time reversal of the desired character as pointed out in A. From the table, one reads further that, e. g., the operator $\varphi(\vec{r})$ of a scalar or pseudoscalar field is to be replaced by $\varepsilon_0 \varphi(\vec{r})$ and its time derivative $\dot{\varphi}(\vec{r})$ by $-\varepsilon_0 \dot{\varphi}(\vec{r})$[1], the quantity ε_0 being a complex number of modulus one[2]. The quantity ε_0 is restricted to ± 1 for real fields. In the same way, ε_1 and $\varepsilon_{1/2}$ have modulus one and $\varepsilon_1 = \pm 1$ for real fields (e. g., $\varepsilon_1 = -1$ for the Maxwell field, c. f., A). The restrictive conditions for Majorana fields, the spin $1/2$ analogue to real fields, will be discussed in connection with the particle-antiparticle conjugation. For different kinds of fields one may, of course, choose different factors ε.

. The matrix U entering into the substitutions for Dirac fields was defined in A by the conditions[3,4]

$$U^{-1} \vec{\alpha}^* U = -\vec{\alpha}, \qquad U^{-1} \beta^* U = +\beta, \tag{1.1}$$

and the postulate of unitarity (in order to preserve the commutation relations)

$$U U^\dagger = 1. \tag{1.2}$$

This definition is unique up to a factor of modulus one[5].

The *particle-antiparticle conjugation*[6] can be formulated in a quite similar way (third column of the table). The essential feature, which is characteristic of this operation, is the replacement of all field operators by the Hermitian adjoint operators or, in the spin $1/2$ case, by a linear combination of these adjoint operators. The quantities η_0, η_1, $\eta_{1/2}$ are again complex numbers of modulus one, and η_0, η_1 are restricted to ± 1 for real fields

[1] Because of the identity (0.1) and the prescription that i is to be replaced by $-i$, it is only in this way that one can hope to obtain an invariant Hamiltonian H.

[2] This is a consequence of the commutation relations, which shall be preserved.

[3] There was a mistake in the corresponding equation (1.9) of A.

[4] $\vec{\alpha}$, β are the usual Hermitian Dirac matrices. It is $\gamma_k = i\beta\alpha_k$, $\gamma_4 = \beta$, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\alpha_1\alpha_2\alpha_3$, $\bar{\psi}(\vec{r}) = \psi^*(\vec{r})\beta$; further * = complex conjugated, ^{T} = transposed, † = Hermitian adjoint of a four-rowed matrix.

[5] Some of the properties of the matrices U, C, T are summarized in Appendix 1.

[6] Kramers, H. A., Proc. Acad. Sci. Amsterdam **40**, 814 (1937); Pauli, W., Rev. Mod. Phys. **13**, 203 (1941); Schwinger, J., Phys. Rev. **74**, 1439 (1948). The definition of the matrix C in the present paper is identical with that given by Pauli.

(e. g., $\eta_1 = -1$ for the Maxwell field). For real fields one has, of course, to identify $\varphi^*(\vec{r})$, $\varphi_k^*(\vec{r})$, $\varphi_0^*(\vec{r})$ with $\varphi(\vec{r})$, $\varphi_k(\vec{r})$, $\varphi_0(\vec{r})$ in the prescriptions for the substitutions.

The matrix C is defined by

$$C^{-1}\vec{\alpha}^* C = +\vec{\alpha}, \qquad C^{-1}\beta^* C = -\beta \tag{1.3}$$

and

$$C C^\dagger = 1 \tag{1.4}$$

uniquely up to a factor of modulus one. From a comparison of (1.1) and (1.3), one sees that the matrix

$$T = C^* U \tag{1.5}$$

anticommutes with the Dirac matrices $\vec{\alpha}$, β

$$T^{-1}\vec{\alpha}\, T = -\vec{\alpha}, \qquad T^{-1}\beta\, T = -\beta. \tag{1.6}$$

As T is furthermore a unitary matrix, one may put

$$T = \alpha_x\alpha_y\alpha_z\beta = i\,\gamma_4\gamma_5, \tag{1.7}$$

which gives a relation between the phase factors of U and C.

For a Majorana field[1] the anticommutator between $\psi^*(\vec{r})$ and $\psi(\vec{r}')$ holds unchanged, but one has the subsidiary condition[2]

$$\psi^*(\vec{r}) = C\,\psi(\vec{r}), \qquad \psi(\vec{r}) = C^*\,\psi^*(\vec{r}). \tag{1.8}$$

The operators $\psi(\vec{r})$ and $\psi(\vec{r}')$ do not anticommute any longer but, as a consequence of the foregoing equation, one finds

$$\{\psi_\alpha(\vec{r}),\ \psi_\beta(\vec{r}')\} = C^*_{\alpha\beta}\,\delta(\vec{r} - \vec{r}'). \tag{1.9}$$

From a special row of the table one sees that, for Majorana fields, $\varepsilon_{1/2}$ is restricted to $\pm i$ and $\eta_{1/2}$ to ± 1.

The *time reversal of the second kind* is obtained if one, first[3], performs a particle-antiparticle conjugation and, subsequently, a time reversal of the first kind. The result of this sequence of

[1] MAJORANA, E., Nuovo Cimento **14**, 171 (1937).

[2] These two equations are compatible because of eq. (A 1.2) in Appendix 1.

[3] This special order of the operations was chosen only to make the prescription unique.

operations is summarized in the fourth column of the table. In all cases, one has

$$\varepsilon \eta \delta = 1 \tag{1.10}$$

(Note that $\delta\delta^* = 1$!). The matrix T has been defined by eq. (1.7).

All these symmetry operations can be applied to state vectors by the prescription given in A: Write any state vector Ψ in the form

$$\Psi = \Omega \Psi_0, \tag{1.11}$$

where Ψ_0 is the vacuum of the free fields, and perform the substitutions on the creation operator Ω. This prescription is unique in spite of the various possibilities of writing down Ω as pointed out in A. The prescription just formulated does not lead to contradictions, as all three operations transform creation operators into creation operators and annihilation operators into annihilation operators.

A given field theory is invariant under a symmetry operation if, in the Schrödinger representation, the commutation relations between the field operators, the Hamiltonian H, and possible subsidiary conditions are preserved. One easily checks that this is true for all symmetry operations considered so far if one has non-interacting fields. If we restrict ourselves to interaction Hamiltonians without derivatives, the commutation relations hold unchanged and one has only to examine the interaction part, H_J, of the Hamiltonian. That means that we have to investigate whether such a choice of the c-number factors in the substitutions can be made so that H_J is simply multiplied by $+1$. Therefore, it is quite clear that a non-trivial problem occurs only if the interaction Hamiltonian is a sum of elementary interactions, where the interaction density, $\mathcal{H}_J(\vec{r})$, is given by a simple product of field operators, supplemented, if necessary, by the Hermitian adjoint expression. If one allows for first derivatives of the Bose fields in the interaction, one has to examine both commutation relations and Hamiltonian; we shall, however, avoid this difficulty by constructing a "nucleus of the interaction representation" which gives commutation relations as in the case without interaction.

A substitution which does not affect c-numbers and the order of factors in products can be generated by a canonical transformation. Consequently, the particle-antiparticle transformation is the only transformation considered in this section, which is equivalent to a canonical transformation.

2. Invariance of relativistic field theories under time reversal of the second kind.

In order to prove, for relativistic field theories, the invariance under time reversal of the second kind it is, primarily, necessary to give a specified definition of these field theories. Only the case of *non-derivative couplings* will be treated in some detail. The modifications of the considerations for derivative couplings (first order derivatives of Bose fields) are discussed at the end of the section.

A relativistic field theory with non-derivative coupling, constituted by fields of spin 0, 1, $^1/_2$, will be defined by the following two postulates.

I. The commutation relations are identical with those for the free fields.

II. The interaction part, H_J, of the Hamiltonian is a Hermitian operator containing no derivatives of the field operators, and the corresponding localized density transforms like a scalar under the orthochronous Lorentz group (including reflections in space, but not in time).

It would certainly be more satisfactory to give a more fundamental definition of relativistic field theories, using the Lagrangian formulation, etc. But, for the present purpose, this would involve the introduction of comparatively complicated general considerations. We are convinced that our results hold also for wider classes of relativistic field theories which are not covered by the postulates I, II and I′, II′, respectively.

In addition to postulate I, we impose a restriction on the relation between different Dirac fields which, in principle, might either commute or anticommute[1]. We explicitly assume

[1] In contrast to opinions occasionally expressed in the literature, there seems not to exist a simple correspondence between theories with anticommuting Dirac fields and those with commuting Dirac fields if one has more than two such fields.

Ia. Kinematically independent Dirac fields *anti*commute.

This is a necessary condition for the general validity of the proof to be given below.

As one of the steps in our proof, viz. the formal reflection in time, leads to a reversal of the order of factors in products, we assume that all products in $\mathcal{H}_J(\vec{r})$ are symmetrized in the same way as they are in the so-called "charge symmetrical" formulation[1].

IIa. Each product of m Bose fields and $2n$ Dirac fields is to be replaced by the sum, divided by $(m+2n)!$, of all permutations of the factors, each of the terms being multiplied by $+1$ or -1 for an even or odd permutation of the Dirac fields, respectively.

Making use of postulate Ia, it is seen that this symmetrized product has the simple property that it is multiplied by $(-)^n$ if the order of all factors is reversed.

Before entering into the proof that the invariance under time reversal is a mathematical consequence of the postulates I, Ia, II, IIa, we have first to make clear the way in which Dirac fields can appear in $\mathcal{H}_J$ and then to formulate in more detail the reflection in space. The form in which Dirac fields enter could be restricted by a further postulate, but this is not necessary if we make use of a lemma proved in Appendix 2. According to this lemma, every covariant quantity consisting of products of $2n$ spinors can be represented as a linear combination of products of the well known bilinear covariant quantities

$$\bar{\psi}\psi, \quad i\bar{\psi}\gamma_\mu\psi, \quad i\bar{\psi}\gamma_\mu\gamma_\nu\psi, \quad i\bar{\psi}\gamma_\mu\gamma_5\psi, \quad i\bar{\psi}\gamma_5\psi. \tag{2.1}$$

Consequently, the spin 0 and spin 1 fields are to be combined with these covariant quantities in such a way that one formally obtains a scalar (under the proper Lorentz group) for the interaction density $\mathcal{H}_J(\vec{r})$.

In postulate II, the invariance of H_J under *reflections in space* was stated. The substitutions corresponding to a reflection at the origin of the coordinate system are summarized in column 5

[1] Heisenberg, W., Z. Physik **90**, 209, **92**, 692 (1934); Schwinger, J., Phys. Rev. **74**, 1439 (1948).

of Table 1. The expressions given there are more general than necessary, as one usually restricts ξ_0, ξ_1 to the values ± 1. Fields with $\xi_{0,1} = +1$ are denoted as proper fields and those with $\xi_{0,1} = -1$ as pseudofields. We shall, however, not restrict $\xi_{1/2}$ apart from having modulus one. One easily checks that the quantities (2.1), constructed from Dirac fields, transform just like ordinary scalars, vectors, tensors of rank two, pseudovectors, and pseudoscalars, if one disregards for the moment the factors $\xi_{1/2}$. Invariance under reflection in space means that the factors ξ can be chosen in such a way that the interaction density, apart from the substitution of $\vec{r}$ by $-\vec{r}$, is simply multiplied by $+1$. Then the integrated interaction Hamiltonian H_J is evidently invariant.

After these preliminaries we are able to give the proof of the invariance under time reversal of the second kind. This proof proceeds in two steps. First, we shall show that a field theory covered by the postulates given above is invariant under a "formal reflection in time", which essentially is the Lorentz transformation of the operators corresponding to a reversal of the direction of the time axis. Subsequently, we shall demonstrate that this reflection in time is equivalent to just the time reversal of the second kind, to which one can go over by the process of Hermitian conjugation.

The substitutions corresponding to the *formal reflection in time* are summarized in column 6 of the table. It is an essential feature of this reflection that the matrix T (eq. (1.7)) plays the role of the corresponding spinor transformation. It should, perhaps, be mentioned that this operation is the only one of all substitutions considered in this paper which cannot be applied to state vectors in a simple manner, as it transforms creation operators into annihilation operators, and vice versa. But this does not matter in our connection, as this reflection enters only as an intermediate step in our proof.

This formal reflection in time is to be accompanied by a reversal of the order of factors in products in order to preserve the commutation relations of the Bose fields. The symmetrization postulate IIa was made in order to have simple behaviour under this reversal of the order of factors. Under this reversal, symmetrized products of operators, among them $2n$ Dirac fields, are

multiplied by $(-)^n$ as already pointed out. This result may be expressed by saying that each covariant quantity (2.1) takes up an additional factor -1. Consequently, one can easily check that the bilinear covariants transform under formal reflection in time, just as ordinary scalars, vectors, etc., and as it was postulated in the table for Bose fields. Thus, as the interaction density $\mathcal{H}_I(\vec{r})$ is formally a scalar under the proper Lorentz group, and as we choose for the reflection in time the same factors ξ which made the Hamiltonian invariant under reflection in space (cf. Table 1), the field theory in question is also invariant under formal time reflection[1].

In the second step of the proof, we go over from the Hamiltonian reflected in time to that obtained by time reversal of the second kind by means of *Hermitian conjugation*[2]. The Hamiltonian or, more explicitly, its interaction part, was assumed to be a Hermitian quantity (postulate II) and is, therefore, not changed by this operation[3]. On the other hand, all field operators are now replaced by the Hermitian adjoint operators, all c-numbers by the complex conjugated numbers, and the order of factors in all products is reversed. Therefore, the original order of factors in all individual, not symmetrized, products is restituted. In this way one gets in fact that Hamiltonian which one can obtain from the original one by a time reversal of the second kind if one chooses the factors δ, entering in the time reversal of the second kind, in such a way that

$$\xi\delta = 1. \tag{2.2}$$

This can be seen from a detailed study of the table. The condition (2.2) can be fulfilled also for real Bose fields and for Majorana fields.

[1] Perhaps it should be emphasized that this reflection in time is treated as a purely formal operation. The problem whether Dirac fields are measurable quantities or not does, therefore, not occur.

[2] This connection between the formal reflection in time and the time reversal of the second kind throws some light on a discrepancy between results obtained by SCHWINGER (Phys. Rev. 82, 914 (1951)) and by WATANABE (l. c.). The time reversal applied by SCHWINGER is, in our language, a formal reflection in time; on the other hand, WATANABE pointed out that only time reversal of the second kind (his standpoint I), but not time reversal of the first kind, leads to a determination of the commutation relations.

[3] In this connection, it should perhaps be noted that proofs of the invariance under time reversal or under particle-antiparticle conjugation, for a given field theory, as a rule make use of the Hermitian character of the Hamiltonian.

In this way the proof is finished; the invariance under time reversal of the second kind is a mathematical consequence of the postulates I, II, Ia, IIa. Then, the equivalence of time reversal of the first kind and of particle-antiparticle conjugation is expressed by eq. (1.10), which allows us to go over from one operation to the other.

Finally, the proof has to be extended to couplings involving first *derivatives* of the Bose fields. To this purpose we make a transformation which can be considered as going over to a "nucleus of the interaction representation". We postulate[1] that it is possible to express the operators $\varphi(\vec{r})$, $\dot{\varphi}(\vec{r})$, $\varphi_k(\vec{r})$, etc. by other operators $\tilde{\varphi}(\vec{r})$, $\dot{\tilde{\varphi}}(\vec{r})$, $\tilde{\varphi}_k(\vec{r})$, in such a way that

I′. The commutation relations for the fields $\tilde{\varphi}(\vec{r})$ etc. are formally identical with those for the original free fields (without $\sim$).

II′. The Hermitian Hamiltonian becomes a sum of a part $\tilde{H}_0$, which is formally identical with the free field Hamiltonian for the original fields, and an interaction part $\tilde{H}_J$, the density of which is the 00-component of a relativistic tensor (if expressed by the fields $\tilde{\varphi}(\vec{r})$ etc.!).

We further retain the postulates Ia, IIa, but formulate them now, of course, for the fields $\tilde{\varphi}(\vec{r})$ etc. Then, the whole proof runs as in the case with no derivatives if one applies the substitutions given in the table on the fields $\tilde{\varphi}(\vec{r})$, etc.

Usually, the fields with and without $\sim$ are different only for time derivatives of Bose fields. From

$$\varphi(\vec{r}) = \tilde{\varphi}(\vec{r}) \tag{2.3}$$

and the identity (0.1), it then follows that one actually has the right substitution for $\tilde{\varphi}(\vec{r})$ if one simply applies the substitutions on the original fields. But, for the argument of relativistic covariance, the transition to the fields $\tilde{\varphi}(\vec{r})$ etc. is a rather essential step.

[1] Cf. the remarks on the notion of relativistic field theories succeeding postulates I and II.

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Appendix 1

Properties of the matrices U, C, and T

The four-rowed matrices U and C are uniquely defined up to a factor of modulus one by eqs. (1.1), (1.2), or (1.3), (1.4), respectively. For the transposed matrices one has[1]

$$U^T = -U, \qquad C^T = +C. \tag{A 1.1}$$

From (1.2) or (1.4) and (A 1.1), it follows that

$$UU^* = -1, \qquad CC^* = +1. \tag{A 1.2}$$

For the matrix T (eq. (1.7)) one finds

$$UTU^\dagger = CTC^\dagger = -T^*. \tag{A 1.3}$$

It is, according to eq. (1.5),

$$T = C^*U = -U^*C \tag{A 1.4}$$

where use was made of

$$T = T^\dagger \tag{A 1.5}$$

and (A 1.1). Finally, one has

$$TT^\dagger = 1. \tag{A 1.6}$$

[1] Proof either explicitly using a special representation or, more generally, following a method by HAANTJES and PAULI. Cf. PAULI, W., Ann. Inst. H. Poincaré **6**, 137 (1936).

Appendix 2

Covariant quantities constructed from Dirac spinors

Lemma: Each covariant quantity constructed from linear combinations of products of n spinors with star (ψ^*) and n spinors without star (ψ)[1] can be represented as a linear combination of products of n bilinear covariant quantities (eq. (2.1)). It is further possible to build these bilinear covariants for a special pairing, i. e., a special correspondence between the $\overline{\psi} = \psi^*\beta$ and ψ operators so that each pair is connected by a general γ matrix (1, γ_μ, $\gamma_\mu\gamma_\nu$, etc.).

Additional remark: A special case of this lemma was proved by PAULI and FIERZ[2]: For four spinors, all scalars constructed from the bilinear covariants for *any* pairing can be expressed as a linear combination of products of bilinear covariants for a *special* pairing. Our lemma is more general in several respects. It asserts that, e. g., any expression $\psi^*_{1\alpha}\,\psi_{2\beta}\,\psi^*_{3\gamma}\,\psi_{4\delta}\,\Gamma_{\alpha\beta\gamma\delta}$ transforming like a scalar can be written as a linear combination of $(\overline{\psi}_1\psi_2)\,(\overline{\psi}_3\psi_4)$, $(\overline{\psi}_1\gamma_\mu\psi_2)\,(\overline{\psi}_3\gamma_\mu\psi_4)$, etc. Further, the proof is not restricted as regards number of spinors and rank of the tensor to be constructed. The wider validity of our lemma is counterbalanced by the fact that we use a more abstract tool for our proof than PAULI and FIERZ did.

Preliminaries to the proof: Every finite irreducible representation of the proper Lorentz group can be characterized by two integral or half integral numbers. One denotes such a representation by the symbol $D(j_1, j_2)$. A Dirac spinor (without star as well as with star) transforms according to the representation $D(^1/_2, 0) + D(0, ^1/_2)$, which is reducible under the proper Lorentz group, but irreducible under the orthochronous Lorentz group. To the construction of all possible covariant quantities from a pair of Dirac spinors corresponds the decomposition of the Kronecker product of the representations, which can be done according to general rules

$$\left.\begin{aligned}&\left(D(^1/_2, 0) + D(0, ^1/_2)\right)^2 = \\ &2\,D(0, 0) + 2\,D(^1/_2, ^1/_2) + D(1, 0) + D(0, 1).\end{aligned}\right\} \quad \text{(A 2.1)}$$

[1] This assumption means no loss of generality, as the matrix C always allows us to go over from one Dirac spinor to another which behaves like the Hermitian adjoint spinor.

[2] FIERZ, M., Z. Physik **104**, 553 (1937).

On the right hand side, one has just the well known bilinear covariants 2 scalars $D(0, 0)$, 2 four vectors $D(^1/_2, ^1/_2)$, and one general six vector $D(1, 0) + D(0, 1)$. Note that we here classify only with respect to the proper Lorentz group, where no difference between proper tensors and pseudo-tensors exists.

Proof of the lemma: For 2 n Dirac spinors, the representation

$$(D(^1/_2, 0) + D(0, ^1/_2))^{2n} \qquad \text{(A 2.2)}$$

gives, if decomposed into irreducible constituents,

(1) the linear independent covariant quantities which can be constructed from these $2n$ spinors (as (A 2.2) is the $2n^{\text{th}}$ power of $D(^1/_2, 0) + D(0, ^1/_2)$),

(2) the covariant quantities which can be constructed from the products of n bilinear covariants for a special pairing (as (A 2.2) is the n^{th} power of (A 2.1)).

Consequently, the number of linearly independent tensors of given rank which can be constructed from n bilinear covariants for a given pairing is not less than the number of linearly independent tensors of that rank which can be constructed from $2n$ spinors.

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ON THE RELATION BETWEEN THE TIME-DEPENDENT AND STATIONARY TREATMENTS OF COLLISION PROCESSES

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Introduction.

In recent years, the notion of the S-matrix has found an increasing application in the treatment of collision processes, both in nuclear problems and field theory. In the numerous papers on this subject, the definition of the S-matrix itself has, however, not always been the same, and the connection between the different definitions has not always been quite clear.

When the treatment of the collision process is based on the Schrödinger equation, different definitions of the S-matrix suggest themselves, according as the treatment is based on the time-independent or the time-dependent Schrödinger equation. In the first case, one is led to the original Heisenberg definition; in the latter case, to DYSON's definition of the S-matrix. On the other hand, if one starts from the equations of motion (the field equations) in a Heisenberg representation instead of using the Schrödinger equation, another method of defining the S-matrix suggests itself, which was developed by KÄLLÉN and by YANG and FELDMAN and which has proved very convenient in various field theories.

In the present paper, an attempt has been made to correlate the various methods and to discuss which results can be derived without use of expansions in powers of the interaction, which results are valid as long as such expansions converge, and which results can be valid only as long as the possibility of bound states may be ignored.

I. Notation.

In order to facilitate derivations, and to condense formulas of n'th order perturbation theory to a printable, inspectional, and manageable size, we introduce a symbolic notation, as follows.

Let A, B, C, ... be q-numbers, represented by matrices with a left-hand label L and a right-hand label R, indicating rows and columns, respectively. In DIRAC's bracket notation,

$$A = BC \text{ means } \langle L|A|R\rangle = \int \langle L|B|k\rangle\, dk\, \langle k|C|R\rangle. \quad (1.1)$$

In most cases, we shall use for k a set of q-numbers commuting with each other and with the unperturbed ("free particle") energy E of the particles considered. For instance, k may be the momenta of the particles, or a set of occupation numbers.

In the following,

$$A = Bf(E_L, E_R) \text{ means } \langle L|A|R\rangle = \langle L|B|R\rangle \cdot f(E_L, E_R) \quad (1.2)$$

in any representation in which E may be considered a function of the variables used for labels L and R. Between braces, however, L and R refer to the positions farthest to the left and to the right between such braces; thus,

$$A\{B \cdot f(E_L, E_R)\} = C \text{ means}$$

$$\int \langle L|A|k\rangle\, dk\, \langle k|B|R\rangle\, f(E_k, E_R) = \langle L|C|R\rangle. \quad (1.3)$$

We shall put

$$E_{ij} \equiv E_i - E_j = -E_{ji}. \quad (1.4)$$

Often we shall write a product of q-numbers and insert between or beside the factors one symbol $\|$ (pronounced "gage") and several symbols δ ("delt"), $'$ ("dash"), § ("scat"), $\flat$ ("flat"), and ? (pronounced "slash"). If G denotes the position of the gage between or beside the q-numbers, these symbols are to be interpreted as follows:

$$\left.\begin{aligned} &\text{Each delt at position } \Delta \text{ stands for a} \\ &\quad \text{factor } -i\pi\delta(E_G - E_s) = (\pi/i)\,\delta(E_{G\Delta}). \\ &\text{Each dash at position } d \text{ stands for a} \\ &\quad \text{factor } D_{Gd} \equiv E_{Gd}/(E_{Gd}^2 + a^2) = -D_{dG}. \end{aligned}\right\} \quad (1.5)$$

$$\left.\begin{aligned}&\text{Each scat at position } S \text{ stands for a}\\&\quad\text{factor } \S\,(E_{GS}) \equiv (E_G - E_S + ia)^{-1}.\\&\text{Each flat at position } F \text{ stands for a}\\&\quad\text{factor } \flat\,(E_{GF}) \equiv (E_G - E_F - ia)^{-1}.\end{aligned}\right\} \tag{1.5}$$

In (1.5), a is an infinitely small, real, positive number. At the end of each calculation we take $a \to 0$. (Occasionally, a has a physical meaning and the convenient mathematical limit $a \to 0$ is only a good *approximation*, as the physical a is small, but not really zero).

The dash, flat, and scat at position K are easily recognized as three ways of dividing by $E_{GK} \equiv E_G - E_K$; if the quantity divided does not vanish for $E_G = E_F$, the results of these three different divisions differ by delta functions, as discussed below (see Eq. (1.9)). If we do not want to specify which of these three methods of division we have in mind, we indicate division by $(E_G - E_K)$ by a gage ($\|$) at position G and a slash (?) at position K.

If $\dagger$ denotes the Hermitian conjugate of a matrix.

$$\langle L \,|\, A^\dagger \,|\, R\rangle = \langle R \,|\, A \,|\, L\rangle^*, \tag{1.6}$$

then the following rules are obvious:

1. *Rules valid if A and B are any expressions containing any dashes, scats, flats, slashes or delts:*

$$\left.\begin{aligned}{}'A\| &= {}^1\!/_2\,(\S\, A\| + \flat\, A\|),\\ (A\|)(\delta B\|) &= A\delta B\|,\\ \delta A\| &= \| A\delta,\\ ('A\|)^\dagger &= \|\,(A^\dagger)',\\ (\delta A\|)^\dagger &= -\|\,(A^\dagger)\delta = -\delta A^\dagger\|,\\ (\S\, A\|)^\dagger &= \|\,(A^\dagger)\,\flat,\\ (\flat\, A\|)^\dagger &= \|\,(A^\dagger)\,\S.\end{aligned}\right\} \tag{1.7}$$

In general, the rule, that the Hermitian conjugate of a product equals the product in reversed order of sequence of the Hermitian conjugates of the factors, applies also to products interjected with a gage and various delts, dashes, etc., as long as we treat these symbols as factors themselves and as long as we put

$$(')^\dagger = ', \qquad (\delta)^\dagger = -\delta, \qquad (\S)^\dagger = \flat, \qquad (\flat)^\dagger = \S. \tag{1.8}$$

A further general rule is

$$\S A\| = 'A\| + \delta A\|; \qquad \flat A\| = 'A\| - \delta A\|. \tag{1.9}$$

This rule is easily verified by multiplying these equations by an arbitrary function of E_{RL} continuous along the real axis, and integrating them in the complex E_{RL} plane along the real axis, taking the limit $a \to 0$ at the end. If $C(a)$ is a path of integration from the minimum value of E_{RL} to a point $(E_{RL})_{\min} - ia$, then parallel to the real axis to the point $-(i+1)a$, then along a semi-circle through 0 to $+(1-i)a$, and from there again at a constant distant a below the real axis to $+\infty$, then

$$\left.\begin{aligned} \int_{(E_{RL})\,\min}^{\infty} (\S A\|)\, f(E_{RL})\, dE_{RL} &= \lim_{a\to 0} \int_{C(a)} A\, \frac{f(E_{RL})}{E_{RL} + ia}\, dE_{RL} \\ &= P\int A\, \frac{f(E_{RL})}{E_{RL}}\, dE_{RL} - i\pi f(0)\, A\,\big|_{E_R = E_L}, \end{aligned}\right\} \tag{1.10}$$

where P is the Cauchy principal value of the integral along the real axis, and where the last term derives from the integral in clockwise (negative) direction around the pole at $-ia$. Similarly, we prove

$$\left.\begin{aligned} \int_{(E_{RL})\,\min}^{\infty} (\flat A\|)\, f(E_{RL})\, dE_{RL} &= P\int A\, \frac{f(E_{RL})}{E_{RL}}\, dE_{RL} \\ &\quad + \pi i f(0)\, A\,\big|_{E_R = E_L}. \end{aligned}\right\} \tag{1.11}$$

So, by the first equation (1.7),

$$\int ('A \,\|\,) f(E_{RL})\, dE_{RL} = P \int A \frac{f(E_{RL})}{E_{RL}}\, dE_{RL}, \qquad (1.12)$$

and by (1.5),

$$\int (\delta A \,\|\,) f(E_{RL})\, dE_{RL} = -i\pi f(0)\, A\,|_{E_R = E_L}. \qquad (1.13)$$

From the validity of Eqs. (1.10) — (1.13) for *arbitrary* $f(E_{RL})$ we conclude to the relations (1.9). Also, subtracting $\S(E_{RL})$ and $\flat(E_{RL})$, we get from (1.5)

$$\left.\begin{aligned} \S(E_{RL}) - \flat(E_{RL}) &= \frac{1}{E_{RL} + ia} - \frac{1}{E_{RL} - ia} \\ &= \frac{-2ia}{E_{RL}^2 + a^2} \to -2\pi i \delta(E_{RL}) \end{aligned}\right\} \quad (1.9\,a)$$

for $a \to 0$, in accordance with (1.9).

2. *Rules valid only if q is a q-number not containing any other delts, dashes, scats, flats or slashes referring to the same gage*:

$$\left.\begin{aligned} {'q}\,\| &= -\|\, q', \\ \S q\,\| &= -\|\, q \flat, \\ \flat\, q\,\| &= -\|\, q \S, \\ ?\, q\,\| &= -\|\, q\, (?^\dagger), \\ (?\, q\,\|\,)^\dagger &= -?\, (q^\dagger)\,\|. \end{aligned}\right\} \quad (1.14)$$

The latter three equations easily follow from the first one together with (1.8) and (1.7).

We might have started from Eqs. (1.10) — (1.13) as *definitions* of the scat, flat, dash, and delt. This kind of definition, however, easily creates confusion, as shown by the following example which is of importance also for its applications.

Consider the algebraic identity

$$\left.\begin{aligned}&(E_{LR}\,E_{LM})^{-1}+(E_{RM}\,E_{LM})^{-1}+(E_{RM}\,E_{RL})^{-1}\\&=(E_{LR}\,E_{LM}\,E_{RM})^{-1}(E_{RM}+E_{LR}-E_{LM})=0.\end{aligned}\right\} \quad (1.15)$$

If the dash is defined by (1.12), Eq. (1.15) easily creates the wrong impression that

$$F\equiv \| A'\,B'+{}'A\,\|\,B'+{}'A\,{}'B\,\| \quad (1.16)$$

would vanish. (In fact, of course, such conclusion cannot be drawn even from (1.12), as the "principal value" to be taken in the double integrations over the energies of the intermediate state M (due to Eq. (1.1)) and of the final L in $P\int dE_L\,f(E_L)\,F$ would be defined differently for each of the three terms in (1.16).) In the Appendix A it is shown that, instead of the vanishing of F, we have the important relation

$$D_{LR}\,D_{LM}+D_{RM}\,D_{LM}+D_{RM}\,D_{RL}=\pi^2\,\delta(E_{LM})\,\delta(E_{MR}), \quad (1.17)$$

which means

$$\left.\begin{aligned}&\|A'\,B'+(\|A'\)\,(\'B\|)+{}'A\,{}'B\|\\&=\|A'\,B'+{}'A\|\,B'+{}'A\,{}'B\|=-\delta A\|\,B_\delta\end{aligned}\right\} \quad (1.18)$$

on account of (1.5).

If A is any matrix, we shall frequently in the following use the notation $A(t)$ for the time-dependent matrix

$$A\,(t)=A\exp(E_{RL}\,t/i\hbar). \quad (1.19)$$

If A is the time-independent matrix representing an observable in Schrödinger representation, the $A\,(t)$ is the matrix representing this same observable in interaction representation. Obviously,

$$A\,(t)\;B\,(t)=(A\,B)\exp(E_{RL}\,t/i\hbar), \quad (1.20)$$

since $E_{RM}+E_{ML}=E_{RL}$ if M is the position between the factors A and B. Thence, any algebraic relation between time-independent matrices remains valid if in both members of the equation all

matrices are replaced by the corresponding time-dependent matrices (1.19). Also by (1.6)

$$A(t)^{\dagger} = A^{\dagger} \exp(E_{RL}\, t/i\hbar) \equiv A^{\dagger}(t). \tag{1.21}$$

Obviously, all matrices of the form $\delta A \,\|$ are time-independent in interaction representation; that is

$$\delta A(t) \,\| = \delta A \,\| = \text{constant in time.} \tag{1.22}$$

It is sometimes useful to calculate the value of a time-dependent matrix $A(t)$ at $t \to \pm\infty$. If A has no singularities for $E_{RL} \to 0$, we may reason that $A(\pm\infty)$ vanishes, unless we prefer to maintain that $A(\pm\infty)$ has no well-defined value as $A(t)$ remains oscillatory. If the matrix considered is of the form $\S\, A(t) \,\|$, the result depends on the order of sequence of the limits $a \to 0$ and $t \to \pm\infty$. In this section, we consider the case that the limit $a \to 0$ is taken first. Then:

$$\left.\begin{aligned} &\lim_{t \to \pm\infty} \int dE_{RL}\, f(E_{RL})\, \S\, A(t) \,\| \\ &= \lim_{t \to \pm\infty} \lim_{a \to 0} \int dE_{RL}\, \frac{f(E_{RL})}{E_{RL} + ia}\, A \exp(E_{RL}\, t/i\hbar) \\ &= \lim_{t \to \pm\infty} \lim_{a \to 0} \int_{-\infty}^{\infty} \frac{dx}{x + i\,|\,t\,|/\hbar}\, f(a\hbar x/|\,t\,|)\, A \exp(\pm ax/i). \end{aligned}\right\} \tag{1.23}$$

Before we take the limits any further, we perform the integration by closing the contour through $x = \mp i\infty$, where $\exp(\pm ax/i)$ vanishes. The pole at $x = -i\,|\,t\,|/\hbar$ is enclosed by the contour in the case of the upper signs only, so that

$$\lim_{t \to -\infty} \int dE_{RL}\, f(E_{RL})\, \S\, A(t) \,\| = 0, \tag{1.24}$$

$$\left.\begin{aligned} &\lim_{t\to+\infty}\int dE_{RL}\, f(E_{RL})\,\S A(t)\,\| \\ &= \lim_{t\to+\infty}\lim_{a\to 0}\left\{-2\pi i f(-ia)\,A\,\exp(-at/\hbar)\right\} \\ &= -2\pi i\, f(0)\,A\,|_{E_{RL}=0} = -2\pi i\int dE_{RL}\,\delta(E_{RL})f(E_{RL})\,A. \end{aligned}\right\} \quad (1.25)$$

As this is true for arbitrary functions $f(E_{RL})$, we conclude that

$$\S A(t\to-\infty)\,\| = 0; \quad (1.26)$$

$$\S\ A(t\to+\infty)\,\| = 2\,\delta A\,\|, \qquad (at \ll 1). \quad (1.27)$$

Similarly we show

$$\flat\, A(t\to-\infty)\,\| = -2\,\delta A\,\|, \qquad (a\,|\,t\,| \ll 1); \quad (1.28)$$

$$\flat\, A(t\to+\infty)\,\| = 0. \quad (1.29)$$

By (1.9) or

$$\left.\begin{aligned} \delta A\,\| &= {}^1\!/_2\,(\S\, A\,\| - \flat\, A\,\|), \\ {}'A\,\| &= {}^1\!/_2\,(\S\, A\,\| + \flat\, A\,\|), \end{aligned}\right\} \quad (1.30)$$

Eqs. (1.26)—(1.29) give the result

$$\delta A(t\to\pm\infty)\,\| = \delta A\,\|, \quad (1.31)$$

which is trivial on account of (1.22), and

$${}'A(t\to\pm\infty)\,\| = \pm\,\delta A\,\|, \qquad a\,|\,t\,| \ll 1. \quad (1.32)$$

Because of the ambiguity of the mathematical method used for arriving at the results (1.24)—(1.32), one should justify this method on physical grounds whenever these results are formally used. The main point in the above derivation is that the limit $t\to\pm\infty$ is taken last of all, while a in (1.25) had already been put equal to zero; otherwise, we would have found zero instead

of $\delta A \parallel$ in all right-hand members of (1.25)—(1.32). In other words, we have kept $a \mid t \mid$ small in the limits $t \to \pm \infty$ for arriving at the results (1.25)—(1.32). A justification or refutation of such procedure is possible only after we give a a physical meaning. Chapter IX deals with a different order of sequence of these limits.

II. Stationary states, scattering matrices, and Heisenberg's *S*-matrix.

In this chapter, we shall use the definition of HEISENBERG's *S*-matrix given earlier by one of us[4], but the quantities there denoted by W, U, and $\delta_+(-x)$ are represented here by the notation E, $-2\pi i F$, and

$$\delta_+(x) = \int_0^\infty \exp(ikx)\, dk/2\pi = {}^1/_2\,\delta(x) - P\left(\frac{1}{2\pi i x}\right). \qquad (2.1)$$

Here, P indicates that one should take the principal value in subsequent integrations over x. With this notation, and $\delta_-(x) = \delta(x) - \delta_+(x)$, the meaning of the scat and of the flat by (1.10)—(1.11) can be represented by

$$\S\, A \parallel = -2\pi i\, \delta_+(E_{RL})\, A; \qquad \flat\, A \parallel = 2\pi i\, \delta_-(E_{RL})\, A. \quad (2.2)$$

H will denote the total Hamiltonian of our system. It is the sum of the free particle energy E and the interaction V. In Schrödinger representation, these quantities are represented by matrices which for a closed system are time-independent, but the state vector (situation function, wave function) $\psi_S(t)$ is time-dependent and satisfies

$$i\hbar\, \partial \psi_S(t)/\partial t = H \psi_S(t) = (E + V)\, \psi_S(t). \qquad (2.3)$$

Let k again be a set of variables commuting with each other and with E. (For instance momenta, or occupation numbers). Starting from an arbitrary Schrödinger representation in q-space, we transform to interaction representation in k-space with wave function $\langle k \mid \psi(t)$ by the method of variation of constants (used also in time-dependent perturbation theory):

$$\psi_S(q, t) = \int \langle q \mid k \rangle \exp(Et/i\hbar)\, dk \langle k \mid \psi(t), \qquad (2.4)$$

where E is considered a function of the variables k. Substitution of (2.4) in (2.3) gives the Schrödinger equation of interaction representation

$$\left.\begin{aligned} & i\hbar\,(\partial/\partial t) \langle k \mid \psi(t) \\ & = \int \langle k \mid V \mid k' \rangle \exp[(E' - E)\, t/i\hbar]\; dk' \langle k' \mid \psi(t) \end{aligned}\right\} \qquad (2.5)$$

or, symbolically,

$$i\hbar\, \partial\, \psi(t)/\partial t = V(t)\, \psi(t) \qquad (2.6)$$

with $V(t)$ derived from V by the definition (1.19).

While in many cases it is useful to treat the system as if it were not closed and to have V itself depend on time (see Chapter IX), we shall now first assume that the system is closed and that the factor $\exp(E_{RL}t/i\hbar)$ of (1.19) constitutes the only time-dependence of $V(t)$. In that case, there are stationary states (labeled by n) for which

$$\psi_{Sn}(t) = \psi_{Sn} \exp(H_n t/i\hbar), \qquad (2.7)$$

so that, by (2.4),

$$\langle k \mid \psi_n(t) = \langle k \mid \psi \mid n \rangle \exp[(H_n - E)t/i\hbar)], \qquad (2.8)$$

where the time-independent coefficients $\langle k \mid \psi \mid n \rangle$ are the time-independent wave functions—eigenfunctions of H—in k-space.

In Dirac fashion we have inserted the label n in a "ket"; the matrix $\langle k \mid \psi \mid n \rangle$ has rows and columns labeled by different sets of variables; and the fact that there is a complete orthonormal set of eigenfunctions of H may be expressed by

$$\int \langle n \mid \psi^\dagger \mid k \rangle\, dk \langle k \mid \psi \mid n' \rangle = \langle n \mid n' \rangle, \qquad (2.9)$$
(orthonormality)

$$\int \langle k \mid \psi \mid n \rangle\, dn \langle n \mid \psi^\dagger \mid k' \rangle = \langle k \mid k' \rangle, \qquad (2.10)$$
(completeness)

or, symbolically, by

$$\psi^\dagger \psi = \mathbf{1}, \qquad \psi\psi^\dagger = 1. \tag{2.11}$$

We shall subdivide the time-independent wave functions into two groups:
(1) Those stationary state eigenfunctions which vanish rapidly for infinite separation of the particles in our system. Such eigenfunctions we call $\langle k \mid \psi_r$ or briefly ψ_r; they correspond to "bound states". They include states in which only part of the system is properly bound, while other particles are going off to infinity (see also Chapter IV).
(2) The remainder of the eigenfunctions $\langle k \mid \psi \mid n \rangle$, in general no longer a complete set of functions. They are the stationary states that are considered in such scattering problems, where all the particles are free after the scattering process. We shall call them "scattering states".

There are various methods for bringing some order in the scattering states. As asymptotically—for infinite separation of the particles—the scattering states satisfy the free-particle Schrödinger equation, one may first, crudely, represent them by a definite free-particle state labeled by the value k_0 of the variables in k-space. However, such free-particle state $\langle k \mid k_0 \rangle$ is of course no exact solution of the time-independent Schrödinger equation which, on account of (2.5) with (2.8), reads

$$(H_n - E) \langle k \mid \psi \mid n \rangle = \int \langle k \mid V \mid k' \rangle \, dk' \, \langle k' \mid \psi \mid n \rangle. \tag{2.12}$$

The scattering state with a plane wave part which asymptotically behaves like $\langle k \mid k_0 \rangle$ must correspond to a total energy $H_n = E_0$. If we succeed in further specifying the scattering state we have in mind, we use from then on k_0 as label of this scattering state. This further specification can be given in many different ways, each giving the scattering state labeled by k_0 a different meaning. Some of the most interesting possibilities are:
(a) By k_0 we denote a scattering state which asymptotically is a superposition of the free-particle state k_0 and of outgoing scattered waves. (By "outgoing" we mean that in xyz-space the scattered wave gives in a given direction (ϑ, φ) asymptotically

($r \to \infty$) for each scattered particle a probability density depending on the scattered wave in k-space for momentum of such particle in that same direction (ϑ, φ), but not depending on the scattered wave for momentum in the opposite direction).
(b) Or, by k_0 we might denote a scattering state which asymptotically is a superposition of the free state k_0 and of incoming additional waves.
(c) Or, by k_0 we could denote a scattering state which besides the free state k_0 asymptotically contains incoming and outgoing waves in some symmetric way.

The scattering state satisfying the description (a) we shall denote by $\langle k \mid \Psi \mid k_0 \rangle$, a scattering state of type (b) we shall call $\langle k \mid \Omega \mid k_0 \rangle$, and one of the kind described under (c) we shall call $\langle k \mid Q \mid k_0 \rangle$. Let $\langle k \mid Y \mid k_0 \rangle$ mean any of these three types of scattering states, specification still to be given. When k_0 takes all possible values, Ψ, Ω, Q, and Y become matrices in k-spaces, the scattering matrices. Ψ is identical with the "wave matrix" introduced in reference 4. Contrary to ψ, both labels are now values of the same set of variables k, but the scattering states do no longer form a complete orthonormal set and therefore Ψ, Ω, Q need not satisfy relations like (2.9)—(2.11); consequently, Ψ, Ω, Q in general are no unitary matrices. Still we can be sure that scattering states belonging to two different energy levels will be orthogonal.

On account of $H_n = E_0$, the Schrödinger equation (2.12) for the scattering states $\langle k \mid Y \mid k_0 \rangle$ may be written symbolically as

$$(E_R - E_L)\, Y = VY. \tag{2.13}$$

Since $\langle k \mid k_0 \rangle$—the unit matrix 1 in k-space—forms an essential part of the scattering states, we write (see Appendix C)

$$Y = 1 + Z, \tag{2.14}$$

so that, by

$$(E_R - E_L) \langle k_L \mid k_R \rangle = 0 \tag{2.15}$$

(2.13) gives

$$(E_R - E_L)\, Z = VY = V + VZ. \tag{2.16}$$

Further, we define

$$B \equiv VY, \tag{2.17}$$

so that the Schrödinger equation is reduced to two equations for two unknown matrices B and Z:

$$(E_R - E_L)\, Z = B; \tag{2.18}$$

$$V + VZ = B. \tag{2.19}$$

While elimination of B leads back to (2.16) and (2.13), we could eliminate Z by solving for it from (2.18). This gives

$$Z = ?\, B \,\|. \tag{2.20}$$

The uncertainty in the meaning of Y at the beginning is here expressed by the uncertainty of the meaning of the slash in (2.20).

The question therefore arises, which meaning of the slash in (2.20) corresponds to each of the interpretations (a), (b) or (c) of the set of scattering states $\langle k \,|\, Y \,|\, k_0 \rangle$. The answer is well known and is independent of the possibility of expansion. It can be shown[9), 4)] that

$$\S B \,\| = -2\,\pi\, i\, \delta_+ (E_{RL})\, B = \left\{ P(1/E_{RL}) - \pi\, i\, \delta (E_{RL}) \right\} B \tag{2.21}$$

is the only expression of the form (2.20) corresponding to outgoing waves only.

$$\flat\, B \,\| = \left\{ P\,(1/E_{RL}) + \pi\, i\, \delta\,(E_{RL}) \right\} B \tag{2.22}$$

is the only expression of the form (2.20) corresponding to incoming waves only, and both $'B \,\|$ and $\delta B \,\|$ describe waves half incoming and half outgoing, but of these two only $'B \,\|$ has the form (2.20).

The general validity of the above statement, that a combination of the free particle state k_0 with outgoing waves only is the solution of (2.19)—(2.20) with $? = \S$, will become much clearer by the time-dependent methods discussed in Chapters V and VI.

We shall now separately discuss the three cases (a), (b), and (c). In case (a), we shall denote Y, Z, and B by Ψ, T, and F, respectively. In case (b), we shall denote them by Ω, Γ, and G. In case (c), we shall denote them by Q, P, and W. Thus, Eqs. (2.14), (2.20) and (2.17), (2.19) will in these three cases be read as

$$\Psi = 1 + T\,; \quad \Omega = 1 + \Gamma\,; \quad Q = 1 + P; \tag{2.23a-c}$$

$$T = \S\, F \| \quad ; \quad \Gamma = \flat\, G \| \quad ; \quad P = {}'W \|; \tag{2.24a-c}$$

$$\left.\begin{aligned} F = V\Psi = V + VT; \quad G = V\Omega = V + V\Gamma; \\ W = VQ = V + VP. \end{aligned}\right\} \tag{2.25a-c}$$

Substituting (2.20) with (2.17) into (2.14) we also find

$$Y = 1 + ?\, VY \|, \tag{2.26}$$

thence,

$$\left.\begin{aligned} \Psi = 1 + \S\, V\Psi \| \; ; \quad \Omega = 1 + \flat\, V\Omega \|; \\ Q = 1 + {}'VQ \|. \end{aligned}\right\} \tag{2.27a-c}$$

Our definitions (2.23a)—(2.25a) of Ψ, T, and F are equivalent with Eqs. (10), (11) and (15) in reference 4, where we have defined HEISENBERG's characteristic S-matrix by Eqs. (23) and (26). In our present notation, these equations read

$$R = 2\,\delta F \| \; ; \quad S = 1 + R. \tag{2.28}$$

Besides the quantity R expressible in terms of the wave matrix Ψ, we define an analogous quantity $-iK$, expressible in terms of the matrix Q, by

$$K = 2\,i\,\delta W \|. \tag{2.29}$$

Since $V = V^\dagger$ is Hermitian, (2.17) and the Hermitian conjugate of (2.17) read

$$B = VY, \; B^\dagger = Y^\dagger V. \tag{2.30}$$

Hence, for any of our interpretations of the slash we get

$$Y^\dagger B - B^\dagger Y = 0, \tag{2.31}$$

and for any pair of interpretations

$$Y_1^\dagger B_2 - B_1^\dagger Y_2 = 0. \tag{2.32}$$

From the three equations (2.31), which, by (2.23)—(2.25), may be written

$$\left.\begin{aligned} F - F^\dagger + T^\dagger F - F^\dagger T = 0, &\quad \text{a)} \\ G - G^\dagger + \Gamma^\dagger G - G^\dagger \Gamma = 0, &\quad \text{b)} \\ W - W^\dagger + P^\dagger W - W^\dagger P = 0, &\quad \text{c)} \end{aligned}\right\} \tag{2.33}$$

one easily derives the following equations

$$\left.\begin{aligned} \Psi^\dagger \Psi &= 1, \quad \text{a)} \\ \Omega^\dagger \Omega &= 1, \quad \text{b)} \\ Q^\dagger Q &= 1 + \left(\frac{K}{2}\right)^2. \quad \text{c)} \end{aligned}\right\} \tag{2.34}$$

We shall give a detailed proof of the last equation only, the proof of the two other equations (2.34) running in the same way. First we get from (2.33c), by multiplication with $2\pi\delta(E_{RL})$,

$$2\,i_\delta W\| - 2\,i_\delta W^\dagger\| + 2\,i\| W^{\dagger\prime} W_\delta - 2\,i_\delta W^{\dagger\prime} W\| = 0, \tag{2.35}$$

where we have used (2.24c) and one of the rules (1.7). Further, on account of these rules, the two last terms on the left-hand side of this equation cancel and ${}_\delta W^\dagger\| = -({}_\delta W\|)^\dagger$, so that (2.35) gives

$$K = K^\dagger, \tag{2.36}$$

which shows that K as defined by (2.29) is Hermitian. Further, multiplying (2.33c) by $D_{RL} = -D_{LR}$ we get, by means of (2.24c) and (1.7),

$${}'W\| - {}'W^\dagger\| - \| W^{\dagger\prime} W' - {}'W^\dagger\, {}'W\| = 0. \tag{2.37}$$

Using the fundamental relation (1.18), the third and fourth term in this equation give

$$\left.\begin{aligned} &{}'W^{\dagger} \,\|\, W' + {}_{\delta}W^{\dagger} \,\|\, W_{\delta} \\ &= ({}'\|\, W^{\dagger\prime}) \cdot ({}'W\|) + (\|\, W^{\dagger}{}_{\delta}) \cdot ({}_{\delta}W\|) = P^{\dagger} P - \left(\frac{K}{2}\right)^2, \end{aligned}\right\} \tag{2.38}$$

where we have used the definitions (2.24c) and (2.29) of P and K and the general rules (1.7) for taking the Hermitian conjugates of these quantities.

Thence, (2.37) gives

$$P + P^{\dagger} + P^{\dagger} P - K^2/4 = 0$$

or

$$Q^{\dagger} Q = (1 + P^{\dagger})(1 + P) = 1 + K^2/4,$$

i. e. (2.34c). The equations (2.34, a, b) may be proved along similar lines by multiplying (2.33, a, b) by $(E_{RL} + i\alpha)^{-1} = -(E_{LR} - i\alpha)^{-1}$ and using the important relation

$$\|A \flat B \flat + \S A \,\|\, B \flat + \S B\| = 0, \tag{1.18a}$$

which follows from (1.18) by (1.9) and (1.7).

The equation (2.34a) is identical with Eq. (62) in reference 4, but the proof given there was not quite satisfactory.

Like (2.9) or the first equation (2.11), Eq. (2.34 a) expresses the orthonormality of the scattering states $\langle k \,|\, \Psi \,|\, k_0 \rangle$; but the unit matrix $1 = \langle k_L \,|\, k_R \rangle$ occurring in this equation is only a submatrix of the unit matrix $\mathbf{1} = \langle n_L \,|\, n_R \rangle$ of (2.9)—(2.11) and one has to add to the stationary states $\langle k \,|\, \Psi \,|\, k_0 \rangle$ the bound states $\langle k \,|\, \psi_r$ in order to form a complete set of functions. (This under the supposition that the $\langle k \,|\, \Psi \,|\, k_0 \rangle$ form at least a complete set of scattering states). Then, Eq. (2.10) may be written as

$$\left.\begin{aligned} &\int \langle k \,|\, \Psi \,|\, k_0 \rangle \, dk_0 \langle k_0 \,|\, \Psi^{\dagger} \,|\, k' \rangle + \sum_r \langle k \,|\, \psi_r) \cdot (\psi_r^{\dagger} \,|\, k' \rangle \\ &\qquad = \langle k \,|\, k' \rangle \end{aligned}\right\} \tag{2.39}$$

or, symbolically[4],

$$\Psi\Psi^\dagger = 1 - \sum_r \psi_r \cdot \psi_r^\dagger, \tag{2.40}$$

where we have introduced the bound state matrices ψ_r, $\psi_r^\dagger$ with only one column and one row, respectively. This shows that Ψ cannot be unitary if there are bound states.

For the three different pairs of interpretations of the slash, the equations (2.32) take the form

$$\left.\begin{aligned} &\Psi^\dagger G - F^\dagger \Omega = 0, \text{ i. e. } G - F^\dagger + T^\dagger G - F^\dagger \Gamma = 0, && \alpha) \\ &\Psi^\dagger W - F^\dagger Q = 0, \text{ i. e. } W - F^\dagger + T^\dagger W - F^\dagger P = 0, && \beta) \\ &\Omega^\dagger W - G^\dagger Q = 0, \text{ i. e. } W - G^\dagger + \Gamma^\dagger W - G^\dagger P = 0. && \gamma) \end{aligned}\right\} \tag{2.41}$$

By multiplying (2.41, β) by $-2\pi i \delta(E_{RL})$, we find

$$\begin{aligned} 0 &= 2\delta W \| - 2\delta F^\dagger \| + 2 \| F^\dagger \flat W_\delta - 2\delta F^\dagger {}' W \| \\ &= 2\delta W \| + 2(\delta F \|)^\dagger + 2\delta F^\dagger (\flat - ') W \| \\ &= -iK + R^\dagger - \frac{1}{2} iR^\dagger K, \end{aligned}$$

where we have used the definitions of T, P, K and R along with the rules (1.7) and (1.9). The Hermitian conjugate of this equation is

$$\left(1 + \frac{1}{2} iK\right) R = -iK. \tag{2.42}$$

In this way, we have obtained Heitler's integral equation [7) 8)] without the use of series expansions.

Since K is Hermitian, all the eigenvalues of the matrix $1 + \frac{iK}{2}$ are different from zero. Thus, this matrix has a reciprocal $\left(1 + \frac{iK}{2}\right)^{-1}$, and (2.42) gives

$$R = \frac{-iK}{1 + \frac{iK}{2}}. \tag{2.43}$$

For the S-matrix (2.28) we thus get

$$S = 1 + R = \frac{1 - iK/2}{1 + iK/2} \tag{2.44}$$

in terms of the "reaction operator" K (comp. Eq. (1.34) of reference 1).

In this form, the unitarity of the S-matrix is obvious, viz.

$$S^\dagger S = SS^\dagger = 1. \tag{2.45}$$

Multiplying (2.41 α) by $-2\pi i\,\delta(E_{RL})$ we find that the two last terms in the left-hand member cancel and we get

$$R_- \equiv 2\,\delta G\| = -2\,(\delta F\|)^\dagger = -R^\dagger. \tag{2.46}$$

Multiplication of the Eqs. (2.33 a, b) and (2.41 γ) by $-\pi i\,\delta(E_{RL})$ does not lead to new information. In fact, the equations obtained in this way are, on account of Eq. (2.46), equivalent with the equations (2.45) and (2.43).

However, by multiplying the Eqs. (2.41) by $(E_{RL} + ia)^{-1} = -(E_{LR} - ia)^{-1}$, we get by (1.18), by a similar procedure as that used in deriving (2.34) from (2.33),

$$\left.\begin{aligned} &\Psi^\dagger \Omega = S^\dagger, && \alpha) \\ &\Psi^\dagger Q = 1 + \frac{iK}{2}, && \beta) \\ &\Omega^\dagger Q = 1 - \frac{iK}{2}. && \gamma) \end{aligned}\right\} \tag{2.47}$$

These equations determine the connection between the three different types of scattering states Y denoted by Ψ, Ω, and Q. We have assumed above that the set of functions $\langle k \mid \Psi \mid k_0 \rangle$ forms a complete set of *scattering* states. We shall make the same assumption about the set of functions $\langle k \mid Y \mid k_0 \rangle$. Whenever these assumptions are justified, any function expressible as a superposition of scattering states $\langle k \mid \Psi \mid k_0 \rangle$ can also be expressed as a superposition of scattering states $\langle k \mid \Omega \mid k_0 \rangle$ *or*

$\langle k \mid Q \mid k_0 \rangle$, and vice versa. Therefore, matrices X_1 and X_2 must exist, such that

$$\Omega = \Psi X_1, \quad Q = \Psi X_2.$$

From (2.47) and (2.34 a) we then get

$$X_1 = S^\dagger, \quad X_2 = 1 + \frac{iK}{2},$$

i. e. (see Appendix D)

$$\Psi = \Omega S = Q\left(1 + \frac{iK}{2}\right)^{-1}, \tag{2.48}$$

where we have used the unitarity of the S-matrix. With (2.48) the equations (2.47), (2.34, b, c) are easily seen to be consequences of the Eq. $\Psi^\dagger \Psi = 1$. From (2.48) we see that the scattering state matrix Ψ, corresponding to outgoing waves only, is obtained from the matrix Ω corresponding to ingoing waves by multiplication on the right with the S-matrix.

If we define a matrix δ by

$$K = -2 \tan \delta, \tag{2.49}$$

we have, on account of (2.44),

$$S = \frac{1 + i \tan \delta}{1 - i \tan \delta} = \frac{e^{i\delta}}{e^{-i\delta}} = e^{i2\delta} = e^{i\eta}, \tag{2.50}$$

where

$$\eta = 2\delta \tag{2.51}$$

is the Hermitian η-matrix introduced by HEISENBERG. In the simple case of a scattering of particles by a fixed potential in configuration space, the eigenvalues δ' of the matrix δ are the "phase shifts" of the scattered waves.[1) 2)]

III. Time-independent perturbation treatment.

We shall now try to solve the time-independent Schrödinger equation (2.13) or (2.19)—(2.20) for the scattering states by a series expansion in the potential V, starting from $Y \approx 1$. Series expansions of any quantity like Y, Z, B, etc. will be denoted by a subscript asterisk, i. e. by Y_*, Z_*, B_*, etc. It is seen by inspection that the series

$$\left.\begin{aligned} B_* &= \sum_{n=1}^{\infty} V(?V)^{n-1}\,\| = \sum_{n=1}^{\infty} V\underbrace{(?V?V?V\cdots?V)}_{n-1 \text{ factors}}\,\| \\ Z_* &= \sum_{n=1}^{\infty} (?V)^n\,\| = \sum_{n=1}^{\infty} \underbrace{?V?V\cdots?V}_{n \text{ factors}}\,\| \end{aligned}\right\} \tag{3.1}$$

solve the equations (2.19)—(2.20) whenever these expansions converge; and if one assumes that it should be possible to find B and Z from (2.19)—(2.20) by successive approximations starting from $Z \approx 0$, it is easily seen that (3.1) is the only solution. Thus, if the series converge,

$$Y_* = 1 + Z_* = \sum_{n=0}^{\infty} (?V)^n\,\| \tag{3.2}$$

is the solution of (2.26) and represents a scattering state. With the three different meanings of the slash, we thus get for the scattering states (a), (b), and (c)

$$\Psi_* = \sum_{n=0}^{\infty} (\S V)^n\,\|;\ T_* = \sum_{n=1}^{\infty} (\S V)^n\,\|;\ F_* = \sum_{n=1}^{\infty} V(\S V)^{n-1}\,\|, \tag{3.3a}$$

$$\Omega_* = \sum_{n=0}^{\infty} (\flat V)^n\,\|;\ \Gamma_* = \sum_{n=1}^{\infty} (\flat V)^n\,\|;\ G_* = \sum_{n=1}^{\infty} V(\flat V)^{n-1}\,\|, \tag{3.3b}$$

$$Q_* = \sum_{n=0}^{\infty} ('V)^n\,\|;\ P_* = \sum_{n=1}^{\infty} ('V)^n\,\|;\ W_* = \sum_{n=1}^{\infty} V('V)^{n-1}\,\|. \tag{3.3c}$$

For the matrices R, K and S, defined by (2.28)—(2.29), we then get the following series expansions:

$$R_* = 2\sum_{n=1}^{\infty}{}_{\delta} V(\S V)^{n-1}\|, \tag{3.4}$$

$$K_* = 2i\sum_{n=1}^{\infty}{}_{\delta} V('V)^{n-1}\|, \tag{3.5}$$

$$S_* = 1 + 2\sum_{n=1}^{\infty}{}_{\delta} V(\S V)^{n-1}\|. \tag{3.6}$$

The expression (3.5) for K_* shows directly that K_* is Hermitian in accordance with (2.36). In fact, we have, according to the general rules (1.7)—(1.8),

$$\left.\begin{aligned} K_*^\dagger &= 2i\sum_{n=1}^{\infty}\|('V)^{n-1}V_\delta = 2i\sum_{n=1}^{\infty}{}_{\delta}('V)^{n-1}V\| \\ &= 2i\sum_{n=1}^{\infty}{}_{\delta} V('V)^{n-1}\| = K_*. \end{aligned}\right\} \tag{3.7}$$

By direct multiplication of the series it is easily seen that the orthogonality conditions (2.34) are satisfied by the expressions (3.3) (see Appendix B). It would now be interesting also to calculate $\Psi_*\Psi_*^\dagger$ in order to get some information about the bound states through (2.40). Since

$$\Psi_* = \sum_{n=0}^{\infty}(\S V)^n\|; \quad \Psi_*^\dagger = \sum_{n=0}^{\infty}\|(V\flat)^n, \tag{3.7}$$

we get by multiplication of the series

$$\left.\begin{aligned} \Psi_*\Psi_*^\dagger &= \sum_{i,j=0}^{\infty}(\S V)^i\|(V\flat)^j \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(\S V)^l\|(V\flat)^{n-l}\right) \equiv \sum_{n=0}^{\infty}A_n. \end{aligned}\right\} \tag{3.8}$$

From the definitions of scat and flat we get

$$\left.\begin{aligned} &A_0 = 1; \qquad A_1 = \S V\| + \|V\flat = 0; \qquad \langle k_0|A_n|k_n\rangle \\ &= \int\langle k_0|V|k_1\rangle dk_1\langle k_1|V|k_2\rangle\cdots dk_{n-1}\langle k_{n-1}|V|k_n\rangle B(n), \end{aligned}\right\} \tag{3.9}$$

$$B(n) = \sum_{l=0}^{n} \prod_{i=0}^{l-1} (E_l - E_i + ia)^{-1} \prod_{j=l+1}^{n} (E_l - E_j - ia)^{-1} \text{ for } n \geqslant 2, \quad (3.10)$$

where $\prod_{i=0}^{-1} (E_0 - E_i + ia)^{-1}$ and $\prod_{j=n+1}^{n} (E_n - E_j - ia)^{-1}$ both mean 1.

In Appendix B, it is shown that all $B(n)$ contain a as a factor, so that $A_n \to 0$ when the limit $a \to 0$ is taken after the integration in (3.9). Hence,

$$\left.\begin{aligned} A_n &= 0 \quad \text{for} \quad n \geqslant 1 \\ \text{and} \qquad \Psi_* \Psi_*^\dagger &= 1, \end{aligned}\right\} \quad (3.11)$$

i. e. Ψ_* has $\Psi_*^\dagger$ as its inverse, and therefore is unitary.

By comparison with the exact equation (2.40) this is seen to be possible only if the system has no bound states. We therefore have come to the interesting conclusion *that the series expansions (3.3) of the scattering state matrices must diverge whenever the system has bound states,* even if the coupling constants entering in V are small. This, of course, does not mean that the series expansion of every scattering wave function $\langle k \mid \Psi \mid k_0 \rangle$ will diverge for a system with bound states; it only means that the series $\langle k \mid \Psi_* \mid k_0 \rangle$ cannot be convergent for all k_0. It also does not necessarily mean that the expansion (3.6) of the S-matrix diverges, since S is unitary also when the system has bound states. Our result merely shows that no information about possible bound states of the system can be obtained by a perturbation treatment.

If the series converge, the Eqs. (2.48) are easily verified by direct multiplication of the series. Take, for instance, the equation

$$\Psi_* = Q_* \left(1 + \frac{iK_*}{2}\right)^{-1} = Q_* \sum_{n=0}^{\infty} \left(\frac{K_*}{2i}\right)^n. \quad (3.12)$$

By (3.5) and by use of the rules (1.7), (1.9) the right-hand member of (3.12) is

$$\sum_{n=0}^{\infty} \left[\sum_{i=0}^{\infty} ('V)^i\right] \left[\delta V \sum_{j=0}^{\infty} ('V)^j\right]^n \Big\|, \quad (3.13)$$

while the left-hand member may be written

$$\Psi_* = \sum_{p=0}^{\infty} ('V + \delta V)^p \| . \tag{3.14}$$

The equality of (3.13) and (3.14) is obvious if we may change orders of sequence of summations, as both (3.13) and (3.14) represent summations over terms containing arbitrary numbers of $('V)$-factors, with arbitrary numbers of (δV)-factors interspersed at arbitrary positions.

Finally note that, if the expansion in (3.12) converges, not only Ψ_* has an inverse $\Psi_*^\dagger$, but also Q_* has an inverse

$$Q_*^{-1} = \sum_{n=0}^{\infty} \left(-\frac{1}{2} iK_*\right)^n \Psi_*^\dagger . \tag{3.15}$$

Also, by (2.45), (2.48), and (2.34a) with (3.11), $\Omega_*^\dagger = S_* \Psi_*^\dagger$ is then the inverse of $\Omega_* = \Psi_* S_*$.

IV. Interpretation of the scattering matrix Ψ and of the characteristic matrix S.

The scattering state $\langle k | \Psi | k_0 \rangle$ obviously can be interpreted as representing incident particles in free-particle state k_0, and scattered particles described by the asymptotic behaviour of the (outgoing) wave $\langle k | T | k_0 \rangle$. (Absorption from the incident beam is described by interference of incident and scattered waves). Values for differential cross sections follow directly from such interpretation. (See section 2 of reference 4). They are found by calculating the value of the probability density for one of the scattered particles for large radial distance in a given direction in xyz-space, and are found to be proportional to the absolute square of the matrix element of the "effective scattering potential" F for a transition to a final state, in which the momentum of the particle considered is directed in the direction into which the scattering probability of that particle was to be calculated and has the magnitude corresponding to energy conservation.

Since

$$F\big|_{E_{RL}=0} = \frac{i}{2\pi}\int_{-}^{+}(S-1)\,dE_{RL}, \tag{4.1}$$

the cross sections for processes in which all the particles are free after the collision are uniquely determined by the matrix elements of the S-matrix. It should be noted, however, that the S-matrix and the wave matrix Ψ defined in II and in references 2 and 4 do *not* account for collision processes in which part of a system is in a bound state before and after the collision. This follows at once from the fact that the function

$$\langle x\,|\,\Psi\,|\,k_0\rangle = \int\langle x\,|\,k\rangle\,dk\,\langle k\,|\,\Psi\,|\,k_0\rangle$$

in configuration space represents plane waves superimposed by outgoing waves for *all* the particles. Such processes must therefore be described by state functions ψ_r which belong to the group of states classified as "bound states" in II.

On the other hand, it is clear that we could have started from a different division of the total Hamiltonian in (2.3). We could, for instance, let E denote the total Hamiltonian of a part of the total system plus the free particle energy of the rest of the system, while V is the rest of the potential not included in E. If k now is a set of variables commuting with each other and with this new operator E, we could formally proceed in the same way as in the preceding chapters, using a kind of "partial" interaction representation with the Schrödinger equation (2.6), but with a different interpretation of the quantities occurring in the equation. Also the division into "scattering" states and bound states would be different in such a treatment, and we would arrive at an S-matrix or a "collision" matrix which in general is not simply a different representation of the Heisenberg S-matrix. Such a procedure is often used in the treatment of collisions between elementary particles and atomic nuclei. To each division of the total Hamiltonian in (2.3) we get in this way a corresponding S-matrix, and the connection between these collision matrices is not always simple. Throughout this paper, we shall explicitly consider only the case where E is the free particle energy, but

our considerations can easily be extended to the case where E includes part of the interaction.

Since

$$S = 1 + 2\delta F\| = 1 + \S F\| - \flat F\| = 1 + R, \qquad (4.2)$$

$$\Psi = 1 + \S F\| = 1 + T, \qquad (4.3)$$

we get

$$R = T - \flat F\|. \qquad (4.4)$$

While $T = \S F\|$ in xyz-space asymptotically represents outgoing waves only[2),4)] $\flat F\|$ represents incoming waves only. Hence, while $\langle k | \Psi | k_0 \rangle$ gives a correct direct picture of the scattering phenomenon as it represents a superposition of a plane wave with spherical outgoing waves only, on the other hand $\langle k | S | k_0 \rangle$ represents in k-space a superposition of a plane wave with an incoming and an outgoing spherical wave. This gives rise to a paradox to which we shall come back in the discussion of the time-dependent wave function for $t \to \infty$ (see Chapter VII).

V. Time-dependent scattering theory and Dyson's S-matrix.

Use of time-dependent methods in scattering theory is based on a simple idea: "If very long ago (formally: "in the infinite past, at $t = -\infty$") there were only particles in the initial free-particle state k_0, then by now this non-stationary state will have developed into the corresponding stationary scattering state". Often one adds to this the remark that, if one for establishment of this scattering state waits from $t = -\infty$ to $t =$ finite, one may as well wait till $t = +\infty$ and consider scattering as a process taking place between $t = -\infty$ and $t = +\infty$.

We start by solving the Schrödinger equation (2.6) for the wave-function $\psi(t)$ in interaction representation systematically. As boundary condition, let $\psi(t_0)$ be given. The linear relation between $\psi(t)$ and $\psi(t_0)$ we express by means of the "propagation matrix" $U(t, t_0)$ as follows:

$$\langle k \mid \psi(t) = \int \langle k \mid U(t, t_0) \mid k' \rangle \, dk' \, \langle k' \mid \psi(t_0) \qquad (5.1)$$

or, briefly,

$$\psi(t) = U(t, t_0)\,\psi(t_0). \qquad (5.2)$$

As we assume that the Schrödinger equation (2.6) may be used to find $\psi(t + dt)$ from $\psi(t)$ for $dt < 0$ as well as for $dt > 0$,* we may also interchange t and t_0 in (5.2). Thence,

$$\psi(t) = U(t, t_0)\, U(t_0, t)\,\psi(t), \qquad (5.3\,a)$$

$$U(t_0, t) = U(t, t_0)^{-1}, \qquad (5.3\,b)$$

that is, the propagation matrix U has a reciprocal.

Substituting (5.2) into (2.6) we find

$$\left.\begin{aligned} i\hbar\,\partial\, U(t, t_0)/\partial\, t &= V(t)\, U(t, t_0) \\ &= \{ V \exp(E_{RL} t / i\hbar \} \, U(t, t_0), \end{aligned}\right\} \qquad (5.4)$$

where $V = V^{\dagger}$, and by (1.21)

$$V(t)^{\dagger} = V(t).$$

Let $\langle k \mid \psi(t) \mid n \rangle$ or, briefly, $\psi_n(t)$ form a set of solutions of the Schrödinger equation (2.6), complete and orthonormal at a given time t_0; that is, the relations

$$\left.\begin{aligned} &\int \langle n' \mid \psi^{\dagger}(t) \mid k \rangle \, dk \, \langle k \mid \psi(t) \mid n \rangle = \langle n' \mid n \rangle \\ \text{or} \qquad &\psi_{n'}^{\dagger}(t)\,\psi_n(t) = \langle n' \mid n \rangle \quad \text{or} \quad \psi^{\dagger}(t)\,\psi(t) = \mathbf{1} \end{aligned}\right\} \qquad (5.5)$$

and

$$\sum_n \langle k \mid \psi(t) \mid n \rangle \langle n \mid \psi^{\dagger}(t) \mid k' \rangle = \langle k \mid k' \rangle$$

or

$$\sum_n \psi_n(t)\,\psi_n^{\dagger}(t) = 1 \quad \text{or} \quad \psi(t)\,\psi^{\dagger}(t) = 1 \qquad (5.6)$$

* In "integrocausal" theories, such assumption may be dropped[10].

are valid at $t = t_0$. We shall now first prove conservation of orthonormality (5.5) even if V in (2.3) in Schrödinger representation depends on time, as for a non-closed system.

Eq. (5.5) can in symbolic notation be written by (5.2) as

$$\psi(t_0)^\dagger U(t, t_0)^\dagger U(t, t_0) \psi(t_0) = \mathbf{1}. \tag{5.7}$$

Now, by (5.2),

$$U(t_0, t_0) = 1 = U(t_0, t_0)^\dagger, \tag{5.8}$$

so that the equation

$$U(t, t_0)^\dagger U(t, t_0) = 1 \tag{5.9}$$

is trivial for $t = t_0$. As t becomes different from t_0, Eq. (5.9) remains valid, because

$$\left.\begin{aligned} & i\hbar(\partial/\partial t)\{U(t, t_0)^\dagger U(t, t_0)\} \\ = {} & U(t, t_0)^\dagger V(t) U(t, t_0) - \{V(t)\ U(t, t_0)\}^\dagger U(t, t_0) = 0. \end{aligned}\right\} \tag{5.10}$$

Thence, the left hand members of (5.5) and (5.7) equal $\psi(t_0)^\dagger \psi(t_0)$, which was given to be equal to $\mathbf{1} = \langle n' \mid n \rangle$.

We shall now prove that also (5.6) remains valid as t becomes different from t_0. Indeed, the left-hand member of (5.6) depends on time according to

$$\left.\begin{aligned} & i\hbar(\partial/\partial t)\{\psi(t)\psi(t)^\dagger\} \\ = {} & V(t)\psi(t)\psi(t)^\dagger - \psi(t)\psi(t)^\dagger V(t)^\dagger, \end{aligned}\right\} \tag{5.11}$$

so, whenever $\psi(t)\psi(t)^\dagger = 1$, we find its time derivative to be equal to $\{V(t) - V(t)^\dagger\}/i\hbar = 0$. Therefore, $\psi(t)\psi(t)^\dagger$, once equal to the unit matrix, will always remain the unit matrix, provided that $\psi(t)\psi^\dagger(t)$ is an analytic function of t.

Next, we shall show the unitarity of $U(t, t_0)$. Making use of the existence of a reciprocal (5.3b) of $U(t, t_0)$ we find, by (5.9),

$$UU^\dagger = UU^\dagger UU^{-1} = UU^{-1} = 1. \tag{5.12}$$

Thence, U is unitary, i. e.

$$U^\dagger = U^{-1} \tag{5.13}$$

is the reciprocal of U, and (5.3b) gives

$$U(t, t_0)^\dagger = U(t_0, t). \tag{5.14}$$

We shall now first consider the case of a closed system with V independent of time, so that $V(t)$ depends on time by the factor explicitly given in (1.19) only. Again we may distinguish stationary scattering states and bound states; in interaction representation, the latter depend on time by (2.6)—(2.8), or

$$i\hbar\,\partial\,\psi_r(t)/\partial\,t = (H_n - E_L)\,\psi_r(t) = V(t)\,\psi_r(t), \tag{5.15}$$

while the time-dependence of scattering states $Y(t)$ is given by (1.19), (2.6), (2.13):

$$i\hbar\,\partial\,Y(t)/\partial\,t = E_{RL}\,Y(t) = V(t)\,Y(t). \tag{5.16}$$

In Chapter II we assumed ψ_r and Ψ to form a complete orthonormal set of functions at $t = 0$; therefore, the solutions $\psi_r(t)$ and $\Psi(t)$ of (2.6) always form a complete orthonormal set, and Eqs. (5.5)—(5.6) may be written as

$$\left.\begin{aligned} &\Psi^\dagger(t)\,\Psi(t) = 1; && \Psi^\dagger(t)\,\psi_r(t) = 0, \\ &\psi_r^\dagger(t)\,\Psi(t) = 0; && \psi_{r'}^\dagger(t)\,\psi_r(t) = \delta_{r'r}, \end{aligned}\right\} \tag{5.17}$$

$$\Psi(t)\,\Psi^\dagger(t) + \sum_r \psi_r(t)\cdot\psi_r^\dagger(t) = \mathbf{1}, \tag{5.18}$$

where all matrices may now be taken time-dependent: $\psi_r(t)$ according to (2.8), and $\Psi(t)$ according to (1.19).

The Eqs. (5.4), (5.8) are now obviously solved by

$$\left.\begin{aligned} U(t, t_0) &= \Psi(t)\,\Psi^\dagger(t_0) + \sum_r \psi_r(t)\cdot\psi_r^\dagger(t_0) \\ &= \{\Psi \exp(E_{RL}\,t/i\hbar)\}\{\Psi^\dagger \exp(E_{RL}\,t_0/i\hbar)\} \\ &+ \sum_r \{\psi_r\cdot\psi_r^\dagger \exp[(E_R\,t_0 - H_r\,t_0 + H_r\,t - E_L\,t)/i\hbar]\}. \end{aligned}\right\} \tag{5.19}$$

Eq. (5.8) is satisfied on account of (5.18); Eq. (5.4) is satisfied on account of (5.15)—(5.16).

In fact, on account of Eqs. (5.17), Eq. (5.19) gives, for scattering states and for bound states, respectively,

$$\left.\begin{aligned} U(t, t_0)\,\Psi(t_0) &= \Psi(t), \qquad \text{a)} \\ U(t, t_0)\,\psi_r(t_0) &= \psi_r(t). \qquad \text{b)} \end{aligned}\right\} \tag{5.20}$$

We see that the propagation matrix $U(t, t_0)$ consists of two parts. The first part,

$$S(t, t_0) = \Psi(t)\,\Psi^{\dagger}(t_0), \tag{5.21}$$

changes a scattering state at time t_0 into one at time t, but destroys all bound-state admixtures in the wave function. The second part,

$$s(t, t_0) = \sum_r \psi_r(t)\,\psi_r^{\dagger}(t_0), \tag{5.22}$$

takes care of the propagation of bound-state wave functions.

From (1.22), applied to the matrices R, K, and S, which all contain a delt, we see that

$$R(t) = R, \quad K(t) = K, \quad S(t) = S \tag{5.23}$$

are constant in time. Hence, whenever (2.48) is valid (see Appendix D), Eqs. (5.19) and (5.21) may also be written

$$\left.\begin{aligned} U(t, t_0) &= S(t, t_0) + s(t, t_0), \qquad \text{a)} \\ S(t, t_0) &= \Omega(t)\,\Omega^{\dagger}(t_0) = Q(t)\left[1 + \left(\frac{K}{2}\right)^2\right]^{-1} Q^{\dagger}(t_0), \qquad \text{b)} \end{aligned}\right\} \tag{5.24}$$

where we have used the unitarity of S and the reality of K.

From (5.21), (5.24b), and (2.34a-c) it is seen that then also

$$S(t, t_0)\,\Psi(t_0) = \Psi(t); \quad S(t, t_0)\,\Omega(t_0) = \Omega(t);$$

$$S(t, t_0)\,Q(t_0) = Q(t),$$

so that $S(t, t_0)$ takes care of the propagation of the scattering states $\Omega(t)$ and $Q(t)$ as well as of the scattering states $\Psi(t)$.

We shall now consider the question of what becomes in course of time of a state given at $t_0 = -\infty$ and what happens to the wave function as $t \to +\infty$.

First, we shall formally define

$$U_+(t) = U(t, -\infty) = \lim_{t_0 \to -\infty} U(t, t_0), \tag{5.25a}$$

$$U_-(t) = U(t, +\infty) = \lim_{t_0 \to \infty} U(t, t_0); \tag{5.25b}$$

assuming that these limits exist.

Now, by (2.23a)—(2.24a) with (1.19) and (1.26), (1.27) with (2.28) we find

$$\text{“}\Psi(-\infty) = 1, \quad \Psi(+\infty) = 1 + 2\,{}_\delta F\| = S.\text{”} \tag{5.26}$$

Similarly, by (2.23b)—(2.24b), (1.28). (1.29), (2.46), and (2.28),

$$\text{“}\Omega(-\infty) = S^\dagger, \quad \Omega(+\infty) = 1.\text{”} \tag{5.27}$$

The second formula (5.26) and the first formula (5.27) require justification for the limit $t \to \pm\infty$ after the limit $a \to 0$, so that $a \mid t \mid$ remains small. This is justified, however, since we assumed no time-dependence of V at all, so that we need $a \to 0$ indeed to have (2.14)—(2.20) satisfy (2.13). Moreover, the first Eq. (5.27) follows from the first Eq. (5.26) and the second Eq. (5.26) follows from the second Eq. (5.27) by Eq. (2.48) whenever that one is valid.

As for the bound states, $\psi_r(\pm\infty)$ is somewhat meaningless. Formally, one may reason[5), 6)] that these expressions “vanish”:

$$\text{“}\psi_r(\pm\infty) = 0\text{”}. \tag{5.28}$$

(MA[6)] calls (5.28) a “conditional equality”.) If the convention (5.28) is accepted, we formally find, by (5.22), (5.24a), (5.25),

$$\text{“}s(t, \mp\infty) = 0\text{”}; \qquad \text{“}S(t, \mp\infty) = U_\pm(t).\text{”} \tag{5.29}$$

Thus, by (5.21) and (5.26),

$$\text{“}U_+(t) = \Psi(t)\,\Psi^\dagger(-\infty) = \Psi(t) = Q(t)\left(1 + \frac{iK}{2}\right)^{-1}\text{”} \tag{5.30a}$$

$$\left.\begin{aligned} &\text{“}U_{-}(t) = \Psi(t)\,\Psi^{\dagger}(+\infty) = \Psi(t)\,S^{-1} = \Omega(t) \\ &\qquad = Q(t)\left(1-\frac{1}{2}\,iK\right)^{-1}\text{”}, \end{aligned}\right\} \quad (5.30\,\mathrm{b})$$

where we have used (2.48), which holds also for the time-dependent quantities.

This should not be construed as to mean that, if at $t_0 = -\infty$ the wave function contains some bound state admixtures, then at finite time t such bound state should have died out; for, if $U(t)$ operates, say on $\psi_r(t_0 \to -\infty)$, the time dependence of the last factor $\psi_r^{\dagger}(t_0)$ in $U(t, t_0)$ apparently cancels the time dependence in the wave function $\psi_r(t_0)$ on which U operates, and the limit (5.28) should not be used. That is, while on account of conditional equalities we have “$\psi_r(-\infty) = 0$” as well as “$\psi_r^{\dagger}(-\infty) = 0$”, yet $\lim_{t\to-\infty} \psi_r(t)^{\dagger}\,\psi_r(t) \neq 0$. Therefore, use of (5.28)—(5.30) should always be made with caution.

Dyson[3] defines the S-matrix as $U_{+}(+\infty)$. In fact, (5.30 a) with (5.26) and (5.29) gives

$$\text{“}S(\infty, -\infty) = U_{+}(\infty) = \Psi(\infty) = S\text{”}, \qquad (5.31)$$

so that Dyson's definition is in agreement with the definition of Heisenberg's characteristic matrix in references 2 and 4, as far as (5.28)—(5.30) are justified, that is, as long as this matrix is applied to scattering states only.

In this connection, it should be noted that the limits (5.26) too are only “conditional equalities”; in fact, their validity is closely related to (5.28). For, if (5.26) is true, then $\langle k \,|\, \Psi(-\infty) \,|\, k_0\rangle = \langle k \,|\, k_0\rangle$, that is, at $t = -\infty$, the scattering functions in k-space (labeled by k_0) form by themselves a complete set of functions (the free-particle states k_0); and, since all bound states were to be orthogonal to all scattering states, there could be no bound states at $t = -\infty$. That is, our possibility of writing (5.26) is based on granting the validity of (5.28)—(5.29).

On the other hand, when we meet the necessity of considering wave functions with bound state admixtures, so that we want to drop Eqs. (5.28)—(5.29) for them, then we should also drop Eq. (5.26), thence Eqs. (5.30)—(5.31). In that case, we have a

choice between either completely abandoning the use of interaction representation and using Heisenberg representation, where there are no such difficulties, or we shall have to settle on use of the more complicated Eq. (5.19) instead of the idealization which is the S-matrix, for predicting future states from past states.

We conclude that, as far as the foregoing treatment of the Dyson S-matrix is meaningful at all, we have

$$\text{“}\Psi(-\infty)\,\Psi(-\infty)^{\dagger} = 1\text{”}, \tag{5.32}$$

so that, by (2.34a) and (5.32), $\Psi(-\infty)$ is unitary. Also, to the same extent, $\Psi(+\infty)$ is unitary, by (5.26) with (2.45). Yet, $\Psi(t)$ is not unitary for intermediate values of t, on account of (5.18).

In justifying (5.28)—(5.29), MA[6] excluded the common case that, in $\langle k \,|\, \psi_r(t)$, E might equal H_r. It may be reasoned that (5.28) is still conditionally valid as long as $\langle k \,|\, \psi_r$ has no irregularities at $E = H_r$.

We shall now find explicit expressions for $U_+(t)$ in terms of the interaction V. If the expansions (3.3) and (3.12) converge, we get from (5.30) the following expressions for $U_+(t)$ and $U_-(t)$:

$$U_+(t)_* = \Psi_*(t) = \exp(E_{RL}\,t/i\hbar) \sum_{n=0}^{\infty} (\S\, V)^n \,\|\, = \sum_{n=0}^{\infty} (\S\, V(t))^n \,\|\, \tag{5.33}$$

or

$$U_+(t)_* = Q_*(t)\left(1 + \frac{iK_*}{2}\right)^{-1} = Q_*(t) \sum_{n=0}^{\infty} \left(\frac{K_*}{2\,i}\right)^n, \tag{5.34}$$

$$U_-(t)_* = \Omega_*(t) = \sum_{n=0}^{\infty} (\flat\, V(t))^n \,\|\, = Q_*(t) \sum_{n=0}^{\infty} \left(\frac{iK_*}{2}\right)^n. \tag{5.35}$$

It should be remembered, however, that, according to the conclusion arrived at in Chapter III, the series (5.33)—(5.35) can converge for all of their matrix elements only if the system considered has no bound states. This at the same time will ensure the validity of the conditional equalities Eq. (5.30) from which Eqs. (5.33)—(5.35) were derived. In this same case we finally get for DYSON's S-matrix, by (5.33) with (1.27),

$$S_* = U_+(+\infty)_* = 1 + 2\sum_{n=1}^{\infty} {}_{\delta}V(\S V)^{n-1}\|, \qquad (5.36)$$

which is identical with the expression (3.6) for HEISENBERG's S-matrix, in accordance with (5.31).

VI. Time-dependent perturbation treatment.

An alternative method of solution for Eqs. (5.4) and (5.8) is use of successive approximations. These equations are obviously equivalent with the integral equation

$$U(t, t_0) - 1 = \int_{t_0}^{t} \{V \exp(E_{RL}\, t_1/i\hbar)\}\, U(t_1,\ t_0)\, dt_1/i\hbar. \qquad (6.1)$$

Integrating by parts (hoping the best for contributions with $E_{RL} = 0$ in (6.1)), using (1.19), (5.4), and (5.8), we find (with $a \to 0$ in the definition of the slash),

$$\left.\begin{aligned} U(t, t_0) - 1 &= -\| V(t)\,?\, U(t, t_0) + \| V(t_0)\,? \\ &+ \int_{t_0}^{t} \{\| V\,?\exp(E_{RL}\, t_1/i\hbar)\}\, V(t_1)\, U(t_1, t_0)\, dt_1/i\hbar. \end{aligned}\right\} \qquad (6.2)$$

Repeating this process n times, we get

$$\left.\begin{aligned} &\sum_{i=0}^{n} \| \{V(t)\,?\}^i\, U(t, t_0) - \sum_{i=0}^{n} \| \{V(t_0)\,?\}^i \\ &= \int_{t_0}^{t} \{\| (V\,?)^n\, V \exp(E_{RL}\, t_1/i\hbar)\}\, U(t_1, t_0)\, dt_1/i\hbar. \end{aligned}\right\} \qquad (6.3)$$

The right—hand member contains the interaction energy V in the $(n+1)\,th$ power and must be assumed to go to zero for $n \to \infty$ if an expansion in powers of V is at all allowed.

Hence, we get

$$Y_*^{\dagger}(t)\, U(t, t_0) = Y_*^{\dagger}(t_0), \qquad (6.4)$$

where $Y_*(t)$ is obtained by (1.19) from Y_* as defined by (3.2), but with the meaning of the slash in (3.2) the conjugate by (1.8) of the meaning of the slash in (6.2)—(6.3).

Now, it was shown in Chapter III that the series Y_* and thus also $Y_*(t)$ converge only if the system has no bound states and, in that case, Y_* has a reciprocal and (6.4) can be solved uniquely with respect to $U(t, t_0)$. For instance, for the special choice that, in (6.3) $? = \flat$, so that $Y_* = \Psi_*$, we have, by (3.11),

$$\left.\begin{aligned} \Psi_*(t)\,\Psi_*(t)^\dagger = \Psi_*\,\Psi_*^\dagger \exp(E_{RL}\,t/i\hbar) = 1 \\ \Psi_*(t) = (\Psi_*(t)^\dagger)^{-1}. \end{aligned}\right\} \tag{6.5}$$

Thus, we get from (6.4) the solution

$$U(t, t_0) = \Psi_*(t)\,\Psi_*^\dagger(t_0), \tag{6.6}$$

which is in accordance with (5.19) if there are no bound states. In the limits $t_0 \to -\infty$ and $t \to +\infty$, (6.6) gives results which are identical with (5.33)—(5.36).

On the other hand, if there are bound states, the expansion Y_* cannot generally converge, and Y_* does not have a reciprocal. Therefore, the matrix elements of Eq. (6.4) do no longer define uniquely a propagation matrix $U(t, t_0)$, so that no information on the propagation of the bound states is obtained by a perturbation treatment.

VII. Interpretation of the time-dependent scattering matrix $\Psi(t)$ and of the matrix $\Psi(+\infty)$.

We return now to a problem left at the end of Chapter IV. It was stressed there that the Ψ-matrix gives a correct direct picture of a superposition of free-particle waves and outgoing scattered waves. This is true for the $\Psi(t)$ matrix as well. However, it was remarked that this was not true for the S-matrix which, contrary to the $\Psi(t)$-matrix, gives us a picture of as many incoming as outgoing scattered waves superimposed on the free-particle waves. This makes us wonder: If $\Psi(t)$ gives the correct picture for any large value of t, how is it possible that

$S = \Psi(+\infty)$ gives the wrong picture? We shall show that this is due partly to the fact that (5.26) is only a conditional equality and partly to an un-physical order of sequence of taking limits, which is automatically introduced by making use of the S-matrix in momentum space in interaction representation.

To see this, we have to recapitulate part of the well-known proof that a wave function of the form $\S F(t)\|$ asymptotically consists of outgoing waves only[9), 4)].

For interpretation of $\S F(t)\|$ in (2.23a) as a probability amplitude, we transform it back to xyz-space for one of the particles scattered. If $k = (\vec{k}, \cdots)$, where $\hbar\vec{k}$ is the momentum of the particle considered, we have in Schrödinger representation, omitting a normalization constant:

$$\left.\begin{aligned} T(x, \cdots, t) &= \int d^{(3)}k \, \exp(i\vec{k}\vec{x} + Et/i\hbar) \langle \vec{k} \cdots \,|\, \S F(t) \| \,|\, \vec{k}_0 \cdots \rangle \\ &= \int_0^\infty k^2 \, dk \int_0^{2\pi} d\varphi \int_{-1}^{1} du \, e^{ikru} \, e^{Et/i\hbar} \, \frac{\langle \vec{k} \cdots \,|\, F \,|\, k_0 \cdots \rangle \, e^{[(E_0 \cdots) - (E \cdots)]\, t/i\hbar}}{(E_0 \cdots) - (E \cdots) + ia} . \end{aligned}\right\} \quad (7.1)$$

Here, $u = \cos\theta$, (we have taken $\vec{x}$ in the $+z$ direction) and E_0 and E are energies of the particle considered, while the dots stand for energies of other particles participating in the collision. Let E_1 be the value which E should take for conservation of energy, so that

$$(E_0 \cdots) - (E \cdots) = E_1 - E, \quad (7.2)$$

and let $\hbar k_1$ indicate the absolute value of the corresponding momentum. Further, putting

$$\left.\begin{aligned} \langle 0, 0, \pm k, \cdots \,|\, F \,|\, \vec{k}_0, \cdots \rangle &= F^{(\pm)}, \\ \langle 0, 0, \pm k_1, \cdots \,|\, F \,|\, \vec{k}_0, \cdots \rangle &= F_1^{(\pm)}, \end{aligned}\right\} \quad (7.3)$$

we get, asymptotically (for $r \to \infty$),[9), 4)]

$$T(0, 0, r, t) \approx \frac{2\pi \, e^{E_1 t/i\hbar}}{ir} \int_0^\infty k dk \, \frac{e^{ikr} F^{(+)} - e^{-ikr} F^{(-)}}{E_1 - E + ia} . \quad (7.4)$$

Because of the resonance for $a \to 0$ at $E = E_1$ we put

$$e^{\pm ikr} \approx e^{\pm ik_1 r} \cdot e^{\pm i\xi r/v_1 \hbar}, \tag{7.5}$$

where

$$\xi \equiv E - E_1 \approx v_1 \hbar (k - k_1) \tag{7.6}$$

with

$$v = dE/\hbar dk. \tag{7.7}$$

We also use

$$EdE = c^2 \hbar^2 \, kdk, \tag{7.8}$$

and complete the path of integration in the complex ξ-plane to a contour via $\xi = \pm i\infty$ for the term with $F^{(\pm)}$. Only the contour for $F^{(+)}$ then encloses the pole at $\xi = +ia$, and we find for $a \to 0$

$$T(0, 0, r, t) \approx -(4\pi^2/rc^2\hbar^2) \, F_1^{(+)} E_1 \exp\left[ik_1 (r - w_1 t)\right], \tag{7.9}$$

where we put

$$w_1 = c^2/v_1 = E_1/\hbar k_1. \tag{7.10}$$

Eq. (7.9) indeed represents an outgoing wave, however large t is.

The error made when one uses $S = \Psi(\infty)$for $\Psi(t)$, that is, using $\langle k \,|\, R \,|\, k_0 \rangle$ in (7.1) instead of $\langle k \,|\, T(t) \,|\, k_0 \rangle$, is in the first place in taking the limit $t \to \infty$ in the definition of R and S by

$$\langle k \,|\, R \,|\, k_0 \rangle = \lim_{t \to \infty} \langle k \,|\, T(t) \,|\, k_0 \rangle \tag{7.11}$$

(similar for S and Ψ), *before* this quantity is substituted into (7.1) and the integral over k performed. This amounts to not combining the factors $\exp(Et/i\hbar)$ and $\exp[(E_1 - E)\, t/i\hbar]$ in (7.1) to the harmless factor $\exp(E_1 t/i\hbar)$ of (7.4), but to writing

$$\left.\begin{aligned} T(0, 0, r, t) \approx \frac{2\pi}{ir} \int_0^\infty e^{Et/i\hbar} \int \frac{(\xi + E_1)\, d\xi}{\hbar^2 c^2 (ia - \xi)} \\ \times \left\{ F^{(+)} e^{ik_1 r} e^{i(v_1 t + r)\xi/v_1 \hbar} - F^{(-)} e^{-ik_1 r} e^{i(v_1 t - r)\xi/v_1 \hbar} \right\}, \end{aligned}\right\} \tag{7.12}$$

and then taking the limit $t \to \infty$ in the definition of the S-matrix in the factor on the second line only, while on the first line one puts $E \approx E_1$ on account of the resonance denominator. Thus, replacing $\Psi(\infty)$ by S in the transformation to xyz-space leads, from (7.12), wrongly to

$$R(0,0,r,t) \approx -\frac{4\pi^2 e^{E_1 t/ih} E_1}{r\hbar^2 c^2} \lim_{\substack{r\to\infty \\ t\to\infty}} \Lambda, \tag{7.13}$$

where Λ is obtained by a contour via $\xi = +i\infty$ for the term with $F^{(+)}$, while for the term with $F^{(-)}$ the contour in the complex plane must be taken along $\xi = \pm i\infty$ as $v_1 t \gtrless r$:

$$\Lambda = F_1^{(+)} e^{ik_1 r} - F_1^{(-)} e^{-ik_1 r} \quad \text{if } v_1 t > r, \tag{7.14}$$

$$\Lambda = F_1^{(+)} e^{ik_1 r} \quad \text{if } v_1 t < r. \tag{7.15}$$

We see that even with the error made we still would have found the correct result (7.9) if we had kept $v_1 t < r$, that is, if we study the asymptotic behaviour $r \to \infty > v_1 t$ for some possibly large, but anyhow finite time t. But, in using the S-matrix, the limit $t \to \infty$ has already been taken first inside the brackets on the second line of Eq. (7.12), *before* we take $r \to \infty$, yes, even before we transform at all from k-space to xyz-space. Therefore, the S-matrix (or rather its scattering part R) represents in xyz-space not (7.13) with (7.15), but (7.13) with (7.14), or

$$R(0,0,r,t) = -\frac{4\pi^2 E_1}{rc^2\hbar^2}\{F_1^{(+)} e^{ik_1(r-w_1 t)} - F_1^{(-)} e^{-ik_1(r+w_1 t)}\}, \tag{7.16}$$

which obviously contains incoming as well as outgoing waves.

Therefore, R does not depict the scattered wave. Only $T(t)$ should be used for this purpose.

In the sense of the conditional equalities (5.26), the collision process may be pictured as a steady transition from the state $\langle k \mid k_0 \rangle$ with definite values k_0 for the momenta at $t = -\infty$ to a state $\langle k \mid S \mid k_0 \rangle$ at $t = +\infty$, so that the probability $P(k)$ of finding the system with momenta $k \neq k_0$ after the collision is

$$P(k) = |\langle k|S|k_0\rangle|^2 = |\langle k|R|k_0\rangle|^2 = 4\pi^2[\delta(E-E_0)]^2|\langle k|F|k_0\rangle|^2,$$

where we have used (2.28) and (1.5). Since

$$[\delta(E-E_0)]^2 = h^{-1}\int_{-\infty}^{\infty} e^{i(E-E_0)t/\hbar}dt\cdot\delta(E-E_0) = \frac{\delta(E-E_0)}{h}\int_{-\infty}^{\infty}dt,$$

we have

$$P(k) = \frac{2\pi}{\hbar}T\delta(E-E_0)|\langle k|F|k_0\rangle|^2,$$

where T is the infinite time during which the collision has taken place. The probability density in k-space per unit time for a transition $k_0 \to k$ is thus[1)]

$$P_{k_0\to k} = \frac{2\pi}{\hbar}\delta(E-E_0)|\langle k|F|k_0\rangle|^2, \qquad (7.17)$$

where the δ-function takes care of energy conservation. Equation (7.17) is in accordance with the results quoted in the first part of Chapter IV.

As was shown in reference 5, the same result is obtained if the time derivative of the probability function

$$P(k,t) = |\langle k|\Psi(t)|k_0\rangle|^2$$

at time t is defined as*

$$\begin{aligned} P_{k_0\to k}(t) &= \frac{\partial}{\partial t}[\langle k|\Psi(t)|k_0\rangle^*\langle k|\Psi(t)|k_0\rangle] \\ &= \langle k_0|\Psi^\dagger(t)|k\rangle\frac{\partial}{\partial t}\langle k|\Psi(t)|k_0\rangle \\ &\quad + \left\{\frac{\partial}{\partial t}\langle k_0|\Psi^\dagger(t)|k\rangle\right\}\cdot\langle k|\Psi(t)|k_0\rangle \\ &= \frac{1}{i\hbar}\{\langle k_0|\Psi^\dagger(t)|k\rangle\langle k|V(t)\Psi(t)|k_0\rangle \\ &\quad -\langle k_0|\Psi^\dagger(t)V(t)|k\rangle\langle k|\Psi(t)|k_0\rangle\}, \end{aligned}$$

where we have used the Schrödinger equation (5.16) for the functions $\Psi(t)$ and $\Psi^\dagger(t)$. Using further (1.19), (2.23a)—(2.25a), we get

* The same idea appears in Eq (1.68) of reference 1.

$$P_{k_0\to k}(t) = \frac{1}{i\hbar}\{\langle k|k_0\rangle[\langle k_0|F|k_0\rangle - \langle k_0|F^\dagger|k_0\rangle]$$
$$+\langle k_0|T^\dagger|k\rangle\langle k|F|k_0\rangle - \langle k_0|F^\dagger|k\rangle\langle k|T|k_0\rangle\}$$
$$= \frac{1}{i\hbar}\{\langle k|k_0\rangle[\langle k_0|F^\dagger T|k_0\rangle - \langle k_0|T^\dagger F|k_0\rangle]$$
$$+\langle k_0|\,\|F^\dagger \flat|k\rangle\langle k|F|k_0\rangle - \langle k_0|F^\dagger|k\rangle\langle k|\S F\|\,|k_0\rangle\},$$

where we have used (2.33a) and $\langle k|k_0\rangle = \delta(k-k_0)$. Finally, since by 1.9a

$$\flat(E_0-E) - \S(E_0-E) = +2\pi i\,\delta(E-E_0),$$

we get for the probability density in k-space per unit time of a transition $k_0 \to k$ at any time

$$\left.\begin{aligned} P_{k_0\to k} = \frac{2\pi}{\hbar}\Big\{&\delta(E-E_0)\,|\langle k|F|k_0\rangle|^2 \\ -\delta(k-k_0)&\int\delta(E_0-E')\,|\langle k'|F|k_0\rangle|^2\,dk'\Big\}. \end{aligned}\right\} \quad (7.18)$$

(7.18) is in accordance with "conservation of normalization":

$$\frac{\partial}{\partial t}\int\langle k_0|\Psi^\dagger|k\rangle\,dk\,\langle k|\Psi|k_0\rangle = \frac{\partial}{\partial t}\int P(k,t)\,dk = \int dk\,P_{k_0\to k} = 0$$

and, if $k \neq k_0$, (7.18) is identical with (7.17).

VIII. The methods of Källen and of Yang and Feldman[11].

In the preceding sections, the S-matrix has been defined in terms of the state functions either in Schrödinger representation or in interaction representation. Sometimes, in particular in field theories, it is more convenient to work in a Heisenberg representation and to derive the S-matrix directly from the field equations without an explicit use of the Hamiltonian and the Schrödinger equation.

Let ξ be any dynamical variable of the system in a Schrödinger representation, i. e. ξ is a time-independent matrix. In interaction representation, we then have the matrix

$$\left.\begin{aligned} \xi(t) &= \xi \exp(E_{RL}\, t/i\hbar), \\ \xi(t) &= \xi(t_0) \exp[E_{RL}(t-t_0)/i\hbar], \end{aligned}\right\} \tag{8.1}$$

where the interaction representation has been chosen identical with the Schrödinger representation at the time $t = 0$. Similarly, we may define a Heisenberg representation

$$\xi_H(t) = U^\dagger(t, 0)\, \xi(t)\, U(t, 0), \tag{8.2}$$

where $U(t, t_0)$ is the propagation matrix of Chapter V.

While the variables $\xi_H(t)$ satisfy the usual equations of motion with interaction, the $\xi(t)$ satisfy the corresponding equations of motion for a system with no interaction, i. e.

$$i\hbar\, d\xi(t)/dt = E_{RL}\, \xi(t). \tag{8.3}$$

For $t = 0$, we have, by our special choice of representations,

$$\xi_H(0) = \xi(0) = \xi. \tag{8.4}$$

We now introduce a new interaction representation[12] of the dynamical variables by matrices $\xi_{t_0}(t)$ depending on two time parameters t_0 and t. For a fixed t_0 they are defined as those solutions of the free particle equations of motion (8.3) which for $t = t_0$ coincide with the Heisenberg matrices $\xi_H(t_0)$, i. e.

$$\left.\begin{aligned} i\hbar\, d\xi_{t_0}(t)/dt &= E_{RL}\, \xi_{t_0}(t), \\ \xi_{t_0}(t_0) &= \xi_H(t_0). \end{aligned}\right\} \tag{8.5}$$

The solution of these equations may, by (8.1) and (8.2), be written as

$$\left.\begin{aligned} \xi_{t_0}(t) &= \xi_H(t_0) \exp[E_{RL}(t-t_0)/i\hbar] \\ &= \{U^\dagger(t_0, 0) \exp[E_{RL}(t-t_0)/i\hbar]\} \\ &\quad \times \xi(t) \{U(t_0, 0) \exp[E_{RL}(t-t_0)/i\hbar]\} \\ &= (\{U^\dagger(t_0, 0) \exp[-E_{RL}\, t_0/i\hbar]\} \\ &\quad \times \xi \{U(t_0, 0) \exp[-E_{RL}\, t_0/i\hbar]\}) \exp(E_{RL}\, t/i\hbar). \end{aligned}\right\} \tag{8.6}$$

From the expression (5.19) for the propagation matrix $U(t, t_0)$ we get at once

$$\left.\begin{aligned} U(t_0, 0) \exp(-E_{RL} t_0/i\hbar) &= \Psi \cdot \Psi^\dagger(-t_0) + \sum_r \psi_r \cdot \psi_r^\dagger(-t_0) \\ &= U(0, -t_0). \end{aligned}\right\} \quad (8.7)$$

This equation simply expresses the fact that the propagation matrix in the Schrödinger representation, i. e.

$$U_S(t, t_0) = e^{Et/i\hbar} U(t, t_0) e^{-Et_0/i\hbar}, \quad (8.8)$$

is a function of the difference $t - t_0$ only.

By (8.7) we get from (8.6)

$$\xi_{t_0}(t) = \{ U^\dagger(0, -t_0) \xi U(0, -t_0) \} \exp(E_{RL} t/i\hbar). \quad (8.9)$$

The matrices $\xi_{t_0}(t)$ may also be interpreted as Heisenberg representatives of the dynamical variables of the system without interaction. Further, since each value of t_0 defines a space-like surface σ_0 in Minkowski space, viz. the surface of points with time coordinates equal to t_0, the variables $\xi_{t_0}(t)$ correspond to the field variables $\Psi(\sigma, x)$ in YANG and FELDMAN'S notation[11]. The connection between the variables $\xi_{t_0}(t)$ and $\xi_{t_1}(t)$ corresponding to two such surfaces σ_0 and σ_1, respectively, is by (8.9) given by the unitary transformation

$$\xi_{t_1}(t_0) = W^\dagger_{t_0 t_1}(t) \xi_{t_0}(t) W_{t_0 t_1}(t) \quad (8.10)$$

with

$$\left.\begin{aligned} W_{t_0 t_1}(t) &= \{ U^\dagger(0, -t_0) U(0, -t_1) \} \exp(E_{RL} t/i\hbar) \\ &= U(-t_0, -t_1) \exp(E_{RL} t/i\hbar) = U(t - t_0, t - t_1), \end{aligned}\right\} \quad (8.11)$$

by (5.3b) and arguments similar to (5.2)—(5.3a) and (8.7)—(8.8).

We now define the in- and out-variables by

$$\left.\begin{aligned} \xi_{\text{in}}(t) &= \lim_{t_0 \to -\infty} \xi_{t_0}(t), & \text{a)} \\ \xi_{\text{out}}(t) &= \lim_{t_1 \to +\infty} \xi_{t_1}(t). & \text{b)} \end{aligned}\right\} \quad (8.12)$$

The connection between these variables is obtained from (8.10)—(8.11) by taking the limits $t_0 \to -\infty$, $t_1 \to +\infty$. From the definitions (5.25) of the $U_+(t)$ matrix and Dyson's definition (5.31) of the S-matrix, we get

$$\left.\begin{aligned} \lim_{\substack{t_0 \to -\infty \\ t_1 \to +\infty}} U(-t_0, -t_1) = \lim_{\substack{t_0 \to -\infty \\ t_1 \to +\infty}} U(t_1, t_0) = \lim_{t_1 \to +\infty} U_+(t_1) = S. \end{aligned}\right\} \quad (8.13)$$

Hence, by (8.12), (8.10), and (8.11)

$$\xi_{\text{out}}(t) = S^\dagger \xi_{\text{in}}(t) S, \quad (8.14)$$

i. e. the in- and out-variables are connected by a unitary transformation with the S-matrix as transformation matrix. Since the relation between in- and out-variables may be obtained by solving the equations of motion for $\xi_H(t)$, (8.14) represents a way of determining the S-matrix without the use of Hamiltonian or Schrödinger equation. These methods of KÄLLÉN and of YANG and FELDMAN have proved to be useful in field theory[12], and may be applied also in cases where the system is not a Hamiltonian system. For Hamiltonian systems, it is easy to find the connection between the in- and out-variables and the matrices Ω and Ψ. By (5.25) and (5.30) we get

$$\left.\begin{aligned} \lim_{t_0 \to -\infty} U(0, -t_0) = U(0, +\infty) = {}''U_-(0) = \Omega'', \\ \lim_{t_1 \to +\infty} U(0, -t_1) = U(0, -\infty) = {}''U_+(0) = \Psi''. \end{aligned}\right\} \quad (8.15)$$

Hence, by (8.12) and (8.9),

$$\left.\begin{aligned} {}''\xi_{\text{in}}(t) = \{\Omega^\dagger \xi \Omega\} \exp(E_{RL} t/i\hbar) = \Omega^\dagger(t) \xi(t) \Omega(t)'', \\ {}''\xi_{\text{out}}(t) = \{\Psi^\dagger \xi \Psi\} \exp(E_{RL} t/i\hbar) = \Psi^\dagger(t) \xi(t) \Psi(t)'', \end{aligned}\right\} \quad (8.16)$$

which is in accordance with (8.14) on account of (2.48).

IX. Switching on and off the interaction.

Although the results thus far obtained are satisfactory, the treatment of the limits $t \to \pm\infty$ remains awkward. Trying to define the U_+-matrix as limit for $t_0 \to -\infty$ of the matrix $U(t, t_0)$ we were trying in fact to solve the Schrödinger equation for $U_+(t)$ with initial condition $U_+(t_0) = 1$ without stating the exact value of the time t_0. As, on account of the Schrödinger equation, $U_+(t)$ does not stay constant, this problem does not really make sense. Our procedure of then looking for a solution for $U_+(t)$ which is "unity on the average" for $t \to -\infty$ is only a makeshift, and leads to the complication that items that do not vanish at all (such as $\psi_r(t)$) yet may happen to vanish on the average for $t \to -\infty$.

We shall therefore try to solve this problem now by assuming that $V \to 0$ for $t \to \pm\infty$, so that we may really have $U_+(t) \to 1$ for $t \to -\infty$ and also a well defined limit of $U_+(t)$ for $t \to +\infty$. For this purpose, we replace V in the original Schrödinger equation (2.3) by

$$V \exp(-a|t|/\hbar) \tag{9.1}$$

with extremely small, but finite positive a.

Inserting (9.1) in the definition (1.19) of the matrix $V(t)$ occurring in (2.6), and therefore also appearing in Eq. (5.4), we have for the non-closed system in interaction representation a potential

$$V(a; t) = V \exp[(E_{RL}\, t - ia|t|)/i\hbar]. \tag{9.2}$$

Thence, we have to solve the equations

$$i\hbar\, \partial U_+(a; t)/\partial t = \{V \exp(E_{1,RL}\, t/i\hbar)\}\, U_+(a; t) \text{ for } t < 0 \tag{9.3}$$

and

$$i\hbar\, \partial U_+(a; t)/\partial t = \{V \exp(E_{-1,RL}\, t/i\hbar)\}\, U_+(a; t) \text{ for } t > 0 \tag{9.4}$$

with the initial condition

$$U_+(a; -\infty) = 1. \tag{9.5}$$

Here we have put

$$E_{n,RL} = E_{RL} + nia, \tag{9.6}$$

and we shall also introduce a generalized scat $\S_n$ for a division by $(E_{||} - E_{\S} + nia)$. Thus, $\S_1$ is equal to $\S$, with the difference that now a is not necessarily going to zero. Similarly, we introduce a generalized flat $\flat_n$ for a division by $(E_{||} - E\flat - nia)$.

By $\Phi(a;t)$ we shall denote the solution of the differential equation (9.3) which has the limit 1 for $t \to -\infty$:

$$\lim_{t\to-\infty} \Phi(a;t) = 1. \tag{9.7}$$

Similarly, $X(a;t)$ shall denote the solution of (9.4) satisfying the condition

$$\lim_{t\to+\infty} X(a;t) = 1. \tag{9.8}$$

Assuming that the limits in (9.7)—(9.8) give real (and not only conditional) equality of Φ at $-\infty$ and of X at $+\infty$ to unity, we conclude that

$$\left.\begin{aligned} \Phi^\dagger(a;-\infty)\Phi(a;-\infty) &= \Phi(a;-\infty)\Phi^\dagger(a;-\infty) = 1, \\ X^\dagger(a;+\infty)X(a;+\infty) &= X(a;+\infty)X^\dagger(a;+\infty) = 1. \end{aligned}\right\} \tag{9.9}$$

Now, consider the matrices $\langle k|\Phi(a;t)|k_0\rangle$ and $\langle k|X(a;t)|k_0\rangle$ as two sets of functions of k, labeled by k_0. At $t = -\infty$, the first set of functions is, by (9.9), a complete orthonormal set and, on account of the conservation of orthonormality and completeness of a set of solutions of the Schrödinger equation, which was shown in Eqs. (5.10)—(5.11) to hold also for non-closed systems, we have for any finite time t

$$\Phi^\dagger(a;t)\Phi(a;t) = \Phi(a;t)\Phi^\dagger(a;t) = 1. \tag{9.10}$$

By the same argument, we get from (9.9)

$$X^\dagger(a;t)X(a;t) = X(a;t)X^\dagger(a;t) = 1. \tag{9.11}$$

If the series converge, we get an explicit expression for $\Phi(a;t)$ in terms of the potential V by

$$\left.\begin{aligned}\Phi_*(a;t) &= 1+\sum_{n=1}^{\infty}\left\{\prod_{i=0}^{n-1}(\S_{n-i}V)\,\|\right\}\exp(E_{n,RL}\,t/i\hbar)\\ &= 1+\S_1 V\|\exp(E_{1,RL}\,t/i\hbar)\\ &+\S_2 V\S_1 V\|\exp(E_{2,RL}\,t/i\hbar)+\cdots\\ &+\S_n V\S_{n-1}V\cdots\S_2 V\S_1 V\|\exp(E_{n,RL}\,t/i\hbar)+\cdots.\end{aligned}\right\}\quad(9.12)$$

Indeed, in $i\hbar\,\partial\Phi_*(a;t)/\partial t$ all first factors $\S_n$ are canceled, and by

$$\{VA\exp(E_{n,RL}t/i\hbar)\}=\{V\exp(E_{1,RL}t/i\hbar)\}\{A\exp(E_{n-1,RL}t/i\hbar)\}\quad(9.13)$$

we may factorize out $\{V\exp(E_{1,RL}\,t/i\hbar)\}$ from the resulting expression and just obtain $\{V\exp(E_{1,RL}\,t/i\hbar)\}\Phi_*(a;t)$.

Since $a>0$, all terms but the first in (9.12) vanish for $t\to-\infty$, which takes care of the initial condition

$$\Phi_*(a;-\infty)=1.$$

In the same way, one sees that the series

$$\left.\begin{aligned}X_*(a;t) &= 1+\sum_{n=1}^{\infty}\left(\prod_{i=0}^{n-1}\flat_{n-i}V\right)\|\exp(E_{-n,RL}\,t/i\hbar)\\ &= 1+\flat_1 V\|\exp(E_{-1,RL}\,t/i\hbar)\\ &+\flat_2 V\flat_1 V\|\exp(E_{-2,RL}\,t/i\hbar)+\cdots\\ &+\flat_n V\flat_{n-1}V\cdots\flat_2 V\flat_1 V\|\exp(E_{-n,RL}\,t/i\hbar)+\cdots\end{aligned}\right\}\quad(9.14)$$

is a solution of (9.4) satisfying the condition $X_*(a;+\infty)=1$, provided, of course, that the series is convergent.

Since $U_+(t)$ is that solution of the Schrödinger equation with the potential (9.2) which is continuous at $t=0$ and satisfies the initial condition (9.5), we have

$$U_+(a;t) = \left\{ \begin{array}{ll} \Phi(a;t) & \text{for } t<0, \\ X(a;t)\,X(a)^\dagger\,\Phi(a) & \text{for } t>0, \end{array} \right\} \quad (9.15)$$

where

$$\Phi(a) = \Phi(a;t\,0),\ X(a) = X(a;0) \quad (9.16)$$

denote the values of $\Phi(a;t)$ and $X(a;t)$ for $t=0$. The expression (9.15) for $U_+(a;t)$ is seen to be continuous at $t=0$ and unitary for all t, on account of (9.10), (9.11):

$$U_+^\dagger(a;t)\,U_+(a;t) = U_+(a;t)\,U_+^\dagger(a;t) = 1. \quad (9.17)$$

For an arbitrary state $\langle k \mid \psi(a;t)$ of the non-closed system with $a \neq 0$, we now obviously have

$$\psi(a;t) = U_+(a;t)\,\psi(a;-\infty), \quad (9.18)$$

where $\psi(a;-\infty)$ is the value of $\psi(a;t)$ for $t=-\infty$. Indeed, (9.18) is a solution of the Schrödinger equation of the non-closed system and (9.5) takes care of the initial condition.

From (9.17) and (9.18) we get

$$\psi(a;-\infty) = U_+^\dagger(a;t)\,\psi(a;t). \quad (9.19)$$

Hence,

$$\psi(a;t) = U_+(a;t)\,U_+^\dagger(a;t_0)\,\psi(a;t_0) \equiv U(a;t,t_0)\,\psi(a;t_0), \quad (9.20)$$

i. e.

$$U(a;t,t_0) = U_+(a;t)\,U_+^\dagger(a;t_0) \quad (9.21a)$$

and

$$U_+(a;t) = \lim_{t_0 \to -\infty} U(a;t,t_0) \quad (9.21b)$$

on account of (9.5).

If we now define a matrix $S(a)$ as the limit of $U_+(a;t)$ for $t \to +\infty$, we get, by (9.15) and (9.8),

$$S(a) = \lim_{t\to+\infty} U_+(a;t) = X^\dagger(a)\,\Phi(a) = \lim_{\substack{t_0\to-\infty \\ t\to+\infty}} U(a;t,t_0), \quad (9.22)$$

which is unitary on account of (9.10)—(9.11). Hence, $S(a)$ is the Dyson S-matrix for the non-closed system where the interaction is switched on and off. In contrast to the Heisenberg S-matrix, the matrix $S(a)$, according to (9.22), with (9.16), (9.12), (9.14), (1.7), does not contain a factor $\delta(E_{RL})$. This is connected with the fact that the energy is not conserved in the system with $a > 0$. By (9.18), $S(a)$ is the matrix which connects any state vector at $t = -\infty$ with the state vector at $t = +\infty$ by

$$\psi(a; +\infty) = S(a)\,\psi(a; -\infty). \tag{9.23}$$

In all these expressions we shall now go to the limit of $a \to 0$, but we shall first assume that the series (9.12) and (9.14) for $t = 0$, i. e. the series $\Phi_*(a) \equiv \Phi_*(a; 0)$ and $X_*(a) \equiv X_*(a; 0)$, are uniformly convergent for all a including the value $a = 0$, so that we can take the limit $a \to 0$ term by term in the expansions. Since $\S_n \to \S$ and $\flat_n \to \flat$ for $a \to 0$, we then get

$$\left.\begin{aligned} \Phi_*(0) &= \sum_{n=0}^{\infty} (\S\, V)^n \,\| = \Psi_* \\ X_*(0) &= \sum_{n=0}^{\infty} (\flat\, V)^n \,\| = \Omega_* \end{aligned}\right\} \tag{9.24}$$

on account of (3.3a), (3.3b). As was seen in Chapter III, these series can converge only if the system considered has no bound states. The assumption made above about the uniform convergence of $\Phi(a)$ and $X(a)$ can therefore only be justified when no bound states exist. In this case, however, we get from (9.12) and (9.14), by $E_{\pm n, RL} \to E_{RL}$ for $a \to 0$,

$$\left.\begin{aligned} \Phi_*(0; t) &= \sum_{n=0}^{\infty} (\S\, V(t))^n \,\| = \Psi_*(t) \text{ for } t < 0, \\ X_*(0; t) &= \sum_{n=0}^{\infty} (\flat\, V(t))^n \,\| = \Omega_*(t) \text{ for } t > 0. \end{aligned}\right\} \tag{9.25}$$

Thus, from (9.15), (9.24), (9.25), (2.47a), and using the relation

$$\Psi(t) = \Omega(t)\, S \tag{9.26}$$

following from (2.48), we find that, in this case,

$$U_+(0;t)_* = \lim_{a\to 0} U_+(a;t)_* = \Psi_*(t) \qquad (9.27)$$

for all values of t. This equation is in agreement with the unitarity of U_+, as the system considered had no bound states. For such systems, Eq. (9.27) is a justification of the expectation expressed at the beginning of Chapter V, since the state $U_+(0;t)$, which at $t = -\infty$ represents an initial free particle state k_0, for finite times will have developed into the stationary state $\langle k \mid \Psi(t) \mid k_0 \rangle$.

Further, we get for such systems, from (9. 21 a), (9.27), and (5.19)—(5.21),

$$\lim_{a\to 0} U_*(a;t,t_0) = \Psi_*(t)\,\Psi_*{}^\dagger(t_0) = S_*(t,t_0) = U_*(t,t_0), \qquad (9.28)$$

For these systems, the limit of $S(a)$ defined by (9.22) is also the S-matrix of HEISENBERG, since by (9.22), (9.24), and (2.47 a)

$$\lim_{a\to 0} S_*(a) = X^\dagger{}_*(0)\,\Phi_*(0) = \Omega^\dagger{}_*\,\Psi_* = \Omega^\dagger{}_*\,\Omega_*\,S_* = S_*. \qquad (9.29)$$

Thus, in the case considered, the method of an infinitely slowly switching on and switching off the interaction leads in every respect to the same results as the method used in Chapter V for a strictly closed system with the conventions (1.23)—(1.32) for taking the limits $t \to \pm\infty$. The two methods differ only by the sequence in which the limits $a \to 0$ and $t \to \pm\infty$ are taken and, at least as far as our series expansions are sufficiently convergent, it does not matter which limit is taken first.

It is generally believed that this result holds also in cases, where the series expansions do not converge, for instance, also for systems with bound states. In particular, it is believed that an equation like (9.27) must always be valid on account of the "adiabatic theorem" of quantum mechanics[13] which states that a stationary state of a closed system by an adiabatic, i. e. infinitely slow, change of the potential goes over into the corresponding stationary state of the closed system with the new potential. The proof of the adiabatic theorem given by BORN and FOCK is, how-

ever, based on special assumptions about the potential and the type of energy spectrum, and to our knowledge no general proof of this theorem has ever been given.

X. Conclusion.

The purpose of a theory of collision processes is the calculation of differential and total scattering cross sections. The technique of obtaining these quantities from the matrices discussed in the foregoing was discussed in more detail in reference 4. From the discussions of Chapter VII of the present paper, however, it is clear that scattering is described most directly by the matrix Ψ defined by the integral equation (2.27 a). In fact, in Eq. (7.18) we obtained the density in momentum space of the transition probability per second directly in terms of the "effective potential" F, which by Eq. (2.25 a) is the product of the interaction operator V into the matrix Ψ. In the integration over final states in a given region in k-space, conservation of energy is then ensured by the delta function appearing in Eq. (7.18).

As shown in Chapter VII, the Heisenberg S-matrix in xyz-space for $t \to \infty$ gives an incorrect picture of the scattering phenomenon, since, due to the wrong order of sequence of limits, it describes incoming as well as outgoing spherical waves in such interpretation. When interpreted as a probability amplitude in momentum space, as in the reasoning preceding Eq. (7.17), it is seen to determine only an infinite transition probability during an infinite time. Although, by an artifice, one succeeds in obtaining Eq. (7.17) from such attempt at a direct interpretation of the S-matrix, a more satisfactory treatment was given by Eq. (4.1), in which the right-hand member amounts to factorizing a delta function $\delta(E_{RL})$ out of the matrix $R = S - 1$ by writing

$$S - 1 = R = \delta(E_{RL})\,\overline{R}; \quad \overline{R} = \int R\, dE_{RL} = \int (S - 1)\, dE_{RL}. \quad (10.1)$$

A comparison of (10.1) with Eq. (2.28), in which the delt had the meaning defined in (1.5), shows that

$$(i/\pi)\ \delta\overline{R}\,\| = R = 2\ \delta F\,\| ; \qquad \overline{R} = (2\pi/i)\ F\big|_{E_{RL}=0}. \quad (10.2)$$

Equation (10.2) relates $\overline{R}$ to F "on the energy shell", that is, to the matrix elements $\langle L \mid F \mid R \rangle$ with $E_{RL} = 0$ only. Outside the energy shell, $\overline{R}$ is not really defined by (10.1). Because of the delta functions in Eq. (7.18), F in that equation may now be replaced by $(i/2\pi)\,\overline{R}$, so that

$$\left.\begin{aligned} P_{k_0 \to k} = h^{-1} \{ & \delta(E - E_0) \mid \langle k \mid \overline{R} \mid k_0 \rangle \mid^2 \\ - \delta(k - k_0) & \int \delta(E_0 - E') \mid \langle k' \mid \overline{R} \mid k_0 \rangle \mid^2 d k' \}. \end{aligned}\right\} \tag{10.3}$$

Thus, by (10.1)—(10.3), cross sections for scattering in a finite time interval may be obtained from the S-matrix without need of the consideration of infinite probabilities preceding Eq. (7.17).

This reasoning is, of course, based primarily on combining the definition (2.28) of the Heisenberg S-matrix with the original definition of the Ψ-matrix as a set of probability amplitudes for given scattering states characterized by the momentum k_0 of the incident particles. If any different definition of the S-matrix is used, its equivalence is first to be proved; and, since the matrix Ψ is used, it has to be investigated to what extent every possible state of a given system of interacting particles can be described by it. This was the aim of the preceding chapters. The main results obtained, old and new ones, are the following.

While the S-matrix is always unitary [Eq. (2.45)], the Ψ-matrix in general is not so [Eq. (2.40)], while the scattering matrix Q symmetric in incoming and outgoing waves, which was defined by the integral equation (2.27c), is never unitary [Eq. (2.34c)]. The latter matrix is related to the Hermitian reaction matrix K by Eq. (2.29) with (2.25c) or with (2.23c)—(2.24c). The reaction matrix K, on the other hand, is related to the S-matrix by Eq.(2.44), and can be used for calculating phase shifts by means of (2.49)—(2.51) (cf. reference 1). Further relations between the matrices Ψ, Q, S, and K are found in Appendix D.

The propagation matrix $U(t, t_0)$, defined by Eq. (5.2) and expressed in terms of Ψ by (5.19), has been used by DYSON for defining an S-matrix called $U_+(\infty)$. This Dyson S-matrix relates the "in"- and "out"-variables of KÄLLÉN, YANG, and FELDMAN by Eq. (8.14). These variables may be interpreted as simply two

different interaction representations of the same quantities; the "in" representation coinciding (as far as this is possible) with the Heisenberg representation in the infinite past, and the "out" representation in the infinite future. They also are the Heisenberg representations of those solutions of the "homogeneous" field equations without interaction which describe incoming or outgoing particles. (Cf. references 11 and 12).

The definitions of $U_+(t)$ [Eq. (5.25)], of the Dyson S-matrix $U_+(\infty)$, and of the in and out variables are not quite satisfactory, unless one assumes that the interaction has been switched on since $t = -\infty$, and will be switched off before $t = +\infty$. In Chapter IX, we defined scattering matrices for finite switching-on and-off velocities. Now, $U_+(a; t)$ is given by Eq. (9.15), and $S(a)$ of Eq. (9.22) is the new Dyson S-matrix. The limits $t \to \pm\infty$ this time have been taken first. Little can be said in general about the properties of this matrix $S(a)$, which for finite a does not even ensure conservation of energy.

All these results are quite general; for actual calculations, however, some special assumptions have to be made. The least of these assumptions is the one discussed on pages 20-21 and in Appendix D, that the scattering matrices Ψ, Ω, and Q all three span the same subspace in Hilbert space. This enables us to derive the relations (2.48), (5.24), and (9.26) between the various scattering matrices.

A treatment of the limits $t \to \pm\infty$ without taking refuge to switching on and off the interaction leads to the "conditional equalities" (5.26)—(5.32) and (8.15)—(8.16). These formulas are usually supposed to prove the general equality of the Dyson S-matrix to the Heisenberg S-matrix. However, this treatment makes sense only in so far as bound states can be ignored, as seen from Eq. (5.28), or by comparing (5.32) with (2.40). In the more satisfactory treatment of Chapter IX, progress in this proof could be made only by assuming convergence of the series expansions of perturbation theory. More specifically, we assumed uniform convergence of the expansions (9.12)—(9.14) for all switching-on-and-off velocities of the field including the case where this velocity tends to zero. Then, the "improved" version of the Dyson S-matrix given by (9.22) was shown in Eq. (9.29) to yield the Heisenberg S-matrix in the final limit $a \to 0$.

However, in Chapter III it has been shown in complete generality that the expansions of perturbation theory for Ψ used in Chapter IX cannot possibly generally converge, if any bound states for the system are possible at all. (Any state, in which not all final particles go off to infinity, is here considered a "bound state"). This proof is based on the disagreement between Eqs. (3.11) and (2.40). For the exceptional systems for which no such bound states are possible, however, we have expressed all scattering matrices and related quantities explicitly in the form of expansions. [See Eqs. (3.3)—(3.6), (5.33)—(5.36), (9.12), (9.14)]. In the even more exceptional case that these expansions really converge, these expansions satisfy all equations derived for the quantities which they represent.

Concluding we may say that we have not been able to give a proof of the equality of the Dyson S-matrix and the Heisenberg S-matrix general enough to be valid also for systems with bound states. (We purposely omitted some arguments possible on the basis of an "adiabatic theorem" which we could not rigorously prove for the systems considered). It should, however, be remembered that cross sections are determined by Eqs. (7.18) or (10.3) without any discussion of what is going on at $t \to \pm \infty$. In principle, F in Eq. (7.18) can be found by direct solution of the integral equation (2.27a) and subsequent use of Eq. (2.25a); or $\overline{R}$ in Eq. (10.3) can be found by first solving for Q from Eq. (2.27c) and then finding W, K, R, and $\overline{R}$ by means of Eqs. (2.25c), (2.29), (2.43), and (10.1), respectively. The integral equations (2.27) retain their validity for systems with bound states.

Not discussed in this review of scattering theory are such problems as (1) the use of exact wave functions instead of the nth-order Born approximation involved in our representation of the scattering matrices in k-space and in our treatment of all but kinetic energy as the perturbation causing the scattering phenomenon; (2) explicit solution of collision problems for systems allowing bound states; (3) use of the method of analytic continuation of the S-matrix in the complex energy plane for obtaining additional information on systems of interacting particles; or (4) solution of scattering problems by variational methods.

Appendices.

A. We shall here prove the important relation (1.17):

$$D_{LR}\,D_{LM} + D_{RM}\,D_{LM} + D_{RM}\,D_{RL} = \pi^2\,\delta(E_{LM})\,\delta(E_{MR}) \tag{A.1}$$

with D_{RL} given by (1.5). This equation will be proved if, for an arbitrary function $f(E_L, E_M)$, the relation

$$\left.\begin{aligned} X \equiv \lim_{a\to 0} \iint dE_L\,dE_M\,f(E_L, E_M) \\ \times\left\{D_{LR}\,D_{LM} + D_{RM}\,D_{LM} + D_{RM}\,D_{RL}\right\} = \pi^2 f(E_R, E_R) \end{aligned}\right\} \tag{A.2}$$

can be shown. To this purpose, we introduce new variables of integration $x = E_{LR}/a$ and $y = E_{MR}/a$, so $E_{LM} = a\,(x-y)$.

Let $f(E_L, E_M) \equiv g\,(E_{LR}, E_{MR})$; then,

$$\left.\begin{aligned} X &= \lim_{a\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,g(ax, ay)\left\{\frac{x(x-y)}{(x^2+1)\,[(x-y)^2+1]}\right. \\ &\quad \left. + \frac{y(y-x)}{(y^2+1)\,[(y-x)^2+1]} + \frac{xy}{(x^2+1)\,(y^2+1)}\right\} \\ &= \lim_{a\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,g(ax, ay)\,\frac{x^2 - xy + y^2}{(x+i)(x-i)(y^2+1)(x-y+i)(x-y-i)}. \end{aligned}\right\} \tag{A.3}$$

As the integrand vanishes at infinity as x^{-2}, we may close the contour in the complex x-plane; for instance, in positive direction through $+\,i\infty$ around the poles i and $y + i$. Thus, (A.3) yields

$$\left.\begin{aligned} X &= \lim_{a\to 0} 2\pi i \int_{-\infty}^{\infty} dy \\ &\times \left[\frac{[-1-iy+y^2]\, g(ia, ay)}{2i(y^2+1)(2i-y)(-y)} + \frac{[-1+iy+y^2]\, g(ay+ia, ay)}{(2i+y)\, y\,(y^2+1)\, 2i} \right] \\ &= \pi^2 \lim_{a\to 0} [\, g(ia, 2ia) + g(2ia, ia) - g(ia, ia)] \\ &= \pi^2 g(0, 0) = \pi^2 f(E_R, E_R), \end{aligned}\right\} \quad (A.4)$$

which completes the proof of (A.2) and (A.1) or (1.17).

B. Next, we shall show by direct multiplication of the series (3.7) that $\Psi_*^\dagger \Psi_* = 1$. By (3.7) we have

$$\Psi_*^\dagger \Psi_* = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \| (V\flat)^l \cdot (\S V)^{n-l} \| \right) \quad (B.1)$$

with

$$A_0 = 1, \; A_1 = \S V\| + \| V\flat = 0 \quad (B.2)$$

and, for $n \geq 2$,

$$\left.\begin{aligned} \langle L|A_n|R\rangle &= \int \langle L|V|k_1\rangle\, dk_1 \langle k_1|V|k_2\rangle\, dk_2 \cdots dk_{n-1} \langle k_{n-1}|V|R\rangle S(n), \\ S(n) &= \sum_{l=0}^{n} B_l(n), \\ B_l(n) &= \prod_{i=1}^{l} (E_L - E_i - ia)^{-1} \cdot \prod_{j=l}^{n-1} (E_R - E_j + ia)^{-1}. \end{aligned}\right\} \quad (B.3)$$

Here, $\prod_{i=1}^{0} (E_L - E_i - ia)^{-1}$ and $\prod_{j=n}^{n-1} (E_R - E_j + ia)^{-1}$ both mean 1, further E_i means E_R for $i = l = n$ and E_j means E_L for $j = l = 0$.

From these definitions we get the following connection between the expressions $B_l(n+1)$ and $B_l(n)$:

$$\left.\begin{aligned} B_l(n+1) &= B_l(n)\,(E_R-E_n+ia)^{-1} \text{ for } 0\leq l<n \\ B_n(n+1) &= B_n(n)\,\frac{E_L-E_R-ia}{(E_L-E_n-ia)\,(E_R-E_n+ia)} \\ B_{n+1}(n+1) &= B_n(n)\,(E_L-E_n-ia)^{-1}. \end{aligned}\right\} \quad \text{(B.4)}$$

Hence,

$$\left.\begin{aligned} S(n+1) &= \sum_{l=0}^{n-1} B_l(n+1)+B_n(n+1)+B_{n+1}(n+1) \\ &= \sum_{l=0}^{n} B_l(n)\cdot(E_R-E_n+ia)^{-1} \\ &\quad + B_n(n)\left[-(E_R-E_n+ia)^{-1}+\frac{E_L-E_R-ia}{(E_L-E_n-ia)\,(E_R-E_n+ia)}\right. \\ &\qquad\qquad \left. +(E_L-E_n-ia)^{-1}\right]. \end{aligned}\right\} \quad \text{(B.5)}$$

Since the last term in this expression is equal to

$$B_n(n)\,\frac{ia}{(E_L-E_n-ia)\,(E_R-E_n+ia)},$$

it goes to zero as $a \to 0$, and we get from (B.5) the formula

$$S(n+1) = S(n)\cdot(E_R-E_n+ia)^{-1}. \quad \text{(B.6)}$$

Now, for $n = 2$, we have

$$\left.\begin{aligned} S(2) &= (E_R-E_L+ia)^{-1}\,(E_R-E_1+ia)^{-1} \\ &\quad +(E_L-E_1-ia)^{-1}\,(E_R-E_1+ia)^{-1} \\ &\quad +(E_L-E_1-ia)^{-1}\,(E_L-E_R-ia)^{-1} \\ &= \frac{ia}{(E_L-E_1-ia)\,(E_L-E_R-ia)\,(E_R-E_1+ia)} = 0 \text{ for } a\to 0. \end{aligned}\right\} \quad \text{(B.7)}$$

Hence,

$$S(n) = 0 \text{ for } n\geq 2, \quad \text{(B.8)}$$

i. e., by (B.2), (B.3) and (B.1),

$$\left.\begin{aligned} A_n &= 0 \text{ for } n \geq 1, \\ \Psi_*^\dagger \Psi_* &= 1. \end{aligned}\right\} \tag{B.9}$$

Besides (B.9) we have, however, also

$$\Psi_* \Psi_*^\dagger = 1. \tag{B.10}$$

To complete the proof of this equation, we have, according to the considerations of Chapter III, Eqs. (3.7)—(3.11), only to show that the quantities

$$\left. B(n) = \sum_{l=0}^{n} \prod_{i=0}^{l-1} (E_l - E_i + ia)^{-1} \prod_{j=l+1}^{n} (E_l - E_j - ia)^{-1} \text{ for } n \geq 2 \right\} \tag{B.11}$$

are either identically zero or at least contain a factor a and, hence, go to zero for $a \to 0$. In (B.11), $\prod_{i=0}^{-1} (E_0 - E_i + ia)^{-1}$ and $\prod_{j=n+1}^{n} (E_n - E_j - ia)^{-1}$ both mean 1. A common denominator D of the fractions in the sum (B.11) is

$$D = \prod_{0 \leq i < j \leq n} (E_i - E_j - ia), \tag{B.12}$$

where the product is extended over all pairs of indices $i < j$ among the numbers 0, 1, 2 ... n. Hence,

$$B(n) = \frac{N(a; E_0, E_1, \cdots E_n)}{D}, \tag{B.13}$$

where the numerator N is given by

$$N(a; E_0 \cdots E_n) = \sum_{l=0}^{n} \frac{\prod_{0 \le i < j \le n} (E_i - E_j - ia)}{(-1)^l \prod_{0 \le i < l} (E_i - E_l - ia) \cdot \prod_{l < j \le n} (E_l - E_j - ia)} = \sum_{l=0}^{n} (-1)^l \prod_{0 \le i < j \le n}{}' (E_i - E_j - ia). \quad \text{(B.14)}$$

Here, the prime on $\prod$ means that, in the product over all index pairs $i < j$, the factors with $i = l$ and $j = l$ should be left out. $N(a; E_0 \cdots E_n)$ is an algebraic function of the variables $(a; E_0, E_1, \cdots E_n)$ which has no singularities for any value of these variables. Further, we see that

$$N(a; E_0, \cdots E_n) = N(E_0 \cdots E_n) + N_1, \quad \text{(B.15)}$$

where

$$N(E_0, \cdots E_n) = \sum_{l=0}^{n} (-1)^l \prod_{0 \le i < j \le n}{}' (E_i - E_j) = N(0; E_0 \cdots E_n) \quad \text{(B.16)}$$

is the function obtained by putting $a = 0$ in (B.14), while N_1 is a function which contains at least one factor a.

Thus, our statement will be proved if we can show that $N(E_0, \cdots E_n)$ is identically zero for all values of the independent variables $(E_0, E_1 \cdots E_n)$. Let us first assume that all variables $E_0, \cdots E_n$ are different and different from zero; then we can take out an l-independent factor $\prod_{0 \le i < j \le n} (E_i - E_j)$ and we get

$$N = \prod_{0 \le i < j \le n} (E_i - E_j) \cdot C(n), \quad \text{(B.17)}$$

where

$$C(n) = \sum_{l=0}^{n} (-1)^l \frac{1}{\prod_{0 \le i < l} (E_i - E_l) \cdot \prod_{l < j \le n} (E_l - E_j)} = \sum_{l=0}^{n} \prod_{m=0 \, (m \neq l)}^{n} (E_l - E_m)^{-1}. \quad \text{(B.18)}$$

However, $C(n)$ can be proved to be zero for all non-vanishing values of $(E_0, E_1 \ldots E_n)$ which are different from each other.

For the proof we consider the algebraic equation of degree $n \geq 1$:

$$f(x) \equiv \sum_{l=0}^{n} \left\{ \prod_{m=0}^{n}{}_{(m \neq l)} \left(\frac{x - E_m}{E_l - E_m} \right) \right\} - 1 = 0. \qquad \text{(B.19)}$$

A factor $(x - E_k)$ is found in the numerators of all terms in the sum over l, except in the term with $l = k$. Therefore, substitution of $x = E_k$ leaves only the one term with $l = k$ which gives

$$f(E_k) = 1^n - 1 = 0. \qquad \text{(B.20)}$$

Thence, the algebraic equation (B.19) of degree n has at least $(n+1)$ roots $x = E_0, E_1, E_2 \cdots E_n$. This is possible only if the equation is an identity in x. Then, the coefficient of x^n must vanish. This coefficient is (for $n \geq 1$) just equal to $C(n)$ defined by (B.18). Hence, $C(n)$ and consequently N is zero for all non-vanishing and different values of the variables $(E_0, E_1, \cdots E_n)$. But, since $N(E_0, \cdots E_n)$ is a continuous function of these variables, N must then be zero for all values of $E_0, E_1 \cdots E_n$, which completes the proof of the equation (3.11) in Chapter III, i. e. Ψ_* is unitary if the series is so strongly convergent that the series Ψ_* and $\Psi_*^\dagger$ can be multiplied term by term.

C. Throughout this paper, k_0 means a particular set of values of the free particle variables k and quantities like $\langle k \mid Y \mid k_0 \rangle$ are thus matrices in k-space and $\mid k_0 \rangle$ is an eigenket of the k-variables corresponding to the eigenvalues k_0 (plane waves). On the other hand, k_0 is also used to label some of the stationary states (the scattering states) of the total system. For a reader who is accustomed to Dirac's notation, this may lead to confusion, since he might regard $\mid k_0 \rangle$ as identical with one of the scattering states $\mid n \rangle$ of the total system. To him, the matrix $\langle k \mid Y \mid k_0 \rangle$ in case of bound states may seem to be an incomplete matrix, i. e. a "non-square" matrix, as k labels a complete set of plane waves used in Fourier analysis of the field, while k_0 labels an incomplete set of functions. He may easily remedy this as follows.

First, complete $\langle k \mid Y \mid k_0 \rangle$ to a "square" matrix $\langle k \mid Y_m \mid n \rangle$ by adding columns of zeros: $\langle k \mid Y_m \mid k_0 \rangle = \langle k \mid Y \mid k_0 \rangle$; $\langle k \mid Y_m \mid r \rangle = 0$. The subscript m reminds us of the fact that Y_m is defined

in "mixed" (k and n) representations. In zeroth approximation, then, $\langle k | Y_m | n \rangle$ equals a matrix $\langle k | 1_m | n \rangle$, which consists of the unit matrix 1 in k-space bordered by colums of zeros:

$$\langle k | 1_m | k_0 \rangle = \delta(k - k_0); \quad \langle k | 1_m | r \rangle = 0. \text{ Thus,}$$
$$\langle k | Y_m | n \rangle = \langle k | 1_m | n \rangle + \langle k | Z_m | n \rangle.$$

The matrix 1_m obviously has the property $1_m\, 1_m^\dagger = 1$ in k-space, while $1_m^\dagger\, 1_m$ is a matrix in n-space with matrix elements

$$\langle k_0' | 1_m^\dagger 1_m | k_0'' \rangle = \langle k_0' | k_0'' \rangle, \langle k_0' | 1_m^\dagger 1_m | r'' \rangle$$
$$= \langle r' | 1_m^\dagger 1_m | k_0'' \rangle = \langle r' | 1_m^\dagger 1_m | r'' \rangle = 0$$

(even for $r' = r''$). We may then introduce matrices completely in k-space by $Y = Y_m 1_m^\dagger$ (thence, $Y^\dagger = 1_m\, Y_m^\dagger$). These matrices in k-space are the ones appearing in the text; for instance, Y, Z, B, Ψ, T, F, Ω, Γ, G, Q, P, W, R, K, S, 1 are matrices of this type, as well as their Hermitian conjugates $Y^\dagger$, $Z^\dagger$, etc.

These matrices, therefore, are obtained from the corresponding "mixed" matrices Y_m, $Y_m^\dagger$, etc., by first stripping the latter of their columns or rows of zeros, (thus changing $\langle k | Y_m | n \rangle$ into $\langle k | Y | k_0 \rangle$, $\langle n | Y_m^\dagger | k \rangle$ into $\langle k_0 | Y^\dagger | k \rangle$, or a product like $\langle n' | Y_m^\dagger Z_m | n'' \rangle$ into $\langle k_0' | Y^\dagger Z | k_0'' \rangle$, etc.), and then ignoring the difference in interpretation of the labels k and k_0, treating the latter as if it labels a complete set of plane waves just like k does.

D. It should be noted that, by assuming that each of the three sets $\langle k | Y | k_0 \rangle$ (with $Y = \Psi$ or Ω or Q) forms a set of functions of k sufficiently wide to express any scattering state linearly in terms of them, we did not assume that the $\langle k | Y | k_0 \rangle$ would form a complete set in which to express every function of k. That is, we did not preclude the existence of bound states. Nevertheless, for ensuring the existence of matrices X_1 and X_2 as described at the top of page 21, somewhat special assumptions had to be made about the lack of linear dependence of the differences between the sets of scattering states Ψ, Ω, and Q on the bound states ψ_r.

The necessity of these special assumptions for the validity of (2.48) can be made clearer by deriving the alternative form

which the relations (2.48) will take when these conditions are not fulfilled. In general, we find from (2.45), (2.47), and (2.40),

$$\Psi = \Psi S^\dagger S = \Psi \Psi^\dagger \Omega S = (1 - \sum_r \psi_r \psi_r^\dagger)\, \Omega S,$$

$$\Psi = \Psi \left(1 + \frac{iK}{2}\right)\left(1 + \frac{iK}{2}\right)^{-1} = \Psi \Psi^\dagger Q \left(1 + \frac{iK}{2}\right)^{-1}$$

$$= (1 - \sum_r \psi_r \psi_r^\dagger)\, Q \left(1 + \frac{iK}{2}\right)^{-1}.$$

Thence, also

$$\Psi S^\dagger = (1 - \sum_r \psi_r \psi_r^\dagger)\, \Omega, \qquad \Psi \left(1 + \frac{iK}{2}\right) = (1 - \sum_r \psi_r \psi_r^\dagger)\, Q.$$

These relations differ from the ones on page 21 by the occurrence of the factor

$$(1 - \sum_r \psi_r \psi_r^\dagger) = \Psi \Psi^\dagger.$$

From the discussions on pages 18—19, however, it is clear that this factor $\langle k' \mid \Psi \Psi^\dagger \mid k'' \rangle$ acts as the unit matrix whenever it acts on pure scattering states Ψ. Indeed,

$$(\Psi \Psi^\dagger)\, \Psi = \Psi\, (\Psi^\dagger \Psi) = \Psi \cdot 1 = \Psi,$$

by Eq. (2.34a). Our assumption made in words on page 20 now amounts to assuming that also

$$(\Psi \Psi^\dagger)\, \Omega = \Omega, \qquad (\Psi \Psi^\dagger)\, Q = Q.$$

In as far as this is the case, the above formulas reduce to the ones on page 21.

To the authors it seems quite possible that this assumption may turn out to be superfluous, that is, perhaps one can prove the automatic general validity of these equations. For various simple systems, such as the potential scattering of two particles, the mentioned equations are trivial indeed. However, for more

complicated systems, in which there may be bound states and scattering states belonging to the same degenerate energy level, we have not been able to find a general proof. As long as no proof of sufficient generality is available, we regard these equations as a somewhat special assumption, on which Eqs. (2.48), (5.24), and (9.26) are based.

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ELECTRON CAPTURE AND LOSS BY HEAVY IONS PENETRATING THROUGH MATTER

BY

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CONTENTS

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§ 1. Introduction.

The phenomena connected with the penetration of high speed particles through matter have been a most important source of information about atomic processes. The discovery of nuclear fission, which made it possible to study the behaviour of swift ions with great masses and charges, has in this respect revealed many new interesting features, especially as regards the capture and loss of electrons by such ions. Capture and loss phenomena were, as is well known, first observed for α-rays, and have recently received renewed attention through the study of the tracks in photographic emulsions of highly charged ions of cosmic origin, penetrating into the upper regions of the atmosphere. Still, experiments on the stopping and ionizing effects of fission ions, and especially direct measurements of the charge of the ions during their passage through gases and solids, offer so far the most detailed and varied evidence as regards electron loss and capture by heavy ions.

In a previous treatise[1], a general survey of the theoretical interpretation of the effects accompanying the penetration of atomic particles through matter has been given. In particular, it was attempted to account for the peculiar law which governs the energy loss of fission ions along their path by estimating the ion charge which, on account of the displacement of the balance between electron capture and loss, diminishes gradually with decreasing velocity. While the stopping and ionization effects in the beginning of the path depend primarily on collisions with the electrons in the atoms of the medium penetrated, nuclear collisions become of decisive importance at the end of the path. On the assumption that, irrespective of the substance through

[1] N. Bohr (1948). This paper, in which also a survey of the earlier literature is given, will in the following be referred to as I.

which it passed, the number of electrons carried by a fission ion of given nuclear charge was simply related to its velocity, it seemed possible to account approximately for the experimental evidence then available.

In recent years, however, important new information as regards the charge of fission ions in different materials has been obtained by the continued thorough investigations of N. O. Lassen[1]. Thus, the measurements of deflections of ion beams in magnetic fields disclosed not only a systematic deviation from the charge values previously estimated from stopping and ionization effects in gases (Lassen, 1949), but revealed an unsuspected large difference between the average charge of fission ions when emerging from solids and the charge of the ions of the same velocity when passing through gases. In gases, the magnetic deflections of the ion beams also exhibited a smaller, but distinct increase in the average charge with gas pressure. The detailed study of the gradual adjustment of the charge of the ions emerging from solids into rarefied gases allows, moreover, as shown by Lassen, to derive direct estimates of the cross sections for electron capture in collisions with the gas atoms.

The variation of ion charge with the density of the penetrated material permits several conclusions as to the mechanism of the collision processes determining the balance charge. Thus, the dependence of the average charge of swiftly moving heavy ions on the pressure of the gas through which they pass shows clearly that in the balance between loss and capture we may not, as in previous discussions, consider only processes by which electrons are removed from or captured in the ground state of the ions, but also processes involving excited states of a lifetime comparable with the intervals between successive collisions with the gas atoms. The remarkable difference between the average ion charge in gases and solids further indicates the occurrence of adjustment processes in excited ion states, of lifetimes essentially shorter than those of the radiative transitions.

In the present paper, it is attempted on the basis of simple arguments to give a comprehensive interpretation of the pheno-

[1] A survey of the results of these investigations is given in a dissertation: N. O. Lassen, On the total charges and the ionizing power of fission fragments, Copenhagen 1952.

mena associated with the passage through matter of highly charged ions. To this purpose, we consider first, in § 2, some general features of the balance between the loss and capture processes, with special reference to the fluctuations in ion charge, and the gradual adjustment of the average charge of ions emerging from solids into gases. On the basis of a simplified statistical description of the constitution of ions carrying many electrons we discuss, in § 3, some immediate conclusions to be drawn from the measurements of average ion charge and from stopping and ionization effects. In § 4, main features of the mechanism of electron loss and capture by heavy ions in collisions with atoms are considered, and it is attempted to derive approximate estimates for the cross sections of such processes, especially as regards dependence on ion charge and velocity, and the atomic number of the substance penetrated. On the basis of these estimates, a comparison with experimental evidence as regards the charge of fission ions in gases at low pressures is given in § 5. Finally, in § 6, the effect of the residual ion excitation is considered in connection with the observations on the variation of the ion charge with gas pressure, and its abnormally high value in solids.

On the publication of this paper, which due to various circumstances has been delayed, but parts of which have been reported at various conferences in the last years, we want to acknowledge our indebtedness to Dr. N. O. Lassen for many illuminating discussions during progressive stages of his experimental researches and of our theoretical considerations. We are also indebted to Dr. G. I. Bell, who before publication kindly made the results of his interesting studies of the loss and capture mechanism available to us.

§ 2. General features of the balance between loss and capture by heavy ions.

The problem of electron capture and loss by heavy ions presents features essentially different from those exhibited by swift α-particles or protons where, due to the smallness of the ratio between the cross section for capture by the bare nucleus and the loss cross section for an electron attached to it, the nucleus will carry an electron only during intermediate short intervals

which, together, amount to a small fraction of the path. In the case of heavy ions like fission fragments, however, the nucleus will along the whole path carry a large number of electrons which, due to continual capture and loss, fluctuates around an average value determined by the velocity and nuclear charge of the ion and the properties of the medium.

Let us, for simplicity, consider a beam of ions penetrating through a gaseous medium of a density so low that the ions between collisions will practically all have returned to their ground state. The state of the beam as regards the effects of the collisions is therefore fully specified by the number $N(\tau)$ of ions carrying τ electrons. Disregarding, for the moment, loss and capture processes in which more than one electron is involved, we find thus, for the rate of change of $N(\tau)$, within an interval of the path where the velocity may be regarded as constant,

$$\left.\begin{aligned}\frac{dN(\tau)}{dx} = \varrho\{&N(\tau-1)\cdot\sigma_c(\tau-1)\\ &+N(\tau+1)\cdot\sigma_l(\tau+1)-N(\tau)\cdot(\sigma_c(\tau)+\sigma_l(\tau))\},\end{aligned}\right\} \tag{2.1}$$

where ϱ is the number of gas atoms per unit volume, $\sigma_c(\tau)$ the cross section for capture of an electron by an ion carrying τ electrons before the collision, and $\sigma_l(\tau)$ is the cross section for loss of an electron by such an ion. For the rate of change of the average number of electrons, $\bar{\tau} = \bar{\tau}(x)$, carried by the ions, we get from (2.1) by simple summation

$$\frac{d\bar{\tau}}{dx} = \frac{d}{dx}\cdot\frac{\sum\tau N(\tau)}{N} = \frac{\varrho}{N}\cdot\sum N(\tau)\cdot(\sigma_c(\tau)-\sigma_l(\tau)), \tag{2.2}$$

where N is the total number of ions in the beam.

In a beam of heavy ions carrying many electrons the distribution of τ around the mean value will extend over several units and, therefore, a strict application of (2.2) demands a detailed knowledge of the dependence of the cross sections on the number of electrons in the ion. However, the summation in (2.2) is easily performed on the assumption that, in the interval in question, both σ_l and σ_c vary slowly and linearly with τ. We may then write

$$\left.\begin{aligned} \sigma_c(\tau) &= \Omega\cdot(1+\alpha_c\cdot(\tau-\omega)), \\ \sigma_l(\tau) &= \Omega\cdot(1+\alpha_l\cdot(\tau-\omega)), \end{aligned}\right\} \tag{2.3}$$

where α_c and α_l are constants small compared with unity, and ω is the value of τ for which the capture and loss cross sections have equal magnitude, Ω. Introducing the expressions (2.3) into (2.2) we thus get

$$\frac{d\bar{\tau}}{dx} = -\varrho\Omega\cdot(\alpha_l-\alpha_c)\cdot(\bar{\tau}-\omega), \tag{2.4}$$

and by integration

$$\bar{\tau}(x) = \omega + (\bar{\tau}(x_o)-\omega)\cdot\exp(-\varrho\Omega\cdot(\alpha_l-\alpha_c)\cdot(x-x_o)) \tag{2.5}$$

for the average electron number $\bar{\tau}(x)$ at the point x in a beam with a given value for $\bar{\tau}$ at the point x_o.

In a corresponding way, we derive from (2.1) and (2.3)

$$\left.\begin{aligned} \overline{\Delta\tau^2}(x) = \frac{1}{\alpha_l-\alpha_c} + \left\{\overline{\Delta\tau^2}(x_o) - \frac{1}{\alpha_l-\alpha_c}\right\} \\ \cdot\exp(-2\,\varrho\Omega\cdot(\alpha_l-\alpha_c)\cdot(x-x_o)) \end{aligned}\right\} \tag{2.6}$$

for the average square fluctuation of the electron number at the point x. For large values of $(x-x_o)$, where the second term in (2.6) vanishes, the fluctuations will thus depend only on $\alpha_l-\alpha_c$, and the distribution around the average will be Gaussian with a width at half maximum equal to $2.35\cdot(\alpha_l-\alpha_c)^{-\frac{1}{2}}$.

In these simple calculations it is assumed that in every capture and loss process only one electron is removed from or transferred to the ion. Still, in the actual cases, especially in encounters with heavy atoms, there is a considerable probability that several electrons are lost or captured by the ion. However, such effects can easily be included in the above description by introducing in (2.1) further terms corresponding to cross sections $\sigma_l^n(\tau)$ and $\sigma_c^n(\tau)$ for collisions by which the electron number τ is changed by n units. Writing thus, with the same approximation as in (2.3),

$$\left.\begin{aligned} \sigma_l^n(\tau) &= \Omega_n \cdot (1 + \alpha_l^n \cdot (\tau - \omega_n)), \\ \sigma_c^n(\tau) &= \Omega_n \cdot (1 + \alpha_c^n \cdot (\tau - \omega_n)), \end{aligned}\right\} \quad (2.7)$$

we find by the same procedure formulas for the average charge and for the fluctuations corresponding to (2.5) and (2.6) if only Ω, $\Omega \cdot (\alpha_l - \alpha_c)$, and $\Omega \cdot (\alpha_l - \alpha_c) \cdot \omega$ are replaced by $\sum_n \Omega_n \cdot n^2$, $\sum_n \Omega_n \cdot n \cdot (\alpha_l^n - \alpha_c^n)$, and $\sum_n \Omega_n \cdot n \cdot (\alpha_l^n - \alpha_c^n) \cdot \omega_n$, respectively. Thus, collisions involving a change of the electron number by several units may in particular influence the fluctuations, but as long as the value of n in the frequent collisions remains small compared with the average fluctuations, the equilibrium distribution will still be of approximately Gaussian type.

When considering the balance and fluctuations of the ion charge in media of greater density, where a considerable part of the ions, if not all, will remain in excited states between collisions, further considerations are necessary, since the cross sections for loss and capture may to a considerable extent depend on the excitation of the ion. Reckoning with suitably defined mean values for the loss and capture cross sections, depending on the actual degree of excitation of the ions, it is possible, however, to treat the problem in the same simple manner as above. The question of excited ion states may even have to be taken into account as regards balance between loss and capture for α-rays, but in this case the effect will in general be of minor importance due to the small electron binding in excited states, contrasting with the properties of ions carrying many electrons, where the excitation potentials may be several times smaller than the ionization potentials.

For fission ions escaping into vacuum from a solid surface, magnetic deflections permit measurements of the charge of the individual ions at a definite point of the path. In a gaseous medium, however, the continual change of ion charge, due to electron loss and capture, allows only to determine the average charge over a considerable part of the path. Still, by varying the gas pressure in the deflection chamber, LASSEN was able to study in detail the gradual decline in average ion charge from the values in solids until balance in the gas is reached. The decline in charge is

at first very rapid, showing a preponderance of electron capture over loss, but diminishes gradually and, in agreement with expectations, the average ion charge approaches a flat minimum through an approximately exponential slope (cf. LASSEN, 1950, fig. 2). The experiments on ion deflection in vacuum give not only values of the average charge higher than in gases, but exhibit characteristic charge fluctuations with approximate Gaussian distribution (cf. LASSEN, 1950, fig. 1). Notwithstanding the different conditions for the ions passing through solids, these fluctuations give, as we shall see, information about the dependence of the capture and loss cross sections on ion charge, supplementing the deductions which can be drawn from the gradual adjustment of the average charge of the ions emerging into gases.

§ 3. Approximate description of ion constitution.

A rigorous treatment of the collisions between highly charged ions and atoms presents us with complicated problems. An approximate account of the collision effects may, however, be obtained by means of a simplified description of atomic constitution (cf. I, § 3.5), in which the binding of the electrons is defined by the simple concepts of orbital extensions and velocities, using as a measure

$$a_0 = \frac{\hbar^2}{me^2} \quad \text{and} \quad v_0 = \frac{e^2}{\hbar}, \tag{3.1}$$

representing the "radius" and "velocity" of the electron in the ground state of the hydrogen atom. For an electron in an ion or atom we introduce in a similar way a radius a, characterizing the extension of the orbital region, and a velocity v, defined by

$$I = \frac{1}{2} mv^2, \tag{3.2}$$

where I is the binding energy. For an atom or ion with nuclear charge Z we thus write

$$a = a_0 \cdot \frac{\nu^2}{n}, \quad v = v_0 \cdot \frac{n}{\nu}, \tag{3.3}$$

where ν may be interpreted as the effective quantum number of the binding state, and $Z—n$ is the number of electrons with orbital radius smaller than a and, consequently, velocities larger than v.

For the ground state of an atom, ν will increase from a value close to unity for the most firmly bound electrons to a broad maximum and, finally, for the outermost atomic electrons, decline again to values of the order 1. For atoms containing many electrons, the maximum of ν will with close approximation be equal to $Z^{\frac{1}{3}}$, and from (3.3) we therefore get

$$dn = Z^{\frac{1}{3}} \cdot \frac{dv}{v_o} \tag{3.4}$$

as an approximate expression for the velocity distribution of the larger part of the electrons bound in the ground state of a heavy atom. The excitation of the atom demands the transfer of one or more electrons from the normally occupied states into unoccupied higher energy states. In the neutral atom such processes will for every electron require an energy exchange of the same order as the binding energy I, though in the case of inner electrons, part of this energy may be released in subsequent readjustment processes resulting in the excitation of other electrons and even in their ejection from the atom. In actual collision processes, a separation in well-defined stages is, however, limited and demands a closer comparison of the effective duration of the encounter and the times involved in the dynamics of the atomic processes.

The simplified description applies also approximately to the ground state of heavy ions of a total charge Z^*, corresponding to a considerable fraction of the nuclear charge. Still, since the maximum value of ν is not reached until $Z—n$ exceeds $Z/2$, it is essential for the applicability of formula (3.4) that Z^* is somewhat smaller than half the nuclear charge. As regards the excited states of highly charged ions, the situation is, moreover, in several respects different from that of neutral atoms, due to the presence of numerous unoccupied quantum states with comparatively strong binding. In fact, if by ν^* we denote the effective quantum number for the most loosely bound electrons in the ground state

of the ion, with ionization potential I^*, the energy required for a larger part of the possible excitation processes will only be of the same order as I^*/v^*.

For heavy ions we must in general reckon with a distribution of the excitation over several electrons. Not only will in actual collision processes often more than one electron be initially ex-

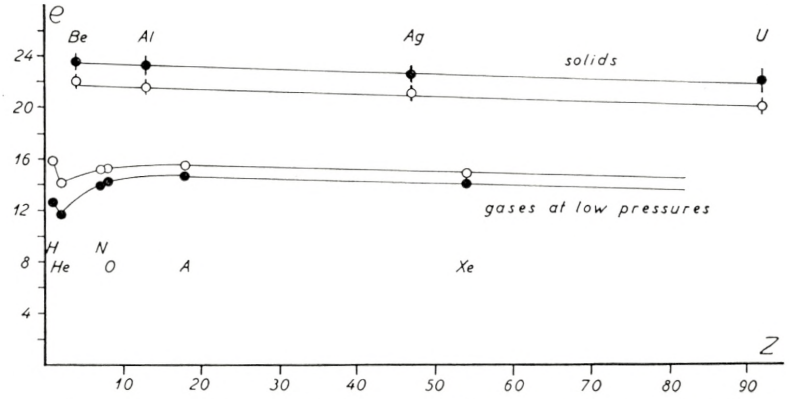

Fig. 1. (LASSEN, 1951a, fig. 12). Average balance charge of fission ions with initial velocities, in solids and in gases at low pressures. The full circles refer to the heavy fragment group ($Z = 54$, $V = 4\,v_0$), and the open circles to the light group ($Z = 38$, $V = 6\,v_0$).

cited, but redistributions of the excitation over the electrons can even in the case of less violent encounters take place in immediate succession of the collisions. If the total surplus energy of the ion exceeds I^*, the result will be electron ejection within an interval very short compared with the limitation of the lifetime due to radiation processes. Also for the estimation of the lifetime of the excited ions and their properties, it is essential to bear in mind that an excitation energy below I^* will ordinarily be distributed over several electrons.

For orientation as regards the values of Z^* of swift heavy ions, a survey of LASSEN's measurements of the charge of fission ions at the beginning of the path in solid materials and in gases at low pressures is given in Fig. 1. It is seen that, apart from some interesting anomalies in the lightest gases, the ion charge is nearly independent of the atomic number of the gas for both groups of fission ions. The same applies to the ion charge in solids, notwithstanding the remarkable difference from gases as regards

absolute values, and the peculiar inversion of the relative charge values of the two groups of fission ions.

The explanation of such specific features will of course demand a closer examination of the collision processes between the ions and the atoms of the penetrated substances and, in particular, of the state of the ions at the beginning of the encounters. For a preliminary discussion it may, however, be reminded that cursory considerations of the competition between electron capture and loss by ions in their ground state lead to the conclusion that, in balance, the velocity v^* of the most loosely bound electrons in this state of the ion should be nearly equal to the ion velocity V. According to (3.4), this gives

$$Z^* = Z^{\frac{1}{3}} \cdot \frac{V}{v_0} \tag{3.5}$$

as a rough estimate of the ion charge in balance (cf. I, § 4.4).

This estimate actually coincides closely with LASSEN's direct measurements of the average charge in gases at low pressures, for the heavier group of fission ions. In fact, for $V = 4\, v_0$ and $Z = 54$, we get from (3.5) the value $Z^* = 15$. For the light group of fission ions ($V = 6\, v_0$, $Z = 38$), however, we would from (3.5) get $Z^* = 20$, while the measured value for the average charge is about 16. Quite apart from the question of the basis for a comparison of absolute charge values, the apparent discrepancy in the relative values is easily explained by remembering that (3.4) is applicable only in cases where Z^* is somewhat smaller than $Z/2$. This condition is amply fulfilled for the heavier ion group, but not for the lighter fission ions, with the consequence that $Z^{\frac{1}{3}}$ in (3.5) must be replaced by a somewhat smaller value of ν.

Such a difference between the two ion groups is also clearly revealed by the stopping and ionization effects of fission ions penetrating through gases. In Fig. 2 are, as an illustration, reproduced LASSEN's results as regards the energy loss along the path in argon for the two groups. As will be seen, the curves are composed of two parts, corresponding to ion velocities large and small compared with v_0, and in which the stopping effects are due mainly to electronic and nuclear collisions, respectively. For

the heavier ion group, the energy loss decreases linearly in the first part of the range. But for the lighter group, anomalies are exhibited at the beginning of the path, and the linear descent appears only after the velocity and the ion charge have decreased considerably from their initial values. As mentioned in I (§ 5.3), it follows from simple theory of energy loss by charged particles[1] that, in heavier gases, a linear decrease in energy along the path

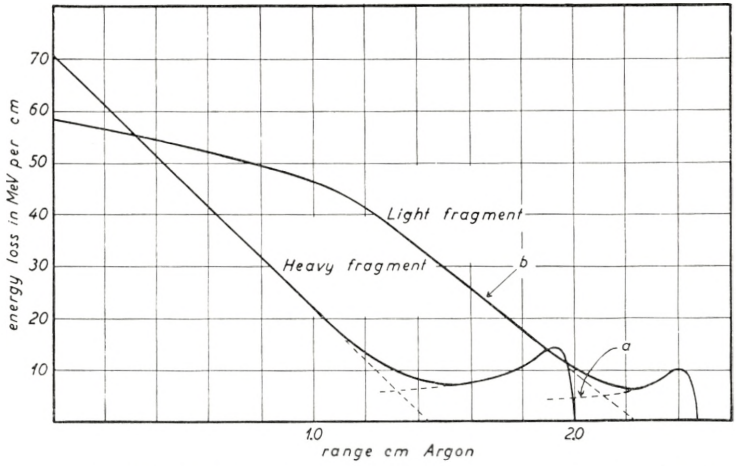

Fig. 2. (LASSEN, 1949, fig. 23). Energy loss along the path of fission ions in argon. In the last part of the path, beyond the minimum, the energy loss by nuclear collisions is dominating. The magnitude of the separate contributions (a and b) from nuclear and electronic collisions is indicated by the dotted lines.

implies a proportionality between Z^* and V, corresponding to (3.5). While, for the heavier group of fission ions, this relation apparently applies for a large part of the range, $Z^{\frac{1}{3}}$ must for the lighter ion group evidently be replaced by a factor which

[1] This theory is especially developed in the case of particle charges and velocities for which quantum mechanical perturbation methods apply with high approximation. Recently, it has been shown (cf. LINDHARD and SCHARFF, 1953) that it is possible, by means of a simple statistical treatment of atomic structures, on this basis to account for the stopping power over a wide region of atomic numbers and particle velocities. As discussed in I, special considerations are necessary in the case of highly charged particles where the conditions of perturbation theory are not fulfilled. The estimate given in I (§ 3.5) of the stopping power of a heavy atom needs, however, a certain correction. In fact, if in this estimate the dynamics of the electron binding is taken into account on the lines used by LINDHARD and SCHARFF, the resulting stopping power will, like for α-rays in the same velocity region, not only be approximately proportional to the square of the ion charge and inversely proportional to ion velocity, but will also vary closely as the square root of the atomic number of the penetrated material.

in the beginning increases significantly with decreasing particle velocity.

As regards more quantitative estimates of the average ion charge from stopping and ionization effects of heavy ions, various complications have to be taken into account. Actually, earlier estimates of the ion charge from ionization in gases, based on the penetration theory of point charges, led to results almost as high as those obtained by direct charge measurements of the ions emerging from solids into vacuum. To explain this discrepancy, it is necessary to take the complex structure of the ion into account. In fact, in close collisions the atomic electrons will penetrate into the interior of the ion, where the effective nuclear charge is considerably higher than Z^*. The correction was in I (§ 4.4 and § 5.3) deemed to be insignificant because the collision diameter b, which in the stopping formula appears as an effective minimum impact parameter, is just equal to the diameter of the ion. However, the contribution of close collisions to the stopping effect is relatively large for fission ions, since the semi-adiabatic limit to the impact parameter in more distant encounters is only a few times larger than b. This circumstance makes an accurate evaluation of the stopping power difficult, but a simple calculation shows the correction due to ion structure to be of the order of magnitude required to explain the differences between the earlier estimates of the charge of fission ions in gases and the direct charge measurements.

§ 4. Mechanism of electron loss and capture.

In encounters between highly charged ions and neutral atoms, considerable changes in electron binding may take place particularly in the atom, where the more loosely bound atomic electrons at an early stage of the collision will be greatly influenced by the strong field around the ion. The transfer of energy accompanying the excitation and ionization of the atoms will, in fact, be the main source of energy loss of the ions. In the collisions, however, also processes can take place resulting in an excitation of the ion, or a change of ion charge due to electron capture and loss. A rigorous treatment of these processes presents a problem of great complication, but, due to the circumstance that the binding states

in the ion involved in electron loss and capture are specified by high quantum numbers, simple mechanical considerations can be used in approximate treatments, and especially for the survey of the essential features of the mechanism of the different processes.

In the loss process, it is a question of a transfer of energy to ion electrons in the collision sufficient for electron escape. Due to the smallness of the forces acting between neighbouring electrons in the ion compared with the total ion field, we may, in considering such energy transfer, in first approximation examine separately the influence on the binding of individual electrons under the action of the forces to which they are exposed during encounters with atoms. In estimating these forces, however, we may only for light atoms compare the collision with separate impacts of the nucleus and the atomic electrons. For heavier atoms, where the orbital velocities of part of the electrons are larger than the particle velocity V, we must take into account that the charges of these electrons during the collision will effectively screen the charge ze of the nucleus, together with which they will act as an atomic core of a total charge number z^*, approximately equal to $z^* = z^{\frac{1}{3}}\,(V/v_0)$, corresponding to (3.5). Since the electrons more loosely bound to the atomic nucleus, due to their small charge and mass, are not able individually to transfer energy to the ion of the magnitude required, the main contribution to the loss process arises from the direct action of the bare nucleus in light atoms, and the atomic core in heavier atoms.

In order to estimate the loss cross section, we recall that the cross section for energy transfer greater than T in a collision between a free electron at rest and a heavy particle with charge z^*e and velocity V is given by the well-known formula (cf. I, § 3.1)

$$\sigma = 2\pi a_0^2 z^{*2} \cdot \left(\frac{v_0}{V}\right)^2 \cdot \left(\frac{mv_0^2}{T} - \frac{mv_0^2}{T_{\max}}\right), \tag{4.1}$$

where $T_{\max} = 2mV^2$ is the upper limit for energy transfer in such a collision.

Introducing for each ion electron $T = mv^2/2$, and summing by means of formula (3.4), we get from (4.1) as a first estimate of the loss cross section

$$\sigma_l = \pi a_0^2 \cdot z^{*2} \cdot Z^{\frac{1}{3}} \cdot \left(\frac{v_0}{v^*}\right)^3, \tag{4.2}$$

where z^* stands for the atomic number z, or core charge, for light and heavy gases, respectively, and where the binding of the most loosely bound electron in the ground state of the ion is characterized by the velocity v^*, close to V.

Such cursory consideration needs, however, essential corrections of different kind. In fact, the neglect of the effect of the electron binding during the encounter is not justifiable, as the orbital velocities are of the same order as V, and especially since the duration of the encounter is comparable with the orbital frequencies. Due to these circumstances, the estimate (4.2) of the cross section for direct removal of the ion electrons is somewhat too large, but in the estimation of the loss cross section it must be taken into consideration that, due to subsequent readjustment of the electron binding in the ion, electron escape will take place if only the total energy transfer to the ion in the encounter exceeds the binding energy I^* of the most loosely bound electron in the ground state. Still, the corrections due to these various effects, which are not strictly separable, may be expected largely to cancel, and it is in this respect interesting that the estimate of the loss cross section for fission ions in several gases, obtained by Bell (1953) by numerical computation, based on a somewhat different simplifying procedure, agrees approximately with the more comprehensive formula (4.2). We may therefore use this formula as a guide in the analysis of the experiments, and especially in the estimation of the variation of the loss cross section with ion charge.

Besides electron loss, the encounters with the atoms will result in excitation of the ion. An estimate based on the simple formula (4.1) gives in fact a cross section for excitation by direct impact of the same order of magnitude as the loss cross section. Even if part of the excitation energy by subsequent readjustment will be spent in electron escape, we must therefore reckon that the collisions will result in excitation of the ion, amounting on the average to about $I^*/2$. In gases at low pressures, this excitation will be dissipated by radiation between collisions, but at higher pressures we have to take into account initial ion excitation in

the encounters, with the result that the total loss cross section is increased. Thus, by a simple estimate based on (4.1), we obtain for an average residual excitation εI^* at the beginning of the encounter a relative increase ε in the loss cross section.

An estimate of the cross section for electron capture by the ion demands a somewhat more detailed consideration of the course of the encounter between the atom and the ion. In fact, the possibility of capture of an electron by the ion will largely depend on the circumstances under which it is released from the atom. Let us consider an atomic electron with orbital velocity v and radius a, as given by (3.3). During the approach of the highly charged ion, the electron will be exposed to a strong field of force, giving rise to an increasing polarization of the binding, which may subsequently lead to its rupture. In order to estimate when electron release takes place, we note that, at a distance R between the two systems given by

$$\frac{Z^* e^2}{R^2} = \frac{mv^2}{a}, \tag{4.3}$$

the force from the ion and the atomic binding force are approximately equal. Still, it has to be taken into account that the possibility for electron release is not only determined by a comparison between the forces, but that the completion of the process will require a time of the order a/v, and that therefore, especially in the case of the more loosely bound atomic electrons, the ion may have travelled a distance comparable with R before the electron is liberated from the atomic field.

After the release from the atom, the electron will be captured if its total energy relative to the ion has a negative value. In his estimate of capture cross sections, on similar lines as followed here, BELL (1953) assumes that an atomic electron is released at a distance R from the ion with velocities corresponding to the momentum distribution in its original binding state. It must, however, be taken into consideration that, under the combined action of the atom and ion fields, the electron velocity distribution will have changed considerably from that in the isolated atom, and that we must expect the velocity of the electron to be largely reduced during the gradual loosening of the atomic binding. At

the completion of the release process, we may thus in first approximation assume that the velocity of the electron relative to the ion will not differ essentially from the ion velocity. On such assumptions, the condition for capture is that the process of electron release is effectively completed at a distance from the ion smaller than R', determined by

$$\frac{Z^* e^2}{R'} = \frac{1}{2} m V^2. \tag{4.4}$$

Assuming, in first approximation, that the release takes place at the distance R, we find that, if $R < R'$, capture occurs with a cross section πR^2, while for $R > R'$ there will be no capture. According to (4.3) and (4.4), it is seen that on this assumption only strongly bound atomic electrons can contribute to capture. Actually, in a heavy atom, the contribution will arise mainly from a comparatively narrow region of orbital velocities around $V/2$. Summing over the electrons in the atom, we obtain by means of formula (3.3) the approximate estimate

$$\sigma_c = \pi a_o^2 Z^{*2} \cdot z^{\frac{1}{3}} \cdot \left(\frac{v_o}{V}\right)^3 \tag{4.5}$$

for the total capture cross section for atoms in which a considerable part of the electrons have velocities comparable with V.

Notwithstanding the cursory character of the description of the capture process, the formula (4.5) may be expected not to be far in error, because the uncertainties introduced by the estimates of R and R' will be largely eliminated by the summation over the atomic electrons. This circumstance was also noted by BELL (1953) in his numerical computation of capture cross sections in several gases by fission ions of various charges and velocities. In spite of the different assumptions used by BELL as regards the kinetic energy of the released electrons, his results for heavier gases also agree approximately with formula (4.5). Moreover, it must be noted that the formulas (4.2) and (4.5) imply that, in a close encounter with a heavy atom, several electrons will be lost and captured by the ion, and that due to subsequent readjustment

of ion excitation the various processes are not strictly separable in the resulting effects.

As regards capture in the lightest gases, we meet with an essentially different situation. In fact, for ions of high charge and velocity, the calculations leading to the capture cross section (4.5) would give no contribution for electrons bound in the lightest atoms, because for these the distance of release, R, would be larger than R', the limit where capture becomes possible. In order to explain the occurrence of capture, we must take into account that the release is a gradual process and, although R may represent the average release distance, escape will take place only with a probability per unit time comparable with (v/a), and thus over a considerable path. Accordingly, there is a small probability that a loosely bound electron will remain with the atom until the distance from the ion is so small that capture can take place.

The detailed analysis of the process presents, of course, a complicated problem, but by an estimate relying on simple mechanical concepts, and assuming that the probability of electron release from the atom within a distance from the ion smaller than R' is of the order $(R'/V)(v/a)$, we get

$$\sigma_c = \pi a_o^2 Z^{*3} \cdot \left(\frac{v_o}{V}\right)^7 \cdot \frac{n'^2}{\nu'^3} \tag{4.6}$$

as a cursory estimate of the capture cross section for a very loosely bound atomic electron, with a binding characterized by a screened nuclear charge $n'e$ and an effective quantum number ν'.

For the discussion of residual ion excitation, we must take into consideration that the electrons will in general be captured in highly excited states. In fact, for heavy atoms to which formula (4.5) applies, the average excitation of an electron after capture by an ion in the ground state will be about $\frac{2}{3} I^*$ while, in the case considered in (4.6), the excitation will in general be still higher and closely approach I^*. As regards the examination in § 6 of the effect of residual excitation on the balance at higher pressures, we further note that, in contrast to the increase in the loss cross section due to residual excitation, discussed above, we must expect a decrease in capture cross section due to subsequent readjustment of the electron binding. Thus, an average residual

excitation at the beginning of the encounter, amounting to εI^*, will give rise to a relative decrease in the estimate (4.5) of magnitude $\frac{3}{2}\varepsilon$ and even more in the estimate (4.6).

§ 5. Discussion of experimental evidence on capture and loss by fission ions in gases at low pressures.

In order to ascertain how far the approximate estimates of loss and capture cross sections, given in § 4, may be used as a guidance for discussion of the experimental evidence, it may be recalled that, while the estimate (4.2), with proper definition of z^*, applies to electron loss in both light and heavy gases, we have as regards the capture problem in the two cases to do with essentially different mechanisms, leading to the estimates (4.5) and (4.6), respectively. In the comparison with the experimental evidence, we shall therefore treat the two cases separately.

In the case of the heavy gases, in which the binding of a major part of the atomic electrons is characterized by orbital velocities comparable with or exceeding the ion velocity V, the formulas (4.2) and (4.5) give simple variations of the capture and loss cross section with ion charge, and in opposite directions. In fact, the capture estimate (4.5) is proportional to Z^{*2}, while the loss cross section (4.2) is inversely proportional to v^{*3} and therefore varies approximately as Z^{*-3}. In particular, we note that the two expressions in all heavier gases become equal for a value of the velocity of the most loosely bound ion electrons closely given by $v^* = V$, in agreement with the cursory estimate of the balance charge used in the discussion in § 3.

From (4.2) and (4.5) we get, with the notation of § 2,

$$\Omega = \pi a_0^2 Z^{\frac{1}{2}} \cdot z^{\frac{1}{2}} \cdot \frac{v_0}{V} \tag{5.1}$$

for the equal loss and capture cross section in balance. As regards the estimates of the mean free path between collisions involving electron capture and loss, and determining for the dependence of balance charge on gas pressure (cf. § 6), it must, however, be taken into consideration that just in heavier gases several

electrons will in general be exchanged in the encounters, and that we must therefore reckon with a somewhat larger value for the mean free path than would correspond to (5.1).

At low pressures, the experimental results, given in Fig. 1, show that the balance charge in heavier gases is nearly independent of atomic number, as also corresponds to the theoretical

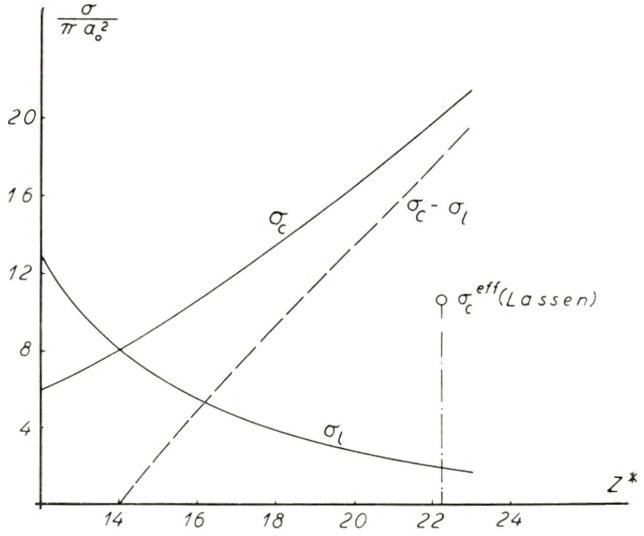

Fig. 3. Capture and loss cross sections for the heavier group of fission ions with initial velocities in argon at low pressures, as functions of ion charge Z*. Comparison with the average effective capture cross section estimated by LASSEN.

expectations. However, it is to be kept in mind that, for such comparison, we are in the first place only dealing with the ratio between the loss and capture cross sections, and that the rapid and opposite variation of the cross sections implies that the balance charge is not very sensitive to this ratio. It is therefore important that an approximate test of the numerical values of the expressions (4.2) and (4.5) can be obtained from LASSEN's studies of the transitional effects observed for ions emerging from solids into gases.

For the heavy group of fission ions with initial velocities in argon at low pressures, the theoretical estimates σ_l and σ_c as functions of Z^* are represented by the two curves in Fig. 3. The intersection point of the curves, corresponding to the balance

charge, agrees closely with the experimental value. The dotted curve in the figure represents the difference between the capture and loss cross section for ion charge higher than the balance value. Further is on the figure indicated LASSEN's estimate of the effective capture cross sections for the average ion charge of the heavy fission fragments emitted from a solid surface. As

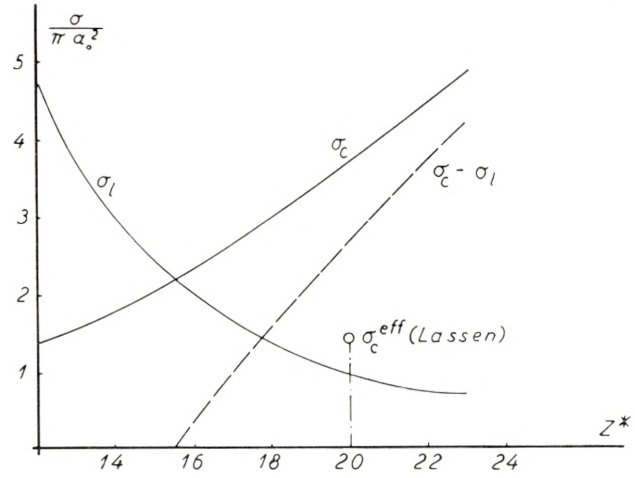

Fig. 4. Capture and loss cross sections for the lighter group of fission ions with initial velocities in argon at low pressures.

mentioned in § 2, this estimate was deduced from the rate of decline in charge for ions emerging into the gas chamber, and it was, for simplicity, assumed that in these effects electron loss could be neglected and the capture cross section considered as constant over the charge interval in question. Considering that, in the transitional effects, we have to do with an averaging over the difference between the capture and loss cross sections within an interval between the charge of the emerging ions and the balance charge in the gas, it is seen that LASSEN's estimate is in quite satisfactory agreement with the cross section curves in Fig. 3.

In a similar way, the curves in Fig. 4 represent theoretical estimates for σ_l, σ_c, and $\sigma_c - \sigma_l$ for the light ion group in argon. Still, in conformity with the considerations in § 3, we have in formula (4.2) introduced, instead of $Z^{\frac{1}{3}}$, a somewhat smaller value of the effective quantum number, so as to obtain coincid-

ence between the intersection point of the curves and the measured balance charge. It will be seen that LASSEN's estimate for the effective capture cross section of light fission ions emerging from a solid surface into the gas is also consistent with the theoretical expectations when considering the run of the curves within the charge region of the transitional effects.

As shown in § 2, the rates of variation with ion charge of the cross sections for loss and capture are determining for the charge fluctuations of the ions along their path. The formulas (4.2) and (4.5) lead to the value $1/(\alpha_l - \alpha_c) = Z^*/5$ in heavier gases at low pressures. While the charge fluctuations in a gas escape direct measurements, it is interesting that this estimate of the average square fluctuation corresponds approximately to the observation of the charge fluctuations of fission ions emerging from solids (LASSEN 1950, 1951 a).

As regards the competition between loss and capture in lighter gases, it is seen that, while the capture cross section for fission fragments in air should be approximately given by the formula (4.5), the charge of the atomic core entering in the loss cross section (4.2) will be somewhat smaller than the value $z^{\frac{1}{3}}\,(V/v_o)$ holding for heavy substances. Thus, the balance charge may be expected to be slightly lower in air than in argon, as was also found by LASSEN (cf. Fig. 1). The anomalies in average ion charge in the lightest gases like helium and hydrogen are, however, of particular interest. Especially the comparatively high value of the ion charge in hydrogen points to a decrease in the capture cross section even more rapid than the decrease in loss cross section, which for the lightest elements is proportional to z^2.

Although the estimate (4.6) may not give accurate numerical results, the relative variations with atomic number, ionic charge and velocity are expected not to be far in error. Such dependence is brought out by comparing (4.6) with the loss cross section (4.2) for the light and heavy fission groups with initial velocities in H_2 and *He*. The measurements show here that for both groups of fission ions the average charge in *He* is about 10 % lower than in H_2. This circumstance is readily explained from (4.2) and (4.6), since σ_l varies as z^2, while σ_c is nearly proportional to z^3 and, accordingly, the charge must remain slightly smaller in *He*.

For a more quantitative comparison with (4.2) and (4.6), we calculate σ_c, σ_l, and the balance charge for the two groups of fission ions in H_2 and He. In order to compute σ_l as a function of Z^*, one must know the value of the effective quantum number, ν^*, for the most loosely bound ion electrons. For the heavy group of fission ions, we may put ν equal to $Z^{\frac{1}{3}}$, while for the light group we can take the somewhat lower value given by the measured balance charge in argon at low pressures, assuming that the velocity of the most loosely bound ion electrons is $v^* = V$. The theoretical estimates of the balance charges are given in Table 1, together with the measured balance charges, and it is seen that the agreement is quite close. In the table is also given the effective capture cross section calculated by LASSEN from the transitional effects for ions emerging into gases from solids. The comparison with the difference of the theoretical estimates of σ_c and σ_l for the charge value of the emerging ions shows agreement, at any rate as regards order of magnitude.

Also the measurements of the balance charge of fission ions with lower velocities (LASSEN, 1951 a) bring out a difference between heavy and light gases, which seems to be in approximate agreement with the theoretical estimates. Thus, the observation that for argon Z^* is closely proportional to V for the heavier ion group, while for the lighter group Z^* varies more slowly with ion velocity, is in conformity with the assumption that in both cases the velocity v^* of the most loosely bound ion electrons is closely equal to V. In the lightest gases, however, an approximate pro-

TABLE 1.

Balance charge and effective capture cross section (in units of πa_0^2) of fission ions with initial velocities in H_2 and He. Comparison between measurements on the ions emerging from uranium (LASSEN, 1951 a, 1954) and theoretical estimates based on (4.2) and (4.6).

	H_2		He	
	heavy	light	heavy	light
Z^*_{bal} (exp.)	12.7	15.8	11.6	14.1
Z^*_{bal} (th.)	12.2	15.7	10.9	14
σ_c^{eff} (exp.)	0.9	0.025	3.2	0.3
$\sigma_c - \sigma_l$ (th.)	0.9	0.02	(7.5)	0.2

portionality between Z^* and V was found both for the heavy and light ion group, corresponding to a different connection between v^* and V, as is also borne out by a comparison between (4.2) and (4.6).

§ 6. Dependence of average ion charge on density of material.

Even if collisions with atoms will generally leave the ion in an excited state, we may in gases at low pressures assume that such excitation will be dissipated by radiation between successive collisions, and that the average charge of the ions simply depends on the cross sections for capture and loss by the ion in the ground state. In gases at higher pressures, or in solid materials, we must, however, take into account that the ions to a greater or smaller extent will remain in an excited state, and for an estimation of the average ion charge the influence of the residual excitation on the balance between loss and capture must be considered.

As already mentioned, the excitation of the ion produced directly by the collisions with the atoms will in general be shared among the ion electrons and, if larger than the minimum ionization potential $I^* = mv^{*2}/2$, give rise to subsequent electron ejection. While in solids such adjustment may not be completed between successive encounters with the atoms, we may in gases, even at comparatively high pressures, assume that the ions at the beginning of each collision have a more or less distributed excitation, never exceeding I^*. In heavier gases, the average excitation after a collision will be about $I^*/2$, but in the lightest gases like hydrogen, and especially for swiftly moving ions, capture will result in a very loose electron binding, and the average excitation may be somewhat higher.

In order to estimate how large a part of the excitation of the ion will remain between collisions in the gas, we shall assume that its dissipation by radiation is characterized by a mean lifetime, τ, corresponding to an ion path τV. Reckoning with a mean free path λ of the ion between collisions, we get therefore, in a well-known manner, that the fraction of the ions which on the average have retained the excitation between collisions will be given by $\tau V/(\tau V + \lambda)$. Assuming that the collisions only

involve capture or loss of a single electron, we may, with the notation of § 2, write $\lambda = 1/(2\,\Omega\,\varrho)$, and we get thus for the ratio $\bar{\varepsilon}$ between the average residual excitation and I^*

$$\bar{\varepsilon} = \frac{\tau V \Omega \varrho}{2\,\tau V \Omega \varrho + 1}. \tag{6.1}$$

In the case of heavy gases, where the relatively large probability for loss and capture of several electrons in close collisions may necessitate the use in (6.1) of a value somewhat smaller than Ω given by (5.1). For the lightest gases, however, due to the considerable probability of collisions giving rise to excitation without electron loss or capture, the value Ω in (6.1) should be replaced by a somewhat larger cross section.

In order to estimate the influence of the residual excitation on the balance between loss and capture, we shall, in direct generalization of (2.3), write

$$\left.\begin{aligned} \sigma_c &= \Omega \cdot (1 - \beta_c \varepsilon + \alpha_c \cdot (\tau - \omega)), \\ \sigma_l &= \Omega \cdot (1 + \beta_l \varepsilon + \alpha_l \cdot (\tau - \omega)), \end{aligned}\right\} \tag{6.2}$$

where Ω and ω as well as the constants α_c and α_l refer to the ground state, while $\beta_c \varepsilon$ and $\beta_l \varepsilon$ are the relative variations in the cross sections for excitation εI^*.

In the absence of excitation, the balance charge of the ion is $Z - \omega$, and the equations (6.1) and (6.2) thus imply a shift in balance charge, of magnitude

$$\Delta Z^* = \frac{\beta_l + \beta_c}{\alpha_l - \alpha_c} \cdot \bar{\varepsilon} = \frac{\beta_l + \beta_c}{\alpha_l - \alpha_c} \cdot \frac{\tau V \Omega \varrho}{2\,\tau V \Omega \varrho + 1}. \tag{6.3}$$

For low densities the shift ΔZ^* is proportional to ϱ, while for high densities it reaches a maximum value, $(\beta_l + \beta_c)/2\,(\alpha_l - \alpha_c)$. Introducing for $\alpha_l - \alpha_c$ values corresponding to (4.2), (4.5) and (4.6), and for $\beta_l + \beta_c$ the estimates in § 4, we get for the maximum value of ΔZ^* about $Z^*/5$ in heavy, and slightly more in light materials. This result is in good agreement with the experiments by Lassen (1951b), where the average charge with increasing pressure seems to reach a constant value about 3 units higher than at the lowest pressures.

While the constant charge value at higher pressures is independent of the emission of radiation, the initial increase in ion charge at low pressures is a direct consequence of the competition between collisions with gas atoms and dissipation of excitation by radiation. The above simple description of the radiative

TABLE 2.

Measured values of p_1, for the two groups of fission ions in various gases (LASSEN 1951 a, b) and the corresponding lifetimes. The uncertainty in p_1 may be a factor ~ 2.

	Heavy group			Light group		
	H_2	He	A	H_2	He	A
p_1 mm	11	12	4	30	15	5
$\tau \cdot 10^{11}$ sec	2.7	1.2	0.2	4	3.5	0.4

decay by an effective lifetime τ is in agreement with the observed approximative linear increase in ion charge with gas pressure. In Table 2, the values of p_1 represent the pressure for which the average ion charge has increased by one unit, estimated from the slope of LASSEN's curves in various gases. The table also gives the corresponding values for the radiative lifetime τ, deduced from (6.3).

As a simple estimate of the radiative lifetime τ of an excited electron state, we may write

$$\tau \simeq \tau_0 \cdot \frac{\nu^5}{Z^{*4}}, \qquad \tau_0 = 0.9 \cdot 10^{-10} \text{ sec}, \tag{6.4}$$

where Z^* is the charge of the ion, and ν an effective quantum number somewhat higher than, but comparable with the quantum numbers of the most loosely bound electrons in the ground state of the ion. The radiative lifetimes to be expected from (6.4) are of the same order of magnitude as those derived from (6.3) and given in Table 2. Moreover, the larger values for τ in hydrogen and helium, compared with argon, may perhaps be explained by the smaller ion charge and the higher excitation states of ion electrons to be expected in the lighter gases. Still, such closer comparison contains much uncertainty, especially in the estimate of Ω, which quantity, as already mentioned, may have to be

considerably increased in the lighter gases, in a way which may at any rate partially account for the larger estimate of τ in hydrogen and helium compared with argon.

While in gases at comparatively low pressures, the time between collisions can, as we have seen, be of the order of the radiative lifetimes of excited electron states on the ion, the passage of ions through solids implies an extremely rapid succession of collisions and, as in gases at high pressures, the dissipation of ion excitation by radiation can be neglected. However, even in solids, the collision frequency, V/λ, will remain smaller than the revolution frequency, $\omega = v/a$, for the orbital motion of the ion electrons. In fact, since the orbital velocities of the ion electrons are comparable with V, the two mentioned frequencies will, for heavier atoms, approximately have the same ratio as the ion radius to the spacing of atoms in the solid, and have an even smaller ratio for lighter atoms. As regards the initial stages, the mechanism of the individual capture and loss processes should thus not differ essentially for gases and solids, and the marked difference in balance charge in the two cases therefore points directly to the importance of subsequent readjustment of the distribution of ion excitation.

Just as regards such readjustment, the rapid succession of the collision processes in solids will restrict the possibility of sharing excitation between electrons on the ion. In fact, the time, τ_{dis}, necessary for distribution over several electrons of an excitation initially confined to one, will be long compared to the revolution frequency, and we may reckon that the time between collisions in solids is shorter than τ_{dis}. The competition between collisions and distribution of energy between ion electrons may thus allow the excitation of the ion to exceed the minimum energy for ionization, I^*. The description of the ion state in balance becomes particularly simple if it may be assumed that there is not sufficient time for redistribution of ion excitations. In this case, an electron captured in an excited state will be lost from the same state, so that for each single electron state there is a direct competition between capture and loss.

Due to the very rapid increase of the cross section for electron loss with decreasing binding energy, the balance between capture and loss will therefore be essentially shifted by suppres-

sion of readjustment of ion excitation, in spite of the circumstance that such readjustment in itself may lead to electron release from the ion. In fact, in collisions with atoms, ion electrons can be removed from states with binding velocity nearly as large as $2V$, and we may reckon that even in solids more strongly bound electron states on the ion are occupied, while in higher states only a few electrons will remain, due to the competition between capture and loss. In a rough estimate on such lines we find that the ion charge will be about $(3/2)\cdot(V/v_0)\,\nu$, where ν is the quantum number of ion electrons with orbital velocity between V and $2V$.

In Figure 1 was shown the measured balance charge of fission ions in various solids and in gases at low pressures. It is seen that, for the heavier group of fission ions, the charge in solids approximately corresponds to the above estimate, since the effective quantum number is $\nu \approx Z^{\frac{1}{3}}$. For the light group, however, we found that already for the charge values in gases, ν was somewhat lower than $Z^{\frac{1}{3}}$, and for the high stripping in solids ν must have decreased even further. This circumstance accounts for the result that the charge of the light group in solids becomes slightly lower than that of the heavy group, opposite to what is the case in gases. Figure 1 also shows a small but marked decrease of ion charge with increasing atomic number of the solid substance penetrated. This effect points to a gradual minor change in the balance between capture and loss, probably connected with the greater average binding of electrons captured in heavier substances and thus reducing the probability for subsequent loss.

Although it thus appears that many of the characteristic features of the difference between average ion charge in dense and dilute materials may be explained by simple mechanical arguments, it must be stressed that we are dealing with a highly complicated problem, the detailed treatment of which needs further experimental and theoretical investigation. In a closer comparison with the empirical results, it must thus be taken into consideration that the high excitation of the ions in dense materials may result in a subsequent emission of electrons from the ions immediately after their escape into vacuum and thus, to a certain extent, increase the measured charge values. It may also be remarked that, in a comparison between the stopping power for

ions in dense and dilute materials, attention must be paid primarily to the considerable difference in the two cases of the charge of ions with given velocity.

Such problems must also be taken into account in comparisons between the phenomena accompanying the penetration of swift heavy ions through gaseous media and the remarkable observation of tracks of highly charged cosmic ions in photographic emulsions (cf., e. g., BRADT and PETERS, 1950). The rich material with which we in such observations are concerned extends, however, over a far wider energy region than the experiments with fission ions. Estimates of cross sections for electron capture and loss, determining for the balance charge of such rapid cosmic ions and its variation on their path through the photographic emulsion, therefore obviously demand considerations beyond the scope of the discussion in this paper.

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ON THE PROPERTIES OF A GAS OF CHARGED PARTICLES

BY

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CONTENTS

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Introduction.

The behaviour of systems having numerous degrees of freedom and endowed with electric charges presents problems of high complexity. Although many of the properties of systems of this kind may be explained within classical physics, a more consistent description was obtained only after the introduction of quantum mechanics.

An example studied with particular care is the so-called electron gas. We shall not attempt here a historical survey of the viewpoints and mathematical treatments of electron gases. It may only be recalled that a striking simplification was obtained when SOMMERFELD introduced the idea of a gas of free electrons, subject only to the exclusion principle, but uninfluenced by the mutual electric forces. In this way, an approximate account was given of many of the properties characteristic of metallic electrons. One of the major results was the smallness of the thermal energy of the gas at usual temperatures. The model could be refined so as to allow for periodic static potentials, which led to a separation of the electronic states into energy bands and to an explanation of essential properties of the different types of solids. At the same time the free electron picture, as applied in the Thomas-Fermi model, could account for properties of atoms in an averaged manner.

It was thus apparent that the Sommerfeld theory in several respects closely represented the properties of, e.g., valence electrons in metals. Still, it seemed difficult to give a direct justification of the model on the basis of more rigorous treatments, because the interaction appeared to have a dominating influence on the behaviour of the electrons. It was not easy to foresee the limitations of the theory, and in problems where it was valid in the first approximation ambiguities arose in the more detailed

treatments. This was the more unsatisfactory since some phenomena, as superconductivity, apparently defied any explanation within the simple theory of metals.

An interesting attempt to describe the dynamics of an electron gas in a systematic way, with inclusion of the interaction between the particles, was made by BLOCH (1933, 1934) on the basis of the Thomas-Fermi model. Although in the treatment further simplifications—introduction of hydrodynamic concepts—were made, it was possible to arrive at a number of simple results as regards the modes of excitation of the system. More recently, TOMONAGA (1950) presented a thorough discussion of the behaviour of a one-dimensional gas, again for fields varying slowly in space. Further, BOHM and collaborators (1949—1953) have published extensive studies of many problems in the three-dimensional case (see especially BOHM and PINES, 1953). The treatments by BLOCH, TOMONAGA, and BOHM indicate the great modification suffered by the system of originally free particles when the interaction is introduced.

This great change in the properties is—as empirically evident—mainly a change in the electromagnetic properties of the system. As in the treatment by BOHM, the interest therefore in the first instance centers on the behaviour of the electromagnetic field. However, we shall here make the further step of writing the equations of motion of the total system as equations for the electromagnetic field only. One would expect this to be possible, since the behaviour of the particles is revealed only through the field to which they give rise. The motion of a particle is then no longer determined by the interaction with numerous other particles, but only by the field. This elimination of the particles in the description of the properties of a medium is nothing but the idea at the basis of Maxwell's field equations in matter. The same idea is found in the static descriptions of THOMAS and FERMI, and of HARTREE. From the present point of view it is less adequate to introduce elaborations of these models of the kind contained in determinant wave functions, or in the description by correlations, where one tries to include energies of order e^2, or higher, by calculating the effect of interaction between two or more particles. The difficulties in a correlation description was evidenced in calculations by HEISENBERG (1947), where

divergencies appeared for long wavelengths. Such kinds of approach become more consistent if one introduces a screened Coulomb interaction between the particles (PINES, 1953; LANDSBERG, 1949; WOHLFARTH, 1950).

We should now make a more definite statement as to the field entering in our description: By the classical field in the substance we mean always the total field acting on and eventually due to classical, or external, charges. We wish to derive the corresponding field equations, which turn out to be of the type of Maxwell's field equations in matter. This attitude might seem to require a detailed justification because fluctuations could destroy the uniqueness of the description. However, the classical field equations define, on the contrary, what one understands by a large system of interacting particles. Thus, it is important to realize that all fluctuations occurring are contained in the present description, simply because it gives the classical equations of motion of the system. A quantization of the equations may be performed by applying the methods familiar from field theory. With such methods it is possible to derive the fluctuations in the system, on similar lines as followed by BLOCH (1934) in his treatment of the Thomas-Fermi gas. In the following, we shall try to amplify these brief remarks, both as regards which is the field considered and as to the uniqueness of the field equations.

Accepting the above, an immediate task is to find what simplifications and modifications can result when the dynamical properties are contained in field equations for the electromagnetic field. Now, even though it can be difficult to derive more exact field equations on explicit form, one may be able to find easily the type of the equations and to give approximate estimates in various limiting cases. One can then, with confidence, attempt an approach to the field equations both from a theoretical and an empirical point of view, and consider the general consequences of the structure of the equations. Of course, at the same time as one has an equation for the electromagnetic field, an equation for the particle field will be obtained (cf. Appendix). The latter equation is of particular interest, for instance, in scattering phenomena. For the present, we are concerned mainly with the electromagnetic field, but one has the choice of developing with higher accuracy the description of either of the two fields.

It will be useful, then, to discuss general field equations of classical type. We limit the treatment to linear field equations, and define these by the dielectric operators of the field. Linear field equations will seem to cover a wide class of phenomena, but in some cases the linear approximation is not sufficient. Terms of higher order appear in, e.g., the Thomas-Fermi atomic model and phenomena as those studied by ALFVÉN (1950) and his school.

In the derivations, in the present paper, of classical linear field equations for a gas of charged particles several further approximations are made. We compute at first the dielectric constants to first order in e^2, starting from the picture of a free gas of particles. One gets then a number of well-known results regarding the properties of free gases, but finds essential improvements in some cases. When the computations are carried out to higher order, one encounters further corrections to the familiar descriptions.* The first order equations contain the main features of the linear field equations of the system. In some cases they can represent the exact solution and yet contain a by no means small polarization (cf. § 5, p. 48). From such first order treatment one sees perhaps best that the field equation concerns the total classical field, since the particle motions—and thus the polarization—are determined only by this field.

As regards the question of the justification of the approximation method applied in the following, where one starts from a free gas and computes the dielectric constant only to first order in e^2, it would seem most appropriate to calculate the linear field equations to a high, or infinite, order and make a comparison with the first order treatment. Employing the powerful technique of quantum field theory, one can indeed get expressions for the linear electromagnetic and particle field equations. We use a more modest approach, and discuss the behaviour of one electron moving in the field calculated to second order. The electron then

* By a computation to higher order we understand a procedure of iteration applied only in deriving the field equations; when these are obtained we do not attempt approximations at a later stage.

In a sense the accuracy of the calculation to first order is dependent on e^2 being small. We can in fact form one dimensionless quantity from e^2 and the density of the gas, ϱ. In the following we use the quantity $\chi^2 = \left[me^2/\hbar^2 (3\pi^5\varrho)^{\frac{1}{3}}\right]$. The value of χ^2 must determine the properties of the gas. The free particle picture is valid if χ^2 is small compared to 1, as in dense gases.

interacts with itself, and we consider in some detail the effect of the damping of motion and of the self-energy. As to the particular example of a one-dimensional gas the present perturbation method, applicable to all wavelengths of the field, gives in the region of long wavelengths precisely the equations deduced by TOMONAGA (1950).

We have not yet commented on the crucial question of the connection with statistical mechanics. In spite of the circumstance that we are concerned, basically, with a conservative system, the field turns out to be of non-conservative type and we encounter dissipation effects of peculiar kind. Thus, absorption of energy and momentum occurs via processes not unlike those resulting from viscosity in a liquid. The absorption effect appears as a finite imaginary operator in the equations for the retarded field. It may be said to derive from an extreme in entropy increase and trend towards statistical equilibrium, and must result in a small thermal excitation of the system. Just the former circumstance implies an unambiguous classical state —and behaviour in time—of the system, where mechanical considerations alone could not suffice. Incidentally, we have therefore employed the word 'classical' in the simple sense: with neglect of statistical or quantum mechanical fluctuations. The statistical fluctuations about the average absorption can be obtained in a straightforward manner from correspondence arguments.

In the following only some of the above general questions are treated in detail. First, in § 1, the field equations in matter are introduced. The equations holding for a nearly free gas are computed classically in § 2, and by quantum mechanical treatment of the particles in § 3. In the static limit these calculations are in line with the method introduced by O. KLEIN, whose point of view was similar to the present one (KLEIN, 1945; LINDHARD, 1946). By way of illustration we consider in § 4 the application to stopping problems, and self-energy, for charged particles passing through matter (cf. also Appendix). The improvement of the free electron picture is discussed in § 5. In a subsequent paper* will be treated the connection between the thermal properties and the field equations, and also the absorp-

* To appear in Dan. Mat. Fys. Medd., and in the following referred to as II.

tion and reflection of a transverse wave by a metallic surface (cf. moreover LINDHARD, 1953).

Before entering on the more formal treatment we may add a few words as to the general formulation given here, and concerning the results of a discussion of the free electron gas. It might perhaps seem that the free gas is a quite specialized example, but it should be appreciated, first, that its applicability to atomic systems containing many electrons is quite wide. The proper methods for a treatment of many-particle systems should just derive from an understanding of the simple example of a free gas.

Second, the electron gas is a system in which the forces between the particles are known. If it is attempted to discuss systems where the forces are less well understood, as in an assembly of nucleons, a detailed description of the simpler case may be of considerable help. It is of special interest that a system of interacting particles exhibits several typical properties, for which the character of the forces is usually not decisive.

One may here particularly mention the connection between the independent-particle model and the collective model of atomic nuclei. These two cases we meet already for electronic systems, which in numerous respects show an extraordinary likeness to a nucleus. The collective and independent-particle features are intimately connected in the present field equations, and though the total effect of the particles, as expressed by the field polarization, can be very large, yet each particle may move nearly as if it were free (in the more general case, the particles have no longer free particle equations, but remain independent). A considerable simplification results since the two descriptions do not appear as entirely different cases, but may be treated as one.

§ 1. Field equations in matter.

In this paragraph are described the more general features of electromagnetic field equations in matter, as a preliminary to the calculations in the following. One of the questions is, simply, the introduction of a suitable notation and terminology; we choose to formulate the field equations by means of the dielectric con-

stants of the field. Actual expressions for dielectric constants are to be found in § 2 and § 3. Further, the discussion here is meant to illustrate characteristic phenomena, as energy dissipation, that are contained in the field equations.

It is desirable to use a description similar to that of the Maxwell equations in matter. In these equations, the properties of a substance are determined by the values of the dielectric constant ε, the permeability μ, and the conductivity σ. Expanding the field in harmonic components in time, the quantities ε and μ may be considered as functions of the frequency ω; the conductivity σ can be included in the dielectric constant and determines then the imaginary part of ε. One distinguishes, on the one hand, between the general equations which do not contain the properties of the medium,

$$\left.\begin{aligned} &\operatorname{div} \vec{D} = 4\pi\varrho_0, \quad \operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \\ &\operatorname{div} \vec{B} = 0 \quad , \quad \operatorname{rot} \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j}_0, \end{aligned}\right\} \quad (1.1)$$

and, on the other hand, the more special equations where the electromagnetic properties enter directly,

$$\vec{B}(\vec{r}, \omega) = \mu(\omega) \cdot \vec{H}(\vec{r}, \omega), \quad \vec{D}(\vec{r}, \omega) = \varepsilon(\omega) \cdot \vec{E}(\vec{r}, \omega), \quad (1.2)$$

$B(\vec{r}, \omega)$, etc., being field components with time dependence $\exp(-i\omega t)$. While the equations (1.1) are quite general, the assumption (1.2) leads directly to linear field equations.

Using a description of the kind (1.2), one is able to give a satisfactory account of many properties of matter. Still, for a number of phenomena not only the time variation of the field, as contained in (1.2), but even its variation in space will have a decisive influence on the values of ε and μ. Moreover, the general interpretation of (1.2) and the transcription of the Maxwell equations into equations for the electromagnetic potentials remain ambiguous if one assumes that ε and μ are functions only of ω. We shall find, however, that when (1.2) is generalized so as to take into account the spatial variation of the fields, such ambiguities disappear.

We may now discuss the general form of linear field equations

in matter, without immediately trying to find the connection with the Maxwell equations (1.1) and (1.2). Let us consider the microscopic behaviour of an electromagnetic field in an extended medium. We may for the present regard the medium as infinite, isotropic, and homogeneous, the latter even over minute regions in space, and with properties independent of time. As will be discussed below, we shall be concerned with the classical, or average, field and at first look apart from fluctuations. In the case under consideration, it will be convenient to describe the fields by their Fourier components in space and time. Indeed, due to the invariance towards displacements in the medium, each Fourier component of the field must be proportional to the same Fourier component of the sources. The factor of proportionality will be some function depending both on the wave vector and frequency of the component in question. Considering, in the first instance, the transverse (divergence-free) part of the field we may express this circumstance by introducing a function which we shall call the transverse dielectric constant. The reasons for this particular formulation and the connection with the Maxwell equations in matter will appear presently. We thus get for the transverse part of the vector potential

$$\left(k^2 - \frac{\omega^2}{c^2}\,\varepsilon^{tr}\right)\vec{A}^{tr}(\vec{k},\omega) = \frac{4\pi}{c}\,\vec{j}_0^{\,tr}(\vec{k},\omega), \tag{1.3}$$

where $\varepsilon^{tr} = \varepsilon^{tr}(\vec{k},\omega)$ is a function of the wave vector k and the frequency ω. The connection between $\vec{A}(\vec{r},t)$ and $\vec{A}(\vec{k},\omega)$ is given by $\vec{A}(\vec{r},t) = \sum_{k,\omega} \vec{A}(\vec{k},\omega)\cdot\exp(i\vec{k}\cdot\vec{r} - i\omega t)$, and in the same way $\vec{j}_0^{\,tr}(\vec{k},\omega)$ is a Fourier component of the transverse current of the sources of the field. The transverse character of the field is expressed by $\vec{k}\cdot\vec{A}^{tr}(\vec{k},\omega) = 0$, so that $\vec{A}^{tr}(\vec{k},\omega) = \vec{A}(\vec{k},\omega) - \vec{k}(\vec{k}\cdot\vec{A}(\vec{k},\omega))/k^2$, and similarly for $\vec{j}_0^{\,tr}(\vec{k},\omega)$. On account of the explicit appearance of the sources equation (1.3) describes a case of forced vibrations of the medium; the solution of the homogeneous equation is discussed more closely in II.

We note that, since $\vec{A}$ and $\vec{j}_0$ are real functions in space and time, their Fourier components $(-\vec{k},-\omega)$ must be complex con-

jugate of the components $(\vec{k}, \omega)$, which implies that $\varepsilon^{tr}(-\vec{k}, -\omega) = \varepsilon^{tr*}(\vec{k}, \omega)$. The dielectric constants in the following will generally be of the type $\varepsilon^{tr}(\vec{k}, -\omega) = \varepsilon^{tr*}(\vec{k}, \omega)$, as is characteristic for retarded and advanced field equations. We shall always be concerned with the retarded dielectric constants, i. e. retarded field equations, except when expressly otherwise stated.

For the electric potential we can choose to write, in a similar way, introducing a new function which we call the longitudinal dielectric constant,

$$\varepsilon^{l} \cdot k^2 \, \Phi(\vec{k}, \omega) = 4\pi \, \varrho_0(\vec{k}, \omega), \tag{1.4}$$

where the notation is as in (1.3) and $\varepsilon^{l} = \varepsilon^{l}(k, \omega)$ depends on the two variables k and ω. The quantity $\varrho_0(\vec{k}, \omega)$ on the right is the Fourier component of the source density of electric charge. When the description (1.4) is used, it is assumed that the gauge is chosen such that the longitudinal vector potential vanishes. It may perhaps already here be stressed that, for fields in material substances, one has directly given a natural system of reference and it is often not convenient to write the field equations in an invariant manner.

The fields derived from the above potentials are

$$\vec{E} = -\operatorname{grad} \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \operatorname{rot} \vec{A} \tag{1.5}$$

and they are the true classical fields acting at a space-time point $(\vec{r}, t)$ in the medium. We consider the retarded ε's so that the fields in (1.5) are the retarded fields. The letters E and B are chosen for these quantities because they can be interpreted as the electric field and the magnetic induction of the Maxwell equations, as we shall now show. We have accounted for the properties of the medium by the two functions ε^{tr} and ε^{l}, instead of ε and μ in the equations (1.2). The connection between the two formulations is apparent when a function $\mu(k, \omega)$ is defined by

$$k^2 \left(1 - \frac{1}{\mu(k, \omega)}\right) = \frac{\omega^2}{c^2} \left(\varepsilon^{tr}(k, \omega) - \varepsilon^{l}(k, \omega)\right), \tag{1.6}$$

for then equation (1.3) becomes, introducing a dielectric constant ε, equal to the longitudinal one, ε^l,

$$\left(\frac{k^2}{\mu}-\frac{\omega^2}{c^2}\,\varepsilon\right)\vec{A}^{tr}(\vec{k},\omega)=\frac{4\pi}{c}\vec{j}_0^{\,tr}(\vec{k},\omega). \tag{1.7}$$

With this notation one can introduce a magnetic field, H, and an electric displacement, D, from the equations

$$\left.\begin{aligned}\vec{H}(\vec{k},\omega)&=\frac{1}{\mu(k,\omega)}\cdot\vec{B}(\vec{k},\omega)\\ \vec{D}(\vec{k},\omega)&=\varepsilon(k,\omega)\cdot\vec{E}(\vec{k},\omega).\end{aligned}\right\} \tag{1.8}$$

One then obtains the Maxwell equations (1.1) in space and time for the quantities $\vec{B}(\vec{r},t)$, $\vec{H}(\vec{r},t)$, $\vec{E}(\vec{r},t)$ and $\vec{D}(\vec{r},t)$. This formulation of the field equations, where μ, H and D are introduced, is often convenient. But in the following we shall usually describe the behaviour of the medium by using only ε^{tr} and ε^l, solving the equations (1.3) and (1.4) with respect to the potentials. In this connection it may also be emphasized that μ, according to (1.6), is determined unambiguously only when ε^l is known. Thus, if one treats exclusively the transverse field it can be convenient to interpret the Maxwell equations (1.8) in the manner that the permeability is 1, or $H = B$, and the dielectric constant is the transverse dielectric constant.

The permeability and the dielectric constants defined above should be considered as complex functions; their imaginary parts are closely connected with the energy absorption by the medium. We note in particular that the connection between the Fourier components of the induced transverse and longitudinal current densities and total electric fields are given by

$$\vec{j}_{\text{ind}}^{\,\{tr}_{l}(\vec{k},\omega)=(\vec{j}-\vec{j}_0)^{\{tr}_{l}=\frac{i\omega}{4\pi}\left(\varepsilon^{\{tr}_{l}(k,\omega)-1\right)\vec{E}^{\{tr}_{l}(\vec{k},\omega), \tag{1.9}$$

the equation holding, as indicated, for the transverse and longitudinal fields separately. Equation (1.9) leads us to define transverse and longitudinal conductivities as, respectively,

$$\sigma^{\mu\nu}(k,\omega) = \frac{\omega}{4\pi} \operatorname{Im}\left(\varepsilon^{\mu\nu}(k,\omega) - 1\right), \tag{1.10}$$

where $\operatorname{Im}(x)$ denotes the imaginary part of x. The real parts of the equations (1.10) correspond to the polarizabilities of the medium.

We now turn to considerations of a more general kind, based only on the Maxwell equations (1.1); we do this primarily in order to give an account of the phenomenon of absorption by the medium. The absorption does not express a lack of conservation of energy of the total system. Indeed, we shall derive it in § 2 and § 3 from conservative equations of motion. The absorption means only that a distribution takes place over a large number of degrees of freedom, and the corresponding motion cannot be regained as an ordered motion. In a more detailed description, the absorption by the medium can be described by thermodynamical and statistical mechanical parameters, and observed as, e. g., a rise in temperature of the system.

The conservation equations for energy and momentum are derived in the familiar manner from the Maxwell equations (1.1). We shall write them down in full so as to show their contents and interpretation in descriptions of the present kind. We find from (1.1) the energy equation

$$\left.\begin{aligned} &-\frac{1}{8\pi}\cdot\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E}+\vec{H}\cdot\vec{B}) - \frac{c}{4\pi}\operatorname{div}(\vec{E}\times\vec{H}) \\ &= \frac{1}{8\pi}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{D}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{B}\cdot\frac{\partial\vec{H}}{\partial t}\right) + \vec{E}\cdot\vec{j}_0, \end{aligned}\right\} \tag{1.11}$$

where the first and second terms on the left are minus the time derivative of the energy density and minus the divergence of the energy current, respectively. The last term on the right is the work done by the field on the sources. The bracketed term on the right is of a peculiar kind; it is often assumed to vanish in homogeneous media. However, applying (1.4), (1.7), and (1.8), we find—by integration over a space-time interval—that it represents the work done by the field on the medium, and it gives the rate dW/dt at which energy dissipates from the field in an

irreversible manner. We may express this result in terms of the potentials, and find directly from (1.8), introducing the conductivities from (1.10),

$$\Delta W = \sum_{k,\omega} \left(\frac{\omega^2}{c^2} \sigma^{tr}(k,\omega) \left| \vec{A}^{tr}(\vec{k},\omega) \right|^2 + k^2 \sigma^{l}(k,\omega) \left| \Phi(\vec{k},\omega) \right|^2 \right) \Delta t, \quad (1.12)$$

for the absorption in the time interval Δt. To be more precise, eq. (1.12) includes the dissipation of energy, if any, through the boundaries of the medium.

As to the equations for the momentum of the field, we apply again (1.1) and get for the component in the direction of the x-axis

$$\left.\begin{aligned} &-\frac{1}{4\pi c}\frac{\partial}{\partial t}(\vec{D}\times\vec{B})_x + \operatorname{div}_i T_{xi} \\ &= \frac{1}{8\pi}\left(\vec{D}\cdot\frac{\partial \vec{E}}{\partial x} - \vec{E}\cdot\frac{\partial \vec{D}}{\partial x} + \vec{B}\cdot\frac{\partial \vec{H}}{\partial x} - \vec{H}\cdot\frac{\partial \vec{B}}{\partial x}\right) \\ &\quad + \varrho_0 \cdot E_x + \frac{1}{c}\left(\vec{j}_0 \times \vec{B}\right)_x, \end{aligned}\right\} \quad (1.13)$$

where

$$T_{si} = \frac{1}{4\pi}\left(E_s D_i + H_s B_i - \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B})\cdot\delta_{si}\right)$$

is Maxwell's stress tensor in matter. From equation (1.13) we find that the momentum density is $(1/4\pi c)\cdot(\vec{D}\times\vec{B})$, which leads to the non-symmetric energy-momentum tensor of MINKOWSKI.* We shall in the discussion of reflection and transmission of light waves by metallic surfaces, in II, attempt to show the direct significance of this property of the energy-momentum tensor. It must be remembered that, if one will ascribe values to quantities as energy and momentum, one shall not primarily look for cases where they remain unchanged, but consider processes as absorption where exchanges of these quantities occur. As to the right-hand side of equation (1.13), the last term is the Lorentz

* Cf. C. MOLLER, The Theory of Relativity, Oxford 1952, p. 203. The Minkowski tensor has been adopted, too, in the 1951-edition of Die Relativitätstheorie by M.v. LAUE.

force density on the sources, while the first term corresponds to the first term on the right of (1.11), and is the rate at which the momentum density is taken up by the medium. The average rate at which momentum density is absorbed can be obtained from (1.12) when multiplying by the factor $\vec{k}/\omega$ inside the brackets.

We are thus able to account for the momentum and energy exchanged between the sources, the field, and the medium. In the case where the sources are outside the medium and the field is determined by boundary conditions, the energy dissipation can be found from (1.11) or (1.12), and the momentum absorption from the corresponding expressions. It may here be noted that, if the sources are inside the medium and not appreciably affected by the above momentum and energy exchange, a detailed treatment of the phenomena as a rule will be much simpler than in boundary problems or cases where the reaction of the sources must be included in the description.

So far the absorption was considered to take place in a continuous manner. When attempting to subdivide the absorption by the medium into elementary processes we must take into account that the quantum of energy corresponding to a frequency ω is $\hbar\omega$. It may be concluded immediately from (1.12) that the probability for the absorption of a quantum of frequency ω and wave vector k is, during a time interval Δt,

$$p(\vec{k},\omega) = \Delta t \cdot \left(\frac{\omega}{\hbar c} \sigma^{tr}(k,\omega) \left| \vec{A}^{tr}(\vec{k},\omega) \right|^2 + \frac{k^2}{\hbar\omega} \sigma^{l}(k,\omega) \left| \Phi(\vec{k},\omega) \right|^2 \right). \quad (1.14)$$

This expression represents a statistical mechanical probability, and in contrast to the previous equations it depends directly on quantum theory, through the explicit appearance of the constant h. From (1.14) can be derived in particular the fluctuations in energy and momentum absorption. We note further that, while the probability for absorption $\Delta p(\vec{k}, \omega)$ for quanta $\hbar\omega$ is positive for positive ω, it becomes negative for negative ω, corresponding to a transfer of negative energy to the field. It is a matter of convention whether one will introduce instead only the numerical value of ω, and accordingly only positive energies. In § 4 will

be discussed a few interesting and simple consequences of equation (1.14).

In the description used here, represented by for instance (1.8) together with (1.1), the quantities ε and $1/\mu$ were considered as numbers in k, ω-space and correspondingly as operators in ordinary space. This formulation will be sufficient for the present purposes. But it is seen that ε and $1/\mu$ in (1.8) may be more general operators, and not necessarily linear. The other formulae in this paragraph can then be similarly interpreted.

The question of measuring the field quantities, like E, D, etc., is a rather subtle one. The dependence of the dielectric constants on wavelength and frequency will evidently play an important role for such measurements. Actual examples of this dependence can be found in the following sections. Even though the description is best illustrated by such examples we shall make here a few remarks on the measuring problem in general. The familiar rule for measurements of the fields is that, for instance, E and D are the fields in crevasses in the medium, cut parallel and perpendicular to the field itself. But let us take into account that at the surface of the medium the fields do not change abruptly (as they would if the ratio between D and E were constant in the medium) and suppose that they vary in some smooth manner over a distance, d, from the surface. If now we cut crevasses of the above kind, though with dimensions smaller than or of the order of d, the fields arising in these crevasses are no longer E and D, because the polarization has changed character. In fact, for sufficiently small crevasses, the measurements must all give E, because, for high values of k, the dielectric constant tends to 1 so that the polarization disappears. This corresponds to the circumstance that the field acting on a classical point charge is E. Exceptions from the mentioned rule for measuring E and D will occur in many other cases; the rule is disobeyed if only short time intervals are allowed for the measurement, and there are cases too where the dielectric constants even for long wavelengths depend strongly on the wavelength of the field.

§ 2. Semi-classical treatment of an electron gas.

We shall first contemplate the behaviour of an electron gas in classical mechanics. The properties of the gas will be described in terms of the dielectric constants of the electromagnetic field equations introduced in § 1. In the first part of the present paragraph, a more formal derivation of the field equations is given; in the second part, a comparison is made with the results of other authors. The question of the justification of the calculation will be taken up in § 5.

The calculations are based on the previously mentioned picture of the dynamics of a system of electrons: There exists a total field in space and time, $\Phi(\vec{r}, t)$, $\vec{A}(\vec{r}, t)$, in which the separate electrons move. This motion gives rise to an induced charge and current density, ϱ_{ind} and $\vec{j}_{\text{ind}}$, which quantities will be functions of the total field. Besides the induced densities there may be source densities, $\varrho_0(\vec{r}, t)$ and $\vec{j}_0(\vec{r}, t)$, distributed within the system. The latter densities can, for instance, correspond to a charged particle passing through the gas. From MAXWELL's equations for empty space, with charge density $\varrho_{\text{ind}} + \varrho_0$ and current density $\vec{j}_{\text{ind}} + \vec{j}_0$, one then derives the field equations for the total field, of the type (1.3) and (1.4). In these equations the source densities are ϱ_0 and $\vec{j}_0$. We shall suppose that the electronic motion differs only slightly from that in the undisturbed state of the system. This implies, generally, that the field equations will become linear.

Quite apart from the question of the applicability of a linear treatment, and of classical theory, the particular approximation of the first order picture is that the motion of the individual particles in the gas gives rise to and derives from a field determined by MAXWELL's equations for empty space. But it will be clear that one should rather assume that the field connected with the motion of the individual electron is governed by the final field equations. Such a treatment on more precise lines is somewhat more complicated and we shall show that it is not always needed. As we shall see in § 5, the corresponding changes to be made in our present computation will imply a velocity-dependent self-

energy and a damping force on the electron from the field induced by it, not to be found in MAXWELL's equations for empty space.

Consider now a gas of free electrons and its behaviour in the presence of an electromagnetic field, the motion of the electrons being governed by classical laws and restricted only by the Pauli exclusion principle in the initial state; from LIOUVILLE's theorem it follows, then, that the exclusion principle is obeyed at any later time. This approximation is conveniently termed semi-classical, and it is based on similar ideas as the Thomas-Fermi treatment of static electric fields. We shall find that the semi-classical treatment in some cases gives other results than the familiar classical treatment, where the electrons are assumed to be at rest.

The distribution function of the electrons over momentum space and ordinary space is $f = f(\vec{p}, \vec{r}, t)$, where $\vec{p}$ is the kinetic momentum conjugate to $\vec{r}$. The initial time-independent distribution function in the absence of external fields is called $f_0(p, \vec{r})$. The behaviour in time of f is given by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\vec{p}} \cdot \operatorname{grad}_p f + \vec{v} \cdot \operatorname{grad}_r f = -\frac{f - f_0}{\tau},$$

where one assumes a trend towards the equilibrium state, determined by the right-hand side of the equation. This damping of the motion may be pictured as due to the resistance arising from collisions with the positive background in the gas. As a rule, we regard $1/\tau$ as infinitely small, in which case the damping term serves to give the retarded solutions of the equations of motion, i. e. of the field equations.

The difference f_1 between f and f_0 is supposed to be small, and further f_0 is assumed to be independent of r. In the equation governing the behaviour of f we neglect second order terms and obtain the simple equation of motion for $f_1 = f_1(\vec{p}, \vec{r}, t)$

$$\frac{\partial}{\partial t} f_1 + e\left(\vec{E} + \frac{1}{c}(\vec{v} \times \vec{B})\right) \cdot \operatorname{grad}_p f_0 + \vec{v} \cdot \operatorname{grad}_r f_1 = -\frac{f_1}{\tau}, \quad (2.1)$$

where $\vec{E}$ and $\vec{B}$ are the total microscopic field strengths, derivable from the potentials. The term containing B in the Lorentz force may be omitted, being perpendicular to $\operatorname{grad}_p f_0$.

We wish to write equation (2.1) in terms of its Fourier components in space and time. Therefore we make the development $f_1(\vec{p}, \vec{r}, t) = \sum_{k,\omega} f_1(\vec{p}, \vec{k}, \omega) \exp(i\vec{k} \cdot \vec{r} - i\omega t)$, and similarly for the fields. We solve with respect to the Fourier components of f_1 and find

$$f_1(\vec{p}, \vec{k}, \omega) = \frac{1}{-i\vec{v} \cdot \vec{k} + i\omega - 1/\tau} e\vec{E} \cdot \operatorname{grad}_p f_0(p). \qquad (2.2)$$

The equation may be applied to, e. g., a Maxwell distribution, but we shall be concerned mainly with the case of complete degeneracy, where $\operatorname{grad}_p f_0 = -(\vec{p}/p) \cdot \delta(p - p_0)$, p_0 being the momentum at the surface of the Fermi distribution. Since f_0 was assumed independent of r, the same holds for p_0. We note that, for temperatures different from zero, the δ-function in $\operatorname{grad}_p f_0$ is replaced by a function of finite width, but as long as the temperature remains low compared to the degeneracy temperature there will be no appreciable change in the calculations below.

The current induced in the system can be expressed now as a function of the field. Multiplying f_1 in (2.2) by ev, and integrating over velocity space, we have for the transverse part of the induced electric current density

$$\vec{j}^{tr}_{\text{ind}}(\vec{k}, \omega) = \frac{2e^2}{h^3 c} \int d^3p \frac{\omega \vec{v}}{\vec{v} \cdot \vec{k} - \omega - i/\tau} \cdot \frac{\delta(p - p_0)}{p} \left(\vec{p} \cdot \vec{A}^{tr}(\vec{k}, \omega)\right).$$

The integration over p amounts to an integration over angles. When this is performed we compare with (1.9) and obtain the expression for the transverse dielectric constant of a degenerate free gas

$$\left.\begin{aligned} \varepsilon^{tr}(k, \omega) &= 1 + \frac{3\omega_0^2}{4\omega} \cdot \frac{m v_0}{p_0} \\ \cdot \left\{ -2 \frac{\omega + i/\tau}{v_0^2 k^2} + \frac{1 - (\omega + i/\tau)^2 / v_0^2 k^2}{v_0 k} \log \frac{v_0 k - \omega - i/\tau}{-v_0 k - \omega - i/\tau} \right\}, \end{aligned}\right\} \qquad (2.3)$$

where ω_0 is the classical resonance frequency of the electron gas,

or $\omega_0^2 = 4\pi e^2 \varrho/m$, ϱ being the density of electrons; v_0 is the velocity at the top of the Fermi distribution.

In quite a similar manner the induced charge density may be calculated from (2.2) as a function of the electric potential. The longitudinal dielectric constant in (1.9) is then found to be

$$\varepsilon^l(k, \omega) = 1 + \frac{3\omega_0^2}{v_0^2 k^2} \cdot \frac{m v_0}{p_0} \left\{ 1 + \frac{\omega + i/\tau}{2 k v_0} \log \frac{v_0 k - \omega - i/\tau}{-v_0 k - \omega - i/\tau} \right\}. \quad (2.4)$$

In these equations, and in the following, the logarithms denote principal values, so that their imaginary parts are between $-i\pi$ and $+i\pi$. It is seen that a change of sign of ω in (2.3) and (2.4) will have the same result as a change of sign in the damping term i/τ. Further, the dielectric constants depend on k only through the square of this vector.

We have distinguished between the momentum p_0 and mv_0. This discrimination is not necessary in the simple calculations in this paragraph, and in the following it is not stressed. However, a discrimination is useful for a relativistic gas, where (2.4) and (2.5) can be applied directly. Moreover, it is important when the self-energy influences the connection between p and v, as discussed in § 5.

As mentioned, a main difference between the treatment by Bohm and co-workers, and the present discussion, arises from a difference in attitude in the description of the field, to which we ascribe a precise classical meaning. While we obtain a common description of all waves of the field, Bohm treats essentially only long waves, for which a simple development in powers of $v_0^2 k^2/\omega^2$ can be made. Bohm asserts that the field equations represent merely collective motions, and not individual particle behaviour.

A simplification of the field equations corresponding to (2.3) and (2.4) is desirable and possible in most problems. We see that the dielectric constants depend on the dimensionless ratio between $|\omega + i/\tau|$ and $v_0 k$. If we are concerned with a field of given frequency the significant wave vectors k will collect around a certain corresponding vector, and the above dimensionless ratio may be said to have an approximate value, determined implicitly by the field equations. We can here distinguish between

two essentially different cases, namely $|\omega + i/\tau|$ larger or smaller than $v_0 k$. If it is assumed that the collision time τ is long, these two cases imply that during one period of the field the path travelled by an electron is, respectively, shorter and longer than the wavelength of the field. In the former case, the electron gets the impression of a time-dependent field while, in the latter case, it reacts nearly as it would in a static field.

Consider first the simpler case of k tending to zero. The transverse dielectric constant (2.3) then approaches the classical one for a homogeneous medium (omitting the factor mv_0/p_0),

$$\varepsilon^{tr} \cong 1 - \frac{\omega_0^2}{\omega\,(\omega + i/\tau)}\,, \quad \text{for} \quad v_0 k < |\omega + i/\tau|, \tag{2.5}$$

the first correction term being $(-\omega_0^2 k^2 v_0^2)/(5\,\omega\,(\omega + i/\tau)^3)$. In this limit, the expression for the longitudinal dielectric constant, (2.4), becomes of similar type,

$$\varepsilon^{l} \simeq 1 - \frac{\omega_0^2}{(\omega + i/\tau)^2}\left\{1 + \frac{3\,k^2 v_0^2}{5\,(\omega + i/\tau)^2} + \cdots\right\}, \quad \text{for} \quad v_0 k < |\omega + i/\tau|, \tag{2.6}$$

where we have included the first correction term in k^2. If one is not concerned with a degenerate gas, one can find this term simply by replacing $3v_0^2/5$ by the average of the velocity squared in the gas in question. It should be mentioned that the dispersion formula (2.6) has been discussed previously by several authors (cf., e. g., Bohm and Pines, 1952).

In the opposite limit of $v_0 k$ large compared with $\omega + i/\tau$ one finds a striking departure from the classical result. As mentioned, the electrons behave nearly as if the field were stationary, but in the dielectric constants finite imaginary terms appear, corresponding to energy absorption by the medium. Using (2.3) and (2.4), we obtain in this case (omitting again the factor mv_0/p_0)

$$\varepsilon^{tr} \cong 1 - \frac{\omega_0^2}{\omega^2}\left\{\frac{3\,\omega^2}{k^2 v_0^2} - i\,\frac{3\,\pi\omega}{4\,k v_0}\left(1 - \frac{\omega^2}{k^2 v_0^2}\right)\right\}, \quad \text{for} \quad v_0 k > |\omega + i/\tau|, \tag{2.7}$$

$$\varepsilon^{l} \cong 1 + \frac{3\,\omega_0^2}{k^2 v_0^2}\left\{1 - \frac{\omega^2}{k^2 v_0^2} + i\,\frac{\pi\omega}{2\,k v_0}\right\}, \quad \text{for} \quad v_0 k > |\omega + i/\tau|. \tag{2.8}$$

In approximate calculations one can use the formulae (2.5), (2.7) and (2.6), (2.8), instead of the more involved ones, (2.3) and (2.4). It is of interest, further, that in a qualitative sense one may apply, e. g., formula (2.5) for all values of k and ω, if only in this equation i/τ is replaced by $i/\tau + i\,4\,kv_0/3\,\pi$.

We may mention briefly some results concerning the non-degenerate gas. For a Boltzmann distribution, an approximate description can be obtained from the above formulae when replacing v_0 by the temperature velocity of the particles. More accurate expressions are found by introducing the distribution function $f_0 = C \cdot \exp(-mv^2/2\,\theta)$ in (2.2). It is apparent that, for ω large compared to $k\,(2\,\theta/m)^{1/2}$, the equations (2.5) and (2.6) are not far wrong; in the series development one need only introduce the proper averages of the particle velocities. In the opposite limit we get, e. g., for the longitudinal dielectric constant, corresponding to (2.8),

$$\varepsilon^l \simeq 1 + \frac{m\,\omega_0^2}{k^2\,\theta}\left\{1 + i\,\frac{\omega}{k}\left(\frac{\pi\,m}{2\,\theta}\right)^{\frac{1}{2}}\right\}, \quad \text{for } (2\,\theta k^2/m)^{\frac{1}{2}} > \left|\omega + i/\tau\right|, \quad (2.9)$$

where the first term in the brackets gives the Debye-Hückel formula for static electric fields, while the second term accounts for the energy absorption.

It is noteworthy that for a Boltzmann gas the imaginary part of the dielectric constants is finite, independently of the values of k and ω. Thus, for transverse fields, one finds directly from (2.2)

$$\mathrm{Im}\,(\varepsilon^{tr}) = \frac{4\,\pi}{\omega}\,\sigma^{tr} = \frac{\omega_0^2}{\omega\,k} \cdot \left(\frac{\pi\,m}{2\,\theta}\right)^{\frac{1}{2}} \cdot \exp(-m\,\omega^2/2\,\theta k^2), \quad (2.10)$$

the formula holding for sufficiently large values of τ.

Spin contribution to magnetic properties.

So far, we have considered charged particles without intrinsic magnetic moments. But already in the semi-classical treatment it is possible to include the electron spin in a simple manner. Consider for this purpose a single Fourier component $\vec{B}\,(\vec{k}, \omega)$ of the magnetic field. An electron with magnetic moment β has the interaction energy

$$\pm\,\beta B\,(\vec{k}, \omega)\,\exp(i\vec{k}\cdot\vec{r} - i\,\omega\,t),$$

corresponding to its moment being either parallel or antiparallel to the field. If the distribution function is f^+ and f^- for parallel and antiparallel spin, respectively, we get to the first order the following Boltzmann equations

$$(-i\omega + \vec{v}\cdot i\vec{k} + 1/\tau)\, f_1^{\pm}(\vec{k},\omega) \pm \beta B(\vec{k},\omega)\, i\vec{k}\cdot \mathrm{grad}_p f_0 \, m = 0, \quad (2.11)$$

where, for instance, $f_1^+(k,\omega) = f^+(k,\omega) - f(k,\omega)$, f being the spin independent distribution function in (2.1). The contributions to the induced magnetic moment of the gas from the parallel and antiparallel distributions are equal. We find readily, for the Fourier components of the spin magnetic moment per unit volume,

$$\vec{M}(\vec{k},\omega) = \sum_p m\beta^2 \vec{B}(\vec{k},\omega) \frac{\vec{k}\cdot \mathrm{grad}_p f_0}{\vec{v}\cdot\vec{k} - i/\tau - \omega}.$$

A magnetic moment $\vec{M}$ is equivalent to a current $(c/4\pi)$ rot $\vec{M}$, and thus we find, by summation, the spin contribution to the transverse dielectric constant of a degenerate gas

$$\delta\varepsilon^{tr}(k,\omega) = -\frac{k^2 p_0^2 c^2 \beta^2}{\pi^2 \omega^2 \hbar^3 v_0}\left\{1 + \frac{\omega + i/\tau}{2kv_0}\log\frac{kv_0 - \omega - i/\tau}{-kv_0 - \omega - i/\tau}\right\}. \quad (2.12)$$

If, more naturally, we describe the effect as a contribution to the permeability μ we can use the relation $\delta(1/\mu) = -(\omega/kc)^2\,\delta\varepsilon^{tr}$. In the limit of low frequencies we obtain

$$\delta\left(\frac{1}{\mu}\right) \simeq -\frac{p_0^2\beta^2}{\pi^2\hbar^3 v_0}\left\{1 + i\frac{\pi\omega}{2kv_0}\right\}. \quad (2.13)$$

For a free electron gas, the first term in the brackets leads to the familiar spin paramagnetism, equal to three times the diamagnetic contribution. Since, generally, the spin contribution to the electromagnetic properties of the system is small compared to the orbital contributions (2.3), it will be neglected in the following.

When comparing our description in this paragraph with treatments by other authors it should be mentioned, first of all, that a hydrodynamical discussion of the motion of a Thomas-Fermi gas has been attempted by Bloch (1933, 1934). The pressure—proportional to $\varrho^{5/3}$—arising from the zero-point kinetic energies was introduced in hydrodynamical equations of motion. Using only the two parameters pressure and velocity of the liquid model for the description of the state of the electron gas, Bloch

obtained an approximate and smoothed-out picture of the motion. It is easily shown that, in the linear approximation, the model of BLOCH gives, for the dielectric constant of the longitudinal motion,

$$\varepsilon^l = 1 - \frac{\omega_0^2}{(\omega + i/\tau)^2 - k^2 u^2}, \tag{2.14}$$

where the constant u^2 is given by $u^2 = (5/9) \cdot v_0^2$ and thus comparable to the velocity, v_0, at the top of the Fermi distribution. The velocity u may be said to represent the sound velocity in the gas without interaction ($\omega_0 = 0$). When kv_0 is small compared to ω we make a series development of (2.14) and compare with (2.6). In this limit, the latter formula leads to $u^2 = (3/5) \cdot v_0^2$, which is close to the above value found by BLOCH (cf. BOHM and GROSS, 1949). For values of kv_0 larger than ω, equation (2.14) gives a screening of the field quite similar to that in (2.8), although the value of u^2 should now be changed to $(1/3) \cdot v_0^2$; moreover, the important finite imaginary part of the dielectric constant (2.8) is not reproduced in (2.14). We shall return, presently, to some of the simple features described by (2.14).

As regards the application of the hydrodynamical model of BLOCH to the transverse field, we note that the transverse motions do not affect the density. The model must therefore give the same result as the simplest classical picture. Accordingly, one finds the transverse dielectric constant (2.5), which formula is obtained from (2.3) when putting the electronic velocity v_0 equal to zero. The hydrodynamical model is not appropriate, apparently, when (2.7) applies, i. e. for $kv_0 > |\omega + i/\tau|$. It fails to describe the orbital diamagnetism and the anomalous skin effect (cf., e. g., LINDHARD, 1953).

As mentioned in the introduction, an interesting attempt to give a more exact solution of the equations of motion for a degenerate gas has been made recently by TOMONAGA (1950). His treatment, however, was limited to the one-dimensional case. On the assumption that the field contained only waves of long wavelength, which in the present formulation means that semi-classical methods apply, TOMONAGA obtained a solution without recourse to usual perturbation theory. The result was a linear field equation for the longitudinal field given by the

dielectric constant (2.14) of BLOCH's three-dimensional model, only now with $u = v_0$, v_0 being the velocity at the top of the one-dimensional Fermi distribution. Now, it so happens that, if we apply the present method to the one-dimensional case where the velocity v and the wave vector can be only parallel and anti-parallel, we find immediately from (2.2) the dielectric constant of TOMONAGA, (2.14) with $u = v_0$. It may appear surprising that a first order treatment gives the same result as a more precise transformation. The reason for the agreement is that in the one-dimensional case the first order calculation is in fact self-consistent if only long waves are considered, as will be shown in § 5. We note, further, that the Tomonaga treatment is not an exact one, because of the omission of the short waves of the field. In § 3 we shall deduce an expression, (3.12), for the one-dimensional dielectric constant, based on quantum mechanical equations of motion for the electrons.

Let us regard, for a moment, the simple picture suggested by the Bloch model (2.14). For long waves the energy of the field quanta is

$$E = (p^2 u^2 + \hbar^2 \omega_0^2)^{\frac{1}{2}} \simeq \mu u^2 + p^2/2\mu ,$$

where $p = \hbar k$ is the momentum, and μ may be described as the mass of the quanta. The value of μ is

$$\mu = \frac{\hbar \omega_0}{u^2} = 2m \left(\frac{e^2}{3\pi \hbar v_0} \right)^{\frac{1}{2}} \cdot \left(\frac{v_0}{u} \right)^2 ,$$

which is usually of the order of the electron mass. If, for instance, a charged particle passes through the gas, such field quanta, equivalent to particles, will be created. In the first approximation, these particles will subsist in the system, but a closer inspection shows that they have a finite life-time (cf. § 5). For increasing values of p we find, according to the more precise field equation (2.4), that the solution of the field equations is considerably more complicated, and large damping terms appear. In this region field quanta similar to particles can not be identified. However, for nearly stationary cases the field equations, (2.8), are of the Yukawa type with screening length $1/\varkappa = v_0/3\,\omega_0$ (equation (2.14) gives $1/\varkappa = u/\omega_0$). In BLOCH's model the screening length

is equal to the Compton wavelength, $\hbar/\mu u$, of the field quanta, and in the more accurate description this is approximately true. The properties of the field quanta, therefore, are not unlike those of mesons. Yet, it should be emphasized that the field quanta in (2.14) account only for part of the properties of the gas, excepting the one-dimensional case. If we consider the thermal properties of the system (2.14), we find that the excitation is vanishing, unless the temperature, θ, is of the order or larger than $\hbar\omega_0$. This differs from the expected result for a free gas, i. e. thermal energy proportional to θ^2. When one uses instead (2.8), together with the particle equations, a θ^2-dependence is found, as will be discussed in II.

In a treatment of stopping problems KRONIG and KORRINGA (1943), and KRONIG (1949), have attempted to describe the motion of a metallic electron gas as that of a charged liquid, subject to friction from the background of positive ions and having a certain internal viscosity. The pressure effects, important in the model of BLOCH, are neglected. For comparison, we shall quote the longitudinal dielectric constant corresponding to the model of KRONIG and KORRINGA.

Let ϱ_m and ϱ_c be the densities of mass and charge of the liquid. The viscosity is η, and the friction with the static background is ξ. From the classical hydrodynamical equations of motion one finds then, in the linear approximation,

$$\varepsilon^l = 1 - \frac{4\pi\varrho_c^2}{\varrho_m\omega^2 + i\omega(\xi + 2\eta k^2)}. \tag{2.15}$$

This differs from the classical ε in (2.5) only by the term involving the viscosity η. If this arbitrary parameter is large, as assumed by KRONIG and KORRINGA, one gets a strong dependence of ε^l on the wave number of the field. However, for large values of k, where the viscosity term is dominating, quantum theory takes over; in § 3 will be given a simple and more appropriate formula for ε^l at short wave lengths (see, for instance, eq. (3.4) and page 32). The model contained in (2.15) is less appropriate than that of BLOCH, and the picture involving a viscosity of the kind met with in ordinary hydrodynamics can be somewhat misleading (cf. p. 39). We note that the transverse dielectric constant in the model of KRONIG and KORRINGA will be quite similar to (2.15).

Already here mention may be made of the field equations proposed by LONDON for superconductors. It can be of interest to compare his equations with those in the present treatment of a free gas. We are then concerned with the equations (I)—(VIII), p. 29. F. LONDON (1950). They are seen to result from the following simple dielectric constants

$$\varepsilon^{tr} = \varepsilon^{l} = 1 - \frac{4\pi}{\Lambda\omega^2} + i\,\frac{4\pi\sigma_n}{\omega}, \tag{2.16}$$

where Λ is a constant which, according to LONDON, defines a density of superconducting electrons, $\varrho_s = (m/e^2\,\Lambda)$. Further, $\sigma_n = \varrho_n e^2 \tau_n/m$ is the conductivity of the remaining normal electrons. It is not easy to understand why ε^{tr} and ε^{l} should be put equal, corresponding to $\mu = 1$. For this reason, and because (2.16) is not specified in detail for long and short waves of the field, it is difficult to make a direct comparison with the dielectric constants derived above.

It can be illustrative to compare (2.16) with the conventional picture of an ideal conductor. The field equations for the latter are usually assumed to be those resulting from (2.5) with $1/\tau = 0$. Therefore, for fields varying not too quickly in time, LONDON's equations are merely the retarded equations for the usual model of an ideal conductor. It appears from (2.7) and (2.8) that the mentioned model of an ideal conductor is inadequate as a picture of an electron gas, which makes more obscure the meaning of equations as (2.16) (cf. II and LINDHARD, 1953).

§ 3. Quantum mechanical treatment of electrons.

We shall now include the quantum mechanical description of the electronic motion in the calculation which, otherwise, follows quite similar lines as the semi-classical treatment in § 2. The main difference is that previously we distinguished only between wavelengths $1/k$ of the field, long or short compared to v_0/ω, while now we must further compare $1/k$ with the wavelengths $1/k_0 = \hbar/mv_0$ of the electrons in the gas. As long as one is concerned with long waves of the field there will be only minor corrections to the semi-classical formulae, but for field waves shorter than $1/k_0$ the field equations are completely changed.

In the calculations we make the same assumption as in § 2 regarding a common field in which the electrons move. This means essentially that time-dependent Hartree equations are used. As to the initial state we take, as before, simply the freely moving electrons of the Sommerfeld model and compute to first order in e^2. None the less, our results go beyond first order when we introduce an effective electron mass (cf. § 5).

Already here mention may be made of one particular circumstance concerning the behaviour in time of a system when disturbed by external fields. It is apparent that the wave functions will not develop independently in time, because each particle moves in a field determined by the others. But, since the originally

orthogonal one-particle wave functions are governed all by the same one-particle Hamiltonian, they will automatically remain orthogonal. This has the particular advantage that the exclusion principle need not be taken into account explicitly in the dynamical treatment, just as in the semi-classical case. It follows, moreover, that for a wide class of phenomena, e. g. in impacts by external charges, the seeming reduction in transition possibilities due to the exclusion principle* does not come into play.

As regards the boundary conditions in the perturbation treatment, one should at the time $t = -\infty$ find the unperturbed state. This condition is fulfilled if an infinitesimal negative imaginary term, $-i\Gamma/2$, is introduced in the energies of the electron states, because one then can obtain the retarded solutions of the equations of motion. It can be convenient to assume that the width Γ of the excited states is finite, since the width is connected with the resistance in simple pictures of metals. However, this manner of describing resistance is sometimes too crude,—it is less justified than in the classical equations (2.1). The way in which the metallic resistance appears in the field equations belongs to the properties of the positive background of charge, with which we are not immediately concerned. In such more complicated cases the use of causal dielectric constants is to be preferred (cf. § 5).

We consider an assembly of electrons having one-particle wave functions $\psi_n^0(\vec{r}, t) = \varphi_n^0(\vec{r}) \cdot \exp(-i\omega_n t + \Gamma_n t/2\hbar)$, the system being initially e. g. in the ground state. The perturbing term in the Hamiltonian is

$$\Omega = e\Phi(\vec{r}, t) + \frac{e}{2mc}\left(\vec{p} \cdot \vec{A}(\vec{r}, t) + \vec{A}(\vec{r}, t) \cdot \vec{p}\right), \qquad (3.1)$$

where for the present we disregard spin contributions.

Equations for longitudinal field.

The perturbation gives rise to an induced charge and current density for each of the electrons. The total induced charge density, $\varrho_i(\vec{r}, t)$, in the system is the sum of the induced densities for the individual electrons. We write, as before, the fields and the

* Explicit reductions of such kind have been applied in nucleon problems by, e. g., Goldberger (1948), and Blatt and Weisskopf (1952).

charge densities in Fourier components in space and time. Choosing the gauge to be such that the longitudinal vector potential vanishes, we find from (1.9) the connection between the longitudinal dielectric constant, $\varepsilon^l(k, \omega)$, and the induced charge density. We specialize to the case of free electrons with wave functions $\varphi_n^0(\vec{r}) = V^{-1/2} \exp(i\vec{k}_n \cdot \vec{r})$ for the initial states, and similarly for the final states. When we choose a Fourier component $\vec{k}, \omega$ of the induced charge density, an electron with momentum $\hbar\vec{k}_n$ can only jump to a state with momentum $\hbar(\vec{k}_n \pm \vec{k})$. The contribution from this electron to the Fourier component of the induced charge density is, assuming a constant imaginary energy difference, $i\gamma$,

$$\varrho_{\text{ind.}}(\vec{r}, t) = -\frac{2me^2}{\hbar^2 V}\left\{\frac{\Phi(\vec{k}, \omega)\cdot \exp(i\vec{k}\cdot\vec{r} - i\omega t)}{k^2 + 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(-\omega - i\frac{\gamma}{\hbar}\right)} + \text{compl. conj.}\right\}. \quad (3.2)$$

The distribution function of the electrons we denote as $f = f(E_n)$, E_n being the unperturbed energies. Summing (3.2) over the electrons and using the definition of the retarded dielectric constant, we find for free electrons

$$\varepsilon^l(k, \omega) = 1 + \frac{2m^2\omega_0^2}{\hbar^2 k^2}\sum_n \frac{f(E_n)}{N}\left\{\frac{1}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right. \left. + \frac{1}{k^2 - 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right\}, \quad (3.3)$$

where $N = \sum_n f(E_n)$ is the total number of electrons, and ω_0 as before is the classical resonance frequency of the gas, $\omega_0^2 = 4\pi e^2 N/mV$.

In (3.3), we sum freely over all electron states, since contributions from jumps between two occupied states cancel, as follows from our previous remarks. In the derivation of (3.3), the technique of quantum field theory is not advantageous, but can be so in higher order computations.

We apply first the formula (3.3) to the extremely simple case of the electrons being initially at rest. Accordingly, the momenta $\hbar k_n$ are neglected in (3.3), which leads to

$$\varepsilon^l(k,\omega) = 1 + \frac{\omega_0^2}{\frac{\hbar^2}{4\,m^2}\,k^4 - \left(\omega + i\frac{\gamma}{\hbar}\right)^2} \quad . \tag{3.4}$$

This formula corresponds to the classical result (2.6) and shows in a direct manner how the inclusion of quantum effects implies a strong dependence on the wavelength of the field.

Take next the case of the Fermi distribution, and assume $f(E_n) = 1$ for $E_n \leq E_0$, while for higher energies $f(E_n) = 0$. Put further $E_0 = mv_0^2/2 = \hbar^2 k_0^2/2\,m$. In the summation in (3.3) we first average over the electrons with the same energy $\hbar^2 k_n^2/2\,m$, that is to say, we integrate over the angle between k_n and the fixed vector k. One then obtains characteristic logarithmic expressions, not unlike those in the semi-classical formulae for ε. Finally, we integrate over the energy E_n, from 0 to E_0. When quoting the resulting formula it is convenient to use a few abbreviations; we write $z = k/2\,k_0$, $\omega' = \omega + i(\gamma/\hbar)$, $u = |\,\omega\,|/kv_0$, and $u' = \omega'/kv_0$. The parameter z, which contains the wavelength of the field and that of an electron in the gas, serves to indicate how far one is away from the classical limit, the latter corresponding to $z = 0$. Further, the quantity u (or u') shows whether an electron during one period of the field travels a distance longer or shorter than one wavelength of the field, these two cases being represented by respectively $u < 1$ and $u > 1$. The formula for the dielectric constant of the degenerate Fermi gas is then

$$\varepsilon^l = 1 + \frac{3\,\omega_0^2}{k^2 v_0^2} \cdot f, \tag{3.5}$$

$$\left.\begin{aligned} f = \frac{1}{2} + \frac{1}{8z}\{1 - (z-u')^2\}\log\frac{z-u'+1}{z-u'-1} \\ + \frac{1}{8z}\{1 - (z+u')^2\}\log\frac{z+u'+1}{z+u'-1}, \end{aligned}\right\} \tag{3.6}$$

where principal values are taken. In the present connection, we

are concerned with very small values of the damping γ. In the limit of small γ we find the following expressions for the real and imaginary parts of $f = f_1 + if_2$,

$$\left.\begin{aligned} f_1(u,z) = \frac{1}{2} + \frac{1}{8z}\{1-(z-u)^2\}\log\left|\frac{z-u+1}{z-u-1}\right| \\ + \frac{1}{8z}\{1-(z+u)^2\}\log\left|\frac{z+u+1}{z+u-1}\right|, \end{aligned}\right\} \quad (3.7)$$

$$f_2(u,z) = \left\{\begin{array}{ll} \dfrac{\pi}{2}u, & \text{for} \quad z+u<1, \\ \dfrac{\pi}{8z}\{1-(z-u)^2\}, & \text{for} \quad |z-u|<1<z+u, \\ 0, & \text{for} \quad |z-u|>1. \end{array}\right\} \quad (3.8)$$

The contents of these formulae for the longitudinal dielectric constant are perhaps best appreciated by finding the approximate values in a few limiting cases. It is seen that, for $z = 0$, the equations (3.5), (3.6) lead to the semi-classical formula (2.4). A similar result is obtained for small values of z. Consider here the case of $z + u < 1$. This corresponds in the semi-classical treatment to formula (2.8), where $u < 1$ and thus the electrons can move through several wavelengths of the field during one period. We assume that the damping is small and develop (3.7) in powers of the small quantities $u \pm z$. This leads to

$$\varepsilon^l(k,\omega) = 1 + \frac{3\omega_0^2}{v_0^2 k^2}\left\{1 + i\frac{\pi\omega}{2v_0 k} - \frac{\omega^2}{v_0^2 k^2} - \frac{k^2}{12k_0^2} + \cdots\right\}, \quad z+u<1 \quad (3.9)$$

a formula equivalent to (2.8), except for the last term in the brackets, giving rise to the subtraction of a constant in ε^l (cf. also LINDHARD, 1946).

Another case with correspondence to the treatment in § 2 is that of $u > 1 + z$. This inequality implies that $\hbar\omega$ is large compared with the energy transfer $\hbar^2(k^2 + 2kk_0)/2m$ to an electron. Therefore the classical formula on the form (2.6), where $u > 1$, must apply, as is also seen from a series development of (3.6)

$$f(u', z) = -\frac{1}{3\,u'^2} - \frac{1}{5\,u'^4} - \frac{z^2}{3\,u'^4} \cdots, \quad |u'| > 1 + z. \tag{3.10}$$

In the more typical quantum mechanical case where $z \gtrsim u + 1$, the longitudinal dielectric constant tends to 1 for increasing z, because the electrons respond only weakly to the short waves of the field. We shall quote only the result for the region where the imaginary part of ε^l is finite. Thus, we find

$$\varepsilon^l(k, \omega) = 1 + \frac{3\,\omega_0^2}{v_0^2\,k^2}\left\{\frac{i\pi}{8\,z}(1 - (z-u)^2) + \frac{z-u}{2\,z} + \cdots\right\}, \tag{3.11}$$

in the case of $|u - z| < 1$, $u + z > 1$. The resulting resonance effect is of the same kind as that in (3.4), for large values of k.

Before finishing this discussion of the longitudinal field we shall deduce the dielectric constant for the case of a Fermi gas in one dimension. In § 2, it was shown that the semi-classical treatment gives just the field equations of TOMONAGA (1950). The present quantum mechanical perturbation method must of course give a somewhat different result for short waves of the field.

The desired equations can be derived immediately from equation (3.3), if there we put equal to zero all vector components in the direction of the y- and z-axes. As before, it is supposed that $f(E_n) = 1$ for $E_n < E_0$, and otherwise $f = 0$. Here, $E_0 = mv_0^2/2 = \hbar^2 k_0^2/2\,m$. Summing over $k_n = k_{xn}$ one finds that

$$\varepsilon^l(k, \omega) = 1 + \frac{m^2\,\omega_0^2}{2\,\hbar^2\,k^3\,k_0}\left\{\log\frac{z - u' + 1}{z - u' - 1} + \log\frac{z + u' + 1}{z + u' - 1}\right\}, \tag{3.12}$$

which formula for large k_0 tends to the semi-classical one, (2.14), with $u = v_0$. As usual, principal values are to be taken in the logarithmic expression in (3.12). In the other extreme of k_0 small compared to the wave vector k of the field, the formula (3.12) leads to the general result (3.4). It is seen that, while in the semi-classical treatment the case of one dimension was much simpler than the three-dimensional case, this is not so in the quantum mechanical treatment.

Equations for transverse field.

For the case of transverse fields we shall apply the same perturbation method as for longitudinal fields, and again it is assumed that the motion of the electrons is non-relativistic. The perturbation term in the Hamiltonian is (3.1), where the spin contribution is neglected. The inclusion of spin would only give rise to small changes in the field equations, as is evident from the result (2.13). In analogy to (3.2) we now find the contribution to the induced transverse current from an electron with wave vector $\vec{k}_n$, in the simple harmonic field $\vec{A}(\vec{k}, \omega) \cdot \exp(i\vec{k} \cdot \vec{r} - i\omega t)$ + compl. conj., corresponding to a single Fourier component,

$$\left.\begin{aligned} &\delta\vec{j}_{ind}(\vec{r}, t) = \\ &-\frac{e^2}{2mcV}\left\{\left(2\vec{k}_n + \vec{k}\right)\frac{\left(2\vec{k}_n + \vec{k}\right)\cdot\vec{A}(k,\omega)}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)} + 2\vec{A}(k,\omega)\right\} \\ &\cdot \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) + \text{compl. conj.} \end{aligned}\right\} \qquad (3.13)$$

We introduce the distribution function of the electrons and, applying (1.3), we can express the transverse dielectric constant on a closed form. The dielectric constant is a three-dimensional tensor which, because of the symmetry in problem, is on diagonal form, with diagonal elements

$$\left.\begin{aligned} &\varepsilon^{tr}(k, \omega) = 1 \\ &\frac{2\omega_0^2}{\omega^2}\sum_n \frac{f(E_n)}{N}\left\{\left(k_n^2 - \left(\vec{k}\cdot\vec{k}_n\right)^2/k^2\right)\left(\frac{1}{k^2 + 2\vec{k}\cdot\vec{k}_n - \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right.\right. \\ &\left.\left. + \frac{1}{k^2 - 2\vec{k}\cdot\vec{k}_n + \frac{2m}{\hbar}\left(\omega + i\frac{\gamma}{\hbar}\right)}\right) - \frac{1}{2}\right\}. \end{aligned}\right\} \qquad (3.14)$$

We pass immediately to the summation for the case of a degenerate Fermi gas, which will give the quantum mechanical equation corresponding to (2.3). With fixed values of ω and the

vector $\vec{k}$ we first average over the angle between $\vec{k}_n$ and $\vec{k}$, and next integrate over the numerical value of $\vec{k}_n$, from 0 to k_0. Using a familiar notation we can then write

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2} f^{tr}(u, z), \tag{3.15}$$

where $f^{tr} = f_1^{tr} + i f_2^{tr}$ is determined from the equations

$$f_1^{tr}(u, z) = \frac{3}{8}(z^2 + 3u^2 + 1) - \frac{3}{32 z}\left\{(1 - (z-u)^2)^2 \log\left|\frac{z-u+1}{z-u-1}\right| + (1 - (z+u)^2)^2 \log\left|\frac{z+u+1}{z+u-1}\right|\right\}, \tag{3.16}$$

$$f_2^{tr}(u, z) = \begin{cases} \frac{3}{4}\pi u (1 - u^2 - z^2), & \text{for } u + z < 1, \\ \frac{3\pi}{32 z}(1 - (u-z)^2)^2, & \text{for } |u - z| < 1 < u + z, \\ 0, & \text{for } |u - z| > 1. \end{cases} \tag{3.17}$$

The equations (3.16), (3.17) are valid in the limit of $\gamma \to 0$. The formula for f, when γ is finite, can easily be obtained from (3.16) by comparing this equation with (3.7) and (3.6).

Let us find the values of the above functions in the simple limiting cases introduced earlier. Suppose first that $u + z < 1$. It is seen that here

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{\frac{k^2}{4 k_0^2} + \frac{3\omega^2}{k^2 v_0^2} + \cdots - i\frac{3\pi\omega}{4 k v_0}\left(1 - \frac{\omega^2}{k^2 v_0^2} - \frac{k^2}{4 k_0^2}\right)\right\}, \quad u + z < 1, \tag{3.18}$$

This result is not unlike that in the semi-classical case, (2.7); only we find a characteristic new term—the first one in the brackets in (3.18). When multiplying by ω^2/k^2c^2, this term is seen to contribute $-(k_0/3\pi)\cdot(e^2/mc^2)$ to the permeability. It represents the weak Landau diamagnetism of the gas and is here equal to one third of the paramagnetic spin contribution (2.13). We note that the Landau term appears only in the field equations when $u + z \ll 1$, i.e. for small values of the frequency ω.

For the case $u > 1 + z$ we have again correspondence to the classical results, and the value of ε^{tr} is found from the series development

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{1 + \frac{1}{5\,u^2} + \frac{3\,z^2}{4\,u^2} + \cdots\right\},\ u > 1 + z. \quad (3.19)$$

Suppose that terms higher than the first in the brackets can be neglected. It is apparent, then, that for finite γ's there remains a small difference from the previous semi-classical equation (2.5), because the damping is not reproduced in (3.19). This difference has its origin in the term $(e^2/mc)\,\vec{A}\,(\vec{r}, t)\cdot|\psi|^2$ in the expression for the quantum mechanical current of a particle, and is due to the crude assumptions made regarding the damping in this paragraph. The difference disappears in a more systematic treatment, as outlined in § 5.

In the quantum mechanical limit the region of interest is $(u-z)^2 < 1 << (u+z)^2$, where the imaginary part of ε is finite,

$$\varepsilon^{tr}(k, \omega) = 1 - \frac{\omega_0^2}{\omega^2}\left\{1 - i\frac{3\,\pi}{32\,z}(1-(u-z)^2)^2\right\},\ |u-z| < 1 << u+z. \quad (3.20)$$

The present calculations of the dielectric constants were non-relativistic. As to a relativistic description we may quote one simple example. It is assumed that the electrons are originally at rest, and the induced charges and currents are computed from the Dirac equation for the electron. From a so-called simple calculation, one then finds that the longitudinal and transverse dielectric constants are equal and given by

$$\varepsilon^{l}(k, \omega) = \varepsilon^{tr}(k, \omega) = 1 + \frac{\omega_0^2}{\dfrac{\hbar^2 k^4}{4\,m^2} - \left(1 + \dfrac{\hbar^2 k^2}{2\,m^2 c^2}\right)\omega'^2 + \dfrac{\hbar^2\,\omega'^4}{4\,m^2\,c^4}}, \quad (3.21)$$

where $\omega' = \omega + i\dfrac{\gamma}{\hbar}$. It is evident that, for small k and ω, the formula leads to (3.4). This result, moreover, proves in a direct manner the equality of ε^{tr} and ε^{l} in the case represented by (3.4), where we calculated only the longitudinal dielectric con-

stant. An application of (3.21) can be found, e.g., in the question of energy loss by a relativistic particle.

§ 4. Damping of motion of charged particles, and self-energy.

The slowing-down of a charged particle moving through the system is a simple and interesting problem, in which the above description can be of great convenience. The motion of a charged particle is of particular interest in the present connection because it relates to the self-consistency of the method of approach employed in this paper, as will be discussed more closely in the next paragraph. We shall derive general formulae for, e.g., the specific energy loss. However, the dielectric constants calculated to first order in e^2, starting from the free particle approximation, will not always lead to very accurate results, and accordingly we do not perform elaborate computations of the stopping.

A treatment of stopping problems based on the dielectric properties of a medium was suggested early by FERMI (1924). As we shall see, a detailed discussion of the problem in this manner will not be confined to the simple dispersion properties of the medium. Still, if one considers only distant collisions, these properties are sufficient, which circumstance was utilized by FERMI (1940) in a treatment of polarization effects in the stopping of relativistic particles. We shall show that our classical field equations are applicable even for the short waves of the field.

It is not the purpose here to discuss the numerous aspects of the problem of energy loss in matter by fast charged particles; a general survey based on simple concepts has been given by N. BOHR (1948). For the present we are concerned with the case where a perturbation treatment can be used. Let it be assumed, further, that the particle is so heavy that its energy loss is comparatively small, and it behaves approximately as a classical point charge, ze, moving with a constant velocity, v, — we may for instance consider a proton. This simplification allows a description by forced classical vibrations, since the source density is given beforehand as a function of space and time.

Before specializing to a free electron gas we give a general

derivation of the energy loss suffered by the particle. In the case considered the charge density is

$$\varrho_0(\vec{r}, t) = z_1 e \delta(\vec{r} - \vec{v}t) = \frac{z_1 e}{V} \sum_{\vec{k}} \exp\left(i\vec{k} \cdot \vec{r} - i\vec{k} \cdot \vec{v}t\right). \quad (4.1)$$

The energy loss per unit path is $z_1 e$ times the electric field strength E at the immediate position $\vec{r} = \vec{v}t$ of the particle, which field is antiparallel to $\vec{v}$. Limiting at first the treatment to the longitudinal field, as is always sufficient if the particle velocity is non-relativistic, we obtain for the time-independent field to which the particle is subjected

$$\vec{E}(\vec{v}t, t) = -\sum_{\vec{k}} i\vec{k} \Phi\left(\vec{k}, \vec{k} \cdot \vec{v}\right), \quad (4.2)$$

which quantity therefore depends on the imaginary, or odd, part of $\Phi\left(\vec{k}, \omega\right)$. The resulting specific energy loss is obtained from (4.1), (4.2), and (1.4); we transform the three-dimensional integration over k to one over its numerical value and one over $\omega = kv \cos\left(\vec{k}, \vec{v}\right)$, and find*

$$\frac{dE}{dR} = -\frac{z_1^2 e^2}{\pi v^2} \operatorname{Im} \left\{ \int_0^{\infty} \frac{dk}{k} \int_{-kv}^{+kv} d\omega \frac{\omega}{\varepsilon^l(k, \omega)} \right\}, \quad (4.3)$$

where $\operatorname{Im}(x + iy) = y$. It is noteworthy that, since the dielectric constant enters in the denominator, the energy loss depends both on the longitudinal conductivity (1.10) and on the polarizability.

For the sake of completeness, we derive the contribution from the transverse field too. The transverse electric field is given by $\vec{E}^{tr}(\vec{k}, \vec{k} \cdot \vec{v}) = i(\vec{k} \cdot \vec{v}/c)\vec{A}^{tr}(\vec{k}, \vec{k} \cdot \vec{v})$. By means of (1.3) the field is obtained from the transverse part of the current corresponding to equation (4.1). It is then an easy matter to deduce the stopping

$$\left.\frac{dE}{dR}\right|_{tr} = -\frac{z_1^2 e^2}{\pi c^2} \cdot \operatorname{Im} \left\{ \int_0^{\infty} k dk \int_{-kv}^{+kv} \omega \, d\omega \frac{1 - \omega^2/k^2 v^2}{k^2 - \frac{\omega^2}{c^2} \varepsilon^{tr}(k, \omega)} \right\}. \quad (4.4)$$

Since the contribution from the longitudinal field is given always by (4.3), the total average energy loss is the sum of (4.3) and

* The energy loss may also be calculated from the energy dissipation term in the conservation equation for the energy of the field (the bracketed term on the right-hand side of (1.11)). This leads again to (4.3).

(4.4). In some cases (4.4) can be simplified considerably. Thus, if ε^{tr} has no finite imaginary part and can be considered as independent of k, the formula reduces to the one deduced by FRANK and TAMM (1937) for the Čerenkov radiation (see further A. BOHR, 1948). As mentioned above, we shall be concerned only with the case where the contribution (4.4) can be neglected.

It is perhaps not out of place to interpose here some further general remarks, as an introductory to the calculations below. In the problem of the stopping power of matter it can be illustrative to consider two extreme states of motion of the electrons in the substance. First, one can regard separated electrons, each bound in, say, a harmonic oscillator of frequency ω. This problem was solved at an early date (N. BOHR, 1913), and it is a characteristic feature that the energy transfers to the electrons become very small at distances exceeding the adiabatic limit v/ω, determined by the velocity of the particle and the frequency of the oscillator. Second, one may consider the opposite case where the electrons move freely in the system. Then, a different kind of reduction must appear in the energy transfer at large distances (if there were no reduction the total energy transfer would be infinite). The proper explanation is to be found in the screening of the field by polarization in the gas, as was shown by KRAMERS (1947), but not in the resistance damping, the latter having been suggested by v. WEIZSÄCKER (1933). Simply for dimensional reasons it is clear that the frequency which for free electrons replaces that of the harmonic oscillator must be the classical resonance frequency of the gas, $\omega_0 = (4\pi e^2 \varrho/m)^{\frac{1}{2}}$, where ϱ is the density of electrons. By combining the above two pictures—the harmonic oscillator and the free gas—in a suitable way, one should be able to account for the stopping effects in atomic systems. We note here that the free electron model—i.e. the Thomas-Fermi treatment—is useful even in the description of atoms. As a matter of fact, the present calculations on a free electron gas were utilized in a simple general discussion of atomic stopping power in a recent paper (LINDHARD and SCHARFF, 1953).

Let us apply the formula (4.3) to the simplest case: a homogeneous gas of electrons at rest. We therefore introduce (3.4) in (4.3) and find by integration

$$\frac{dE}{dR} = \frac{4\pi z_1^2 e^4}{mv^2} \cdot \varrho \cdot L, \tag{4.5}$$

where

$$L = \log \left\{ \frac{mv^2}{\hbar\omega_0} + \left(\left(\frac{mv^2}{\hbar\omega_0} \right)^2 - 1 \right)^{\frac{1}{2}} \right\}. \tag{4.6}$$

In the model used the energy loss therefore vanishes for $mv^2 \leqq \hbar\omega_0$. However, the model is preferably to be applied in the familiar extreme of large v. From a series development of (4.6) one here gets

$$L = \log \frac{2\, mv^2}{\hbar\omega_0}, \tag{4.7}$$

which is just the result of KRAMERS*. One may indeed, as indicated above, derive this formula using a simple qualitative argument (cf. A. BOHR, 1948; LINDHARD and SCHARFF, 1953), but then an undetermined constant remains inside the logarithm.

If one calculates separately the contribution to the energy loss from distant collisions, one can from (2.6) or the Bloch model (2.14) refind the result (4.7), again apart from a constant in the logarithm. Mention should be made here of the model of KRONIG and KORRINGA (1943), KRONIG (1949), introduced in a treatment of stopping by a free gas. A special feature of their liquid picture of the system is the appearance of a viscosity, η, leading to the dielectric constant (2.15). KRONIG and KORRINGA applied this semi-classical model not only for distant collisions, where it does not differ from (2.6) or (2.14), but even for close collisions where the viscosity governs the motion. One gets a formula similar to (4.7), though with $8\,\eta/3\,\varrho$ instead of $\hbar$ in the logarithm. It is hardly desirable in this way to replace quantum theory by classical viscosity. Still, KRONIG and KORRINGA found the screening by polarization at large distances prior to KRAMERS.

We now turn to the treatment of a degenerate Fermi gas, where (3.5), (3.7), and (3.8) apply. Of course, these more involved formulae should be regarded merely as approximate estimates of the behaviour of the gas. The reason for their application here is that we wish to see whether deviations from (4.7) or

* It is to be noted that we have proved here the formula (4.7) only for such dilute gases where the Maxwell distribution applies (cf. § 5).

(4.7) will occur. For instance, one might find a change by a factor inside the logarithm in (4.7), which would be difficult to discover in the treatments of, e. g., KRAMERS (1947) and PINES (1953).

Introduce first the equation (3.5) in (4.3), so that the energy loss is expressed by the functions f_1 and f_2, integrated over the variables u and z. The logarithmic term, L, in (4.5) takes the form

$$L = \frac{6}{\pi}\int_0^{v/v_0} u\,du \int_0^{\infty} z^3\,dz\, \frac{f_2(u,z)}{(z^2+\chi^2 f_1(u,z))^2+(\chi^2 f_2(u,z))^2}, \qquad (4.8)$$

where $\chi^2 = e^2/(\pi\hbar v_0)$. The quantity χ^2 varies only slowly with the density of electrons, and is for metallic electrons somewhat less than unity. The density of the gas enters only through χ^2 and through the upper limit, v/v_0, in the integration over u. The functions f_1 and f_2 are given by (3.7) and (3.8), respectively. The value of L in (4.8) depends most directly on f_2, i. e. the imaginary part of ε, while f_1 is important only for small values of z, which implies that the polarization is of significance for distant collisions.

For the case of velocities v high compared with v_0 one may proceed as follows in evaluating L. Divide the integration in two parts, $u < u^1$ and $u > u^1$, where u^1 is a constant somewhat larger than unity. The integration over $u < u^1$ will give some constant. For $u > u^1$ there are two contributions, one from the region where f_2 is finite, or $|u-z| < 1$, corresponding to close collisions. The other contribution arises from the resonance at longer distances, where $z^2 + \chi^2 f_1(u, z) = 0$. Using the formulae (3.7), (3.8), and (3.10), we find that the two last mentioned contributions are equal to $\log(v/v_0 u^1)$, and adding the result for $u < u^1$ we have

$$L = \log\left\{C(\chi)\,\frac{2mv^2}{\hbar\omega_0}\right\}, \qquad (4.9)$$

where the quantity C is expected to depend on the value of χ. Of course, for small densities, and accordingly thermal velocities of the electrons in the gas, the value of C will tend to unity and equation (4.7) results. In the opposite extreme of high densities, i.e. $\chi^2 \ll 1$, a simple numerical computation indicates that C

again approaches unity. Intermediate densities, $\chi^2 \sim 1$, give values of C not much less than 1. In most cases one will then not be far in error in assuming $C = 1$.

For low velocities of the particle, by which we understand values of v small compared to the maximum velocity v_0 of the electrons in the gas, there is a considerable difference between (4.8) and the simpler picture resulting in (4.6). We can here in the limit replace u by 0 in f_1, and f_2, in the denominator in (4.8), and using (4.5) it is seen that

$$\left.\begin{aligned} \frac{dE}{dR} &= \frac{4 z_1^2 e^4 m^2}{3 \pi \hbar^3} v \cdot C_1(\chi) \\ C_1(\chi) &= \int_0^1 \frac{z^3 \, dz}{(z^2 + \chi^2 f_1(0, z))^2}, \end{aligned}\right\} \tag{4.10}$$

showing that for small v the energy loss is proportional to the velocity. The case of low velocities was treated on semi-classical lines in a paper by FERMI and TELLER (1947). The formula obtained by them results from (4.10) if in C_1 the function $f_1(0, z)$ is put equal to $f_1(0, 0) = 1$, and at the same time $\chi^2 \ll 1$, which gives $C_1 = -\log \chi - \frac{1}{2} \approx \frac{1}{2} \log (v_0 \hbar / e^2)$.

One may replace (4.10) by a simpler and yet approximately correct formula. For this purpose we notice that the function $C_1(\chi)$ over a wide range of densities increases nearly as $\varrho^{1/4}$. It can further be useful to compute numerically some values of the expression (4.8) in the interval $0 < (v/v_0) < 1$, where generally it is not allowed to put $u = 0$ in f_1 and f_2. We find that, instead of (4.10), the following logarithmic term in (4.5) can be used

$$L = \text{const} \cdot \left(\frac{2 m v^2}{\hbar \omega_0} \right)^{\frac{3}{2}}, \tag{4.11}$$

where the constant is of the order 0.1—0.05. The formula should be applicable in the density interval $0.02 \lesssim \chi^2 < 1$ but, because of the difference from the result for electrons at rest, (4.11) is less reliable than (4.9). According to (4.7) and (4.11), the function L is both for high and low velocities a function of the argument $2 m v^2 / \hbar \omega_0$, at least in the first approximation. This result can be of convenience in the handling of more involved problems (LINDHARD and SCHARFF, 1953).

We have supposed above that it is allowed to use a perturbation treatment, even for quite low velocities of the particle. However, when v is sufficiently low—of the order $z_1 e^2/\hbar$—there enters the new feature of capture and loss of electrons, and the particle will carry electrons which to some extent screen the field around it. For this and other reasons it is assumed, usually, that a perturbation treatment is applicable only when $v > z_1 e^2/\hbar$ (cf., e.g., BETHE and LIVINGSTON, 1937). Nevertheless, the present type of perturbation method is not much in error even for lower values of v. In connection with this point it is of importance to notice, first, that in a treatment of the present kind there is actually a screening of the field and an inclusion of capture and loss in so far as such effects can be contained in linear field equations. Moreover, the crucial entity in the collisions is the relative velocity, which for low v is given by the electron velocities in the system, and not by the much smaller quantity v. These remarks are of course not limited to a free gas of electrons, but apply for atomic systems in general.

If one regards the stopping problem from a frame of reference moving with the particle, the formulae (4.10) and (4.11) give the loss of momentum per unit time of an electron gas streaming, with uniform velocity v, past a point charge $z_1 e$. One may therefore make a comparison between the stopping of a slow particle and calculations of residual resistance, and since in the latter case the rate of momentum loss is proportional to the electric current according to Ohm's law, the proportionality to v in (4.10) is not accidental. More quantitatively, if in this formula one allows a reduction to the semi-classical approximation, i.e. $f_1(0, z) = f_1(0, 0)$, there is complete equivalence to a familiar formula for the resistance as caused by foreign scattering centres in a metal.* The approximate empirical justification of the resistance formula, both as regards proportionality to $z_1^2 \cdot v$ and as to absolute values, will again show that for low v it is not unjustified to apply a perturbation approach in the stopping problem.

In the considerations in the previous sections, the damping of the electronic motion by resistance was introduced in a most

* Cf., for instance, MOTT and JONES: Properties of Metals and Alloys, Oxford University Press 1936, p. 294. The approximation used there is of a similar kind as that of FERMI and TELLER (1947).

cursory manner. The above remarks on one aspect of the resistance problem will give an indication as to how the resistance results from the dielectric constant in a linear field equation.

Straggling.

The straggling in energy loss provides an example of the direct application of equation (1.14). We consider, as before, a heavy particle moving through the system. It is then possible to find the probability of absorption of quanta $\hbar\omega$, by use of (1.14), (1.4), (4.1), and (4.2).

We shall not enter on details of the problem and only compute the straggling for the case where the particle has penetrated a distance through the system sufficient to ensure that the distribution around the average is nearly Gaussian. For this case the standard deviation, Ω, in the distribution in energy loss is determined by the equation

$$\Omega^2 = \langle (dE)^2 \rangle - \langle dE \rangle^2 = \int_0^\infty d\omega\, p(\omega)\, \hbar^2 \omega^2, \tag{4.12}$$

where $p(\omega)\, d\omega$ is the differential probability for energy transfer $\hbar\omega$. The formula (4.12) gives the fluctuation for the limit where Bose-Einstein statistics reduces to Boltzmann statistics, i. e. when field quanta are only rarely excited. Therefore, equation (4.3) corresponds to the straggling

$$\Omega^2 = -\, dR\, \frac{z_1^2\, e^2}{\pi v^2}\, \mathrm{Im} \left\{ \int_0^\infty \frac{dk}{k} \int_0^{kv} 2\, d\omega\, \frac{\omega^2}{\varepsilon^l(k, \omega)} \right\}. \tag{4.13}$$

As before, we consider first the case where the electrons in the gas are initially at rest. One then introduces (3.4) in (4.13); we shall quote the resulting expression only in two limiting cases. For high velocities is obtained the well-known formula

$$\Omega^2 = 4\, \pi\, z_1^2 e^4\, \varrho\, dR, \tag{4.14}$$

where relativistic corrections are omitted. Usually, the relativistic correction is a multiplication by the factor $(1 - v^2/2c^2)/(1 - v^2/c^2)$, cf. (3.21). It is often simpler, instead of the absolute value of

the straggling, to give the relative value Ω^2/dE. When (4.14) holds we find $\Omega^2/dE = mv^2/L$. In the extreme of low velocities of the particle, which, however, according to (4.6) must obey $mv^2/\hbar\omega_0 > 1$, the relative straggling can easily be shown to approach a minimum value

$$\frac{\Omega^2}{dE} = 2^{\frac{1}{2}}\,\hbar\omega_0, \tag{4.15}$$

determined by $\hbar\omega_0$, the lowest possible energy transfer to the gas.

For a degenerate Fermi gas one gets of course the result (4.14) for high velocities v. For low velocities we quote the formula corresponding to (4.10). We introduce (3.5) in (4.13) and obtain

$$\left.\begin{aligned} \Omega^2 &= 4\,\pi\, z_1^2\, e^4\, \varrho\, dR \cdot L_\Omega, \\ L_\Omega &= \frac{12}{\pi}\left(\frac{v_0}{v}\right)^2 \int_0^{v/v_0} u^2\, du \int_0^\infty \frac{z^4\, dz\, f_2(u,z)}{(z^2+\chi^2 f_1(u,z))^2+(\chi^2 f_2(u,z))^2}. \end{aligned}\right\} \tag{4.16}$$

Next, u is put equal to zero in f_1 and f_2 in the denominator. On account of (3.8), L_Ω then takes the simpler form

$$L_\Omega = \frac{3}{2}\left(\frac{v}{v_0}\right)^2 \int_0^1 \frac{z^4\, dz}{(z^2+\chi^2 f_1(0,z))^2}, \tag{4.17}$$

showing that for low velocities the straggling, Ω^2, behaves as v^2. As in (4.10) the integral in (4.17) may be approximated by a simple function. Over a wide region of densities it is found that L_Ω is nearly proportional to $\varrho^{-1/2}$, and dividing by (4.11) the following estimate of the relative straggling is obtained for low velocities

$$\frac{\Omega^2}{dE} \simeq (5\, mv^2 \cdot \hbar\omega_0)^{\frac{1}{2}}. \tag{4.18}$$

As was to be expected, the individual energy transfers can thus be interpreted as the velocity v of the particle times an effective momentum of the electrons.

Multiple scattering.

A problem similar to straggling in energy loss is the multiple scattering of the particle. It is well known that the major contribution to the multiple scattering arises from collisions with the atomic nuclei of the substance, because they can have charges high compared with the electron charge. Still, let us briefly resume the results for the multiple scattering contributed by the gas of electrons. The mean of the square of the total angle of deflection of the particle, ψ^2, can be found by summing the squares of the individual momentum transfers perpendicular to the path, and dividing by the momentum squared of the particle. The general formula for the multiple scattering is then

$$\psi^2 = -\frac{z_1^2 e^2 \hbar\, dR}{\pi M^2 v^4} \operatorname{Im} \left\{ \int_0^\infty \frac{dk}{k} \int_0^{kv} 2\, d\omega \, \frac{k^2 - \dfrac{\omega^2}{v^2}}{\varepsilon^l(k,\omega)} \right\}, \tag{4.19}$$

if all angles entering in the formula are small. We apply here the present expressions for the dielectric constant, where the effect of the atomic nuclei is omitted. When evaluating (4.19) for high velocity of the particle, using either (3.4) or (3.5), we find that ψ^2 behaves similarly as the energy loss, but the distant collisions, i. e. the resonance collisions, will now be suppressed. One shows easily that

$$\psi^2 = \frac{m}{M} \cdot \frac{dE}{2E'} \tag{4.20}$$

where $E = Mv^2/2$ is the energy of the particle. In the limit of low velocities the dielectric constant (3.5) — (3.8) may be used. With the same approximation as in (4.17) it is seen that here

$$\psi^2 = \frac{\Omega^2}{4E^2}. \tag{4.21}$$

Width of states.

The width of the particle states, Γ, we introduce as $\hbar$ times the previously mentioned transition probability per unit time,

$$\Gamma = -\frac{2\,z_1^2\,e^2}{\pi v}\,Im\left\{\int_0^\infty \frac{dk}{k}\int_0^{kv} d\omega\,\frac{1}{\varepsilon^l(k,\omega)}\right\}, \tag{4.22}$$

and a similar expression results for the transverse field. It is interesting that the width is a classical property, in the sense that (4.22) does not depend directly on $\hbar$.

For high velocities we get, from (3.4) and (4.22),

$$\Gamma = \frac{z_1^2\,e^2}{2\,v}\,\omega_0 \log\frac{4\,mv^2}{\hbar\omega_0}. \tag{4.23}$$

For low velocities, in a Fermi gas, (3.5) leads to

$$\Gamma = \frac{z_1^2\,e^4\,m}{\pi\hbar^2}\cdot\frac{v}{v_0}\cdot C_\Gamma(\chi), \quad C_\Gamma = \int_0^1 \frac{z^2\,dz}{(z^2+\chi^2 f_1(0,z))^2} \sim \frac{\pi}{4\,\chi}. \tag{4.24}$$

Self-energy.

The self-energy of a charged particle is closely connected with its energy loss. The self-energy may be found from the energy density of the field in equation (1.11); in order to obtain the self-energy due to the medium we must subtract the self-energy in vacuum. In so far as the recoil of the source can be neglected a calculation of the self-energy is straightforward, since the fields are immediately given. We consider the longitudinal contribution, which alternatively may be found as one half of the potential at the position of the particle. Introducing the source density (4.1), as corresponding to a point charge with velocity v, we perform a calculation similar to that resulting in (4.3) and obtain the following expression for the self-energy

$$u(v) = \frac{z_1^2\,e^2}{2\,\pi v}\int_0^\infty \frac{dk}{k}\int_{-kv}^{+kv} d\omega\left\{\frac{1}{\varepsilon^l(k,\omega)} - 1\right\}, \tag{4.25}$$

where the mass M of the particle should be large.

The expression (4.25) is quite similar to that describing the energy loss, (4.3), or the transition probability to other states. We may, in fact, combine the two in one formula for the complex self-energy, U,

$$U = u - i\frac{\Gamma}{2} = \frac{z_1^2 e^2}{2\pi v}\int_0^\infty \frac{dk}{k}\int_{-kv}^{+kv} d\omega \left\{\frac{1}{\varepsilon_C^l(k,\omega)} - 1\right\}, \qquad (4.26)$$

where $\Gamma/\hbar$ is the previously deduced transition probability. Further, $\varepsilon_C^l(k, \omega)$ represents $\varepsilon^l(k, \omega)$ for $\omega > 0$, and $\varepsilon^{l*}(k, \omega)$ for $\omega < 0$. We note that the transverse analogue of (4.26) may be derived directly from (4.4).

For high velocities we get from (4.25) and (3.4) the simple result

$$u = -z_1^2 \frac{\pi e^2}{4\hbar v} \cdot \hbar\omega_0, \qquad (4.27)$$

leading to quite small values when v is large. The contributions to (4.27) arise from polarization at distances about equal to the adiabatic impact parameter, v/ω_0, and not from the very small probability of carrying an electron. Similarly, if the particle is at rest, or moving slowly, we can for a degenerate gas employ the dielectric constant (3.5), (3.6) and obtain

$$u = -\frac{2 z_1^2 me^4}{\pi^2 \hbar^2}\int_0^\infty dz \cdot \frac{f_1(0, z)}{z^2 + \chi^2 f_1(0, z)}, \qquad (4.28)$$

where $f_1(u, z)$ is given by (3.7) and the integral, accordingly, is about equal to $\pi/2\chi$. Thus, we have found an approximate expression for the binding of, e. g., a proton in the gas. In the semi-classical limit one finds for the self-energy, to second power in the velocity of the particle,

$$u = \frac{z_1^2 me^4}{\pi\hbar^2} \cdot \frac{1}{\chi}\left\{-1 + \frac{1}{6}\left(1 - \frac{\pi^2}{16}\right)\frac{v^2}{v_0^2}\right\}. \qquad (4.29)$$

One may attempt to improve eq. (4.26) by including the recoil of the particle. Since in (4.26) the frequency ω is given by $\hbar\omega = \hbar k \cdot v = \Delta p \cdot (\partial E/\partial p)$, it would seem natural in quantum theory simply to replace $\hbar\omega$ by $\hbar k \cdot p/m + \hbar^2 k^2/2m$. The proper formula is, however, slightly more complicated. In the Appendix is derived the term to be added to the free particle Hamiltonian in the wave equation (cf. (A. 12)).

§ 5. Discussion of the approximation method.

In the calculations of the dielectric constants in § 2 and § 3, to first order in e^2, it was assumed that the electrons may be regarded as freely moving particles. This assumption is a central one in the Sommerfeld theory as well as in the Thomas-Fermi model. We shall discuss in how far the first order field equation, starting from free particles, is a good approximation to the correct field equations. Our treatment of this question is based on the pictorial semi-classical ideas. We consider two characteristic quantities; first, the self-energy as implying a momentum-energy relation different from that of a free particle, and second, the damping of motion of a single particle, giving rise to an uncertainty in its energy. The smallness of both damping and self-energy is a sufficient condition for the use of the first order approximation.

But let us first mention one simple argument showing a peculiar self-contained feature of the first order treatment. We notice that the semi-classical first order equations in § 2 depend only on the charge density $e\varrho$, on the ratio e/m, and on the particle velocities $\vec{v}_i = \vec{p}_i/m$. The distribution of particles of finite charge can thus be replaced by a continuum distribution having the same mass and charge density and the same velocity distribution, with the result that the first order equations remain unchanged. However, since the interaction of a charge with itself is at least proportional to the charge squared, all self-energies can be neglected in the continuum description, where charges may be considered infinitesimal, and the first order equations of § 2 become classically exact. This consideration makes apparent a most important feature of the field equations calculated: the first order perturbation treatment can be accurate, and at the same time the resulting induced field is by no means a small quantity.

Returning to an actual gas of particles, let us first explain what we mean by a self-consistent treatment in the semi-classical approximation. In § 2 we deduced the dielectric constants, the approximation being that the wavelengths of the electrons were very small compared to those of the field. On this assumption all contributions were found to arise from electrons at the surface

of the distribution in momentum space, with velocity v_0 and momentum p_0. We made a distinction between mv_0 and p_0, although they were equal in calculations to first order in e^2. In a more accurate classical treatment the only change is that mv differs from p. We now assume that, for a given electron density, the electrons occupy all states below the momentum $p_0 = (3\pi^2\varrho)^{1/3}\cdot\hbar$, with none above. This gives in any case a well-defined approximation to classical self-consistent equations. Since the contributions arise only from the surface of the distribution, our description involves merely one new quantity

$$\alpha = \frac{mv_0}{p_0} = \frac{m}{p_0}\frac{\partial E}{\partial p}\bigg|_{p=p_0}, \tag{5.1}$$

E being the energy of the electron.

This description can be applied when excitations involve states in the neighbourhood of the Fermi surface. Therefore, it can be used even in the quantum mechanical calculation in § 3, but only for fields of long wavelength and low frequency. In fact, if we define an effective electron mass as $m^* = p_0/v_0 = m/\alpha$, we merely need – in the formulae in § 2 and § 3 – replace throughout m by m^*; this should be done in ω_0^2 and χ^2, too. It is seen that then we obtain, e. g., the well-known dependence of spin paramagnetism and orbital diamagnetism (pp. 23 and 34) on effective electron mass. Further, while m^* refers to slowly varying fields, the opposite case of very large wave vectors and frequencies is accurately described by the uncorrected dielectric constants in § 3.

The energy E is the sum of the kinetic energy $mv^2/2$ and the self-energy u, given by (4.25). In order to bring out the essential features we now simplify the description. We can write approximately, for not too high velocities, $u(v) = u_0 + u_2\cdot v^2/2v_0^2$, as in eq. (4.29). It is then found from (5.1) and (4.29) that α is determined by

$$\frac{1}{\alpha} = 1 + C\cdot\chi^3, \tag{5.2}$$

where $C = (\pi/3)\cdot(1 - \pi^2/16)$. The previously introduced quantity $\chi^2 = me^2/\pi\hbar p_0$ is thus a measure of the applicability of the free electron picture and the first order field equations.† The depend-

† In LINDHARD (1946), eq. (25), a somewhat different consideration led to $\chi^2 < 1$ as the condition for the validity of the first order semi-classical picture.

ence on χ^3 in (5.2) is not due to the simplifying assumptions made as regards the self-energy, but holds accurately in the semi-classical picture as long as α is close to unity; a numerical computation shows that C is very nearly 0.5. Clearly, α is close to unity in dense gases, and for densities corresponding to electrons in metals α is slightly smaller than 1. For gases of low densities the approximations made are apparently not justified.

For a dilute gas in temperature equilibrium, we may similarly ask for the limit of applicability of the free electron picture, and of the Maxwell velocity distribution. Since, in the description of such gases, Planck's constant does not enter, the parameter characterizing the gas is, instead of χ^2, a quantity proportional to

$$\chi_1^2 = \frac{e^2 \varrho^{\frac{1}{3}}}{\theta}, \tag{5.3}$$

which just corresponds to the ratio between the binding energy and the temperature energy. It may here be recalled that the field equations for a Boltzmann gas were found roughly to correspond to those of a degenerate gas, only v_0 being replaced by the temperature velocity of the particles. On the same lines as above we then find an equation for α similar to (5.2), but with χ_1^2 instead of χ^2. Thus, the simple Maxwell distribution is only valid for χ_1^2 small compared to unity.

Mention may be made of the curious case of a one-dimensional degenerate gas, where the first order semi-classical dielectric constant was found to be $\varepsilon = 1 - \omega_0^2/(\omega^2 - v_0^2 k^2)$. The contributions to ε arise from particles at the surface of the distribution in momentum space, and the field surrounding such particles moving with constant velocity v_0 has non-zero Fourier components only for $\omega = \pm v_0 k$. For this self-field ε is infinite, so that the self-energy and its derivative with respect to momentum vanishes. Thus, we have proved that the first order treatment is semi-classically exact, in agreement with the result of TOMONAGA (1950).

So far the ratio α and the momenta were considered, for simplicity, as real quantities. However, we found earlier a damping of motion, by no means negligible compared to the self-energy. Let us regard the resulting effect in the case of χ^2 small, so that the free particle picture is approximately valid.

The imaginary part, $\Gamma/2$, of the energy is then given semi-classically by eq. (4.26). If again we use the estimate asymptotically valid for low velocities, we get according to (4.24) proportionality between Γ and the velocity of the particle; the corresponding complex momentum has an imaginary part equal to the constant $(\pi/8)\cdot(me^2/\hbar)\chi$. Accordingly, α is approximately given by

$$\alpha = 1 - C\cdot\chi^3 + i\frac{\pi^2}{8}\chi^3. \tag{5.4}$$

In the present limit of $\alpha \simeq 1$ we could of course easily get estimates more accurate than (5.4). In this connection we note that Γ, while rising at first linearly with the velocity, will reach a maximum for energies somewhat above the top of the Fermi distribution. For still higher energies Γ decreases slowly towards zero. The value of the imaginary part of (5.4) is only an approximate estimate, due to the simplifying assumptions made; but the dependence on χ^3 has general validity in the semi-classical approximation.

For a finite imaginary part of α some interesting consequences result in the formulae, as (2.4), for the dielectric constants, where then v_0 is to be replaced by $\alpha p_0/m = \alpha\hbar\,(3\pi^2\varrho)^{1/3}/m$.† For instance, for long wavelengths, where a phenomenon similar to mesons is observed (cf. p. 25), the inclusion of the imaginary part of (5.4) implies a width of the states of the field quanta, corresponding to a life-time for the field quanta at rest

$$\frac{1}{\tau'} = \frac{\omega_0}{2}\,\mathrm{Im}\,(\alpha) \approx \frac{1}{8\cdot 3^{\frac{1}{2}}}\cdot\frac{me^4}{\hbar^3}, \tag{5.5}$$

independent of the gas density for small χ's.

It is, perhaps, not without interest to make here a qualitative comparison with atomic nuclei. Indeed, in spite of the differences between electromagnetic and mesonic couplings, the

† The resulting improved dielectric constants are of the causal type, where the imaginary part does not change sign with ω. The causal dielectric constants are particularly convenient in higher order treatments.

By introducing in the transverse dielectric constant the effect of the self-energy, as expressed by α, we observe that the term $i\,(3\,\pi\omega_0^2/4\,\omega k v_0)\cdot(mv_0/p_0)$ in (2.7) remains unaffected. This result is important in the so-called anomalous skin effect, which is governed just by the mentioned term (cf. II).

general features of the results obtained above should also apply for a system of nucleons. One will expect the appearance of complex self-energies of the nucleons, of similar type as found above. The semi-empirical proposal by FESHBACH et al. (1953) may be regarded as a simplified description of such kind, although it will be evident that both the real and imaginary parts of the self-energy must vary with particle excitation.

The considerations in this paragraph and in § 4, aiming at improved electromagnetic field equations, at the same time give improved equations for the particle field. The latter equations are also of the type of Maxwell's equations in matter, in being of high order, and involving polarization and absorption effects. The equations for the electromagnetic field are important for the dynamics of the system, e. g. in the case of interaction with external charges. The equations for the particle field are useful for instance when a particle, identical with those in the gas, enters the system from the outside. In general, the particle equations are of importance when one is able to discern approximately the motion of one, or a few, individual particles.

Summary.

The paper treats the behaviour of a gas of charged particles, preferably a degenerate gas. It is pointed out that the dynamic properties of this system are contained in equations for the electromagnetic field merely, of type of Maxwell's equations in matter. By the field is meant, classically, the field induced by and acting on external, classical charges. A systematic treatment on this basis implies great simplifications in the theory.

The interpretation of general field equations is discussed, and the manner in which they account for absorption processes. The dielectric constants, defining linear field equations, are computed in a number of cases, to first and higher order in e^2, using both classical and quantum description of the particle motion. As a demonstration of the method is treated energy dissipation by a charged particle moving through the system, and its self-energy. Further is discussed self-consistent field equations, and the improved electronic equations of motion.

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Appendix.

Improved particle field equations.

In § 4 was deduced, classically, the self-energy and damping of motion of a charged particle moving in a field obeying the equation

$$\Delta \varepsilon \Phi = -4\pi \varrho_0, \tag{A. 1}$$

where the operator ε is the longitudinal dielectric constant. We shall now derive the corresponding quantum effect, to first order in e^2. In this way one gets an improved equation for the particle field, of similar type as the equation for the electromagnetic field, (A. 1). In the derivation we employ a technique more advanced than that used for the deduction of the equation (A. 1) in § 3. Still, the calculation will not be performed in the least cumbersome manner, but is hoped to be the more illustrative as regards the effects involved.

The field equation for the particle is taken to be the non-relativistic one,

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\psi = e\Phi\psi, \tag{A. 2}$$

and at the same time we introduce $\varrho_0 = e\,\psi^*\psi$ in (A. 1). The equations (A. 1), (A. 2) are conveniently written on integral form

$$\psi(x) = \psi^{in}(x) - e\int S^R(x-x')\,\Phi(x')\,\psi(x')\,dx', \qquad \text{(A. 3)}$$

$$\Phi(x) = \Phi^{in}(x) + e\int D^R(x-x')\,\psi^*(x')\,\psi(x')\,dx', \qquad \text{(A. 4)}$$

where x stands for (x, y, z, t). Moreover, $S^R(x-x')$ is the retarded solution of (A. 2) when the right side of the equation is replaced by $-\delta(x-x')$; $D^R(x-x')$ is the analogous solution of (A. 1) with a source $-4\pi\delta(x-x')$ on the right. The incoming fields ψ^{in} and Φ^{in} are solutions of the uncoupled equations.

The coupled equations result in a scattering of the particle, which we calculate to second order from a series development in (A. 3) and (A. 4). For the outgoing field we find

$$\left.\begin{aligned}\psi^{out}(x) &= \psi^{in}(x) + e\int S(x-x')\,\Phi^{in}(x')\,\psi^{in}(x')\,dx' \\ &+ e^2\iint dx'\,dx''\,S(x-x')\cdot D^R(x'-x'')\,\psi^{*in}(x'')\,\psi^{in}(x'')\,\psi^{in}(x') \\ &- e^2\iint dx'\,dx''\,S(x-x')\cdot S^R(x'-x'')\,\Phi^{in}(x')\,\Phi^{in}(x'')\,\psi^{in}(x''),\end{aligned}\right\} \qquad \text{(A. 5)}$$

where $S = S^A - S^R$.

Consider the vacuum expectation value of $\psi^{*in}(x_1)\,\psi^{out}(x)$, which quantity can serve to characterize the scattering. The first order term in e is seen to vanish for symmetry reasons, and we get to second order

$$\left.\begin{aligned}\langle 0|\,\psi^{*in}(x_1)\,\psi^{out}(x)\,|0\rangle &= \langle 0|\,\psi^{*in}(x_1)\,\psi^{in}(x)\,|0\rangle \\ - e^2\iint dx'\,dx''\,S(x-x')\,\{\,&D^R(x'-x'')\langle 0|\,\psi^{*in}(x'')\,\psi^{in}(x')\,|0\rangle \\ + S^R(x'-x'')\langle 0|\,&\Phi^{in}(x')\,\Phi^{in}(x'')\,|0\rangle\,\}\,\langle 0|\,\psi^{*in}(x_1)\,\psi^{in}(x'')\,|0\rangle.\end{aligned}\right\} \qquad \text{(A. 6)}$$

In this equation we introduce

$$\langle 0|\,\psi^{*in}(x'')\,\psi^{in}(x')\,|0\rangle = -i\hbar\,S^-(x'-x''), \qquad \text{(A. 7)}$$

according to ordinary field quantization; the function S^- is the negative frequency part of S. As to the electromagnetic field the method of quantization of Peierls (1952), establishing the general connection with the Green's functions of the field equations, leads to

$$\langle 0|\,\Phi^{in}(x')\,\Phi^{in}(x'')\,|0\rangle = i\hbar\,D^+(x'-x''). \qquad \text{(A. 8)}$$

We wish to eliminate the coupling with the electromagnetic field and attempt to replace it by a new term in the particle field equation

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\Delta\right)\psi(x)+\int K(x-x')\,\psi(x')\,dx'=0. \qquad \text{(A. 9)}$$

The above scattering is obtained if, according to (A. 6), (A. 7), and (A. 8), K is of the form

$$K(x)=ie^2\hbar\,(-D^R(x)\cdot S^-(x)+S^R(x)\cdot D^+(x)). \qquad \text{(A. 10)}$$

It can be convenient to write this formula in terms of D^C and S^C, defined by $D^C=-2i\overline{D}+D^{(1)}$. To this purpose we note that the expression $-D^R(x)\,S^-(x)+S^R(x)\,D^+(x)$ is equal to

$$(-i/2)\left\{D^R(x)\,S^{(1)}(x)+D^{(1)}(x)\,S^R(x)\right\},$$

and a simple manipulation leads to

$$K(x)=\frac{i}{4}e^2\hbar\,S^C(x)\,D^C(x), \qquad \text{(A. 11)}$$

the equation holding only for the positive frequency part of the field, which is usually sufficient, since the incoming particle has positive energy. Substituting (A. 10) or (A. 11) for K in (A. 9) an improved equation for the particle field is obtained. For a Fourier component of the particle wave proportional to $\exp(i\vec{k}\cdot\vec{r}-i\omega t)$ the extra term in the field equation—which could be called a self-energy of the particle—is accordingly

$$\left.\begin{aligned}U(k,\omega)=-\frac{ie^2}{4\pi^3}\iint d^3k'\,d\omega'\,\frac{1}{\omega'+i\delta-\dfrac{\hbar k'^2}{2m}}\cdot\frac{1}{(\vec{k}-\vec{k}')^2}\\ \cdot\left(\frac{1}{\varepsilon_C(|\vec{k}-\vec{k}'|,\omega-\omega')}-1\right),\end{aligned}\right\} \qquad \text{(A. 12)}$$

where we have subtracted the vacuum term with $\varepsilon=1$; further $\varepsilon_C(k,\omega)=\varepsilon(k,|\omega|)$ and δ is a vanishing positive quantity.

In eq. (A. 12) we have of course in first approximation the connection $\omega = \hbar k^2/2\, m$ if the particle energy is approximately that of a freely moving particle.

When the mass m, is large, and for a given particle velocity $v = \hbar k/m$, the formula (A. 12) reduces to the semi-classical one, (4.26).

The above derivation of linear particle field equations from general linear electromagnetic field equations owes its feasibility and simplicity to the use of the quantization rule (A. 8). Evidently, one may give, on the same lines, a derivation of the electromagnetic field equations from general particle equations, and thus obtain mutually consistent equations for the two fields. In the text, this was performed, quantum mechanically, for free particles, but also for a more general case when only long waves or low frequencies were important.

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Added in proof: The reader is further referred to an interesting manuscript by T. Kinoshita and Y. Nambu (Institute for Advanced Study, Princeton, N. J.), The Collective Description of Many-Particle Systems.

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THEORETICAL ANGULAR CORRELATIONS IN ALLOWED BETA TRANSITIONS

BY

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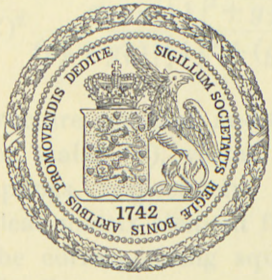

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The angular distribution in beta decay is usually expressed in terms of the beta particle energy and the angle between the directions of emission of the electron and the neutrino. In the present paper, this distribution is transformed into the distribution function for any two of the observable variables, viz. the beta energy, the recoil energy, and the angle between the direction of emission of the recoil and the electron.

Introduction.

In beta transitions, the angular correlation is commonly expressed in terms of the beta particle energy, E, and the angle, θ, between the directions of the momentum of the electron, p, and that of the neutrino, q.

For allowed transitions, $\Delta J = \begin{cases} 0 \\ \pm 1 \end{cases}$ (no), this correlation is given by the probability distribution for E and θ[1), 2)]

$$\left.\begin{aligned} & P(E, \theta)\, dE\, d\Omega_\theta \\ = {} & F(Z, E)\, pEq^2\, [1 + (b/E) + (ap/E) \cos\theta]\, dE\, d\Omega_\theta, \end{aligned}\right\} \quad (1)$$

where $d\Omega_\theta$ is a solid angle interval around θ; $F(Z, E)$ is the Coulomb correction as a function of E and the charge Z of the recoil, and

$$b = 2\sqrt{1-(\alpha Z)^2}\, \frac{g_S g_V \left|\int 1\right|^2 + g_T g_A \left|\int \vec{\sigma}\right|^2}{(g_S^2 + g_V^2)\left|\int 1\right|^2 + (g_T^2 + g_A^2)\left|\int \vec{\sigma}\right|^2}, \quad (2)$$

where α is the fine structure constant, and the g's are the relative coupling constants for scalar (S), vector (V), tensor (T), and axial vector (A) couplings, respectively. Furthermore, $\left|\int 1\right|^2$ is the square of the nuclear matrix element for the Fermi interactions, and $\left|\int \vec{\sigma}\right|^2$ is the corresponding square for the Gamow-Teller interactions. Finally,

$$a = \frac{(g_V^2 - g_S^2)\left|\int 1\right|^2 + \frac{1}{3}(g_T^2 - g_A^2)\left|\int \vec{\sigma}\right|^2}{(g_S^2 + g_V^2)\left|\int 1\right|^2 + (g_T^2 - g_A^2)\left|\vec{\sigma}\right|^2}. \quad (3)$$

The units are as usual in beta decay: $\hbar = 1$, $c = 1$, and $m = 1$. The pseudoscalar interaction has been omitted, since it contributes to allowed transitions only with higher order terms[3].

The expression (1) is mathematically convenient, since E and θ may vary independently in the intervals $1 \leq E \leq E_0$ and $0 \leq \theta \leq \pi$, respectively, and since E and θ completely determine the momentum triangle (apart from orientation in space[4]). This is a result of the fact that the neutrino rest mass has been equated to zero, as can be seen from the conservation of energy and momentum, which shows that E, θ determines one and only one value of q.

Physically, however, (1) is less convenient because it refers to the neutrino which, at the present time, cannot be observed in angular correlation experiments. The transformation of the angular correlation (1) into distribution functions for any two of the three measurable quantities p, φ, and r, where r is the recoil momentum and φ the angle between the directions of r and p, is straightforward. In the present paper, it is the intention to give the three distribution functions corresponding to (1) for the three pairs of variables (E, r), (E, φ), and (r, φ) for reference use.

The essential questions which arise are, firstly, inside which area can the variables so chosen vary and, secondly, do the variables determine the momentum triangle completely so that the transformation establishes a one to one correspondence between the old and the new variables.

The (E, r) Distribution.

For the transformation of (1), the guiding relations are the conservation laws of energy and momentum given by

$$E + q = E_0, \tag{4}$$

$$p^2 + q^2 + 2\,pq \cos\theta = r^2, \tag{5}$$

which are valid when the recoil mass is considered infinite. By differentiation of θ with respect to r, (5) gives

$$|\sin\theta\, d\theta| = 2\, d\Omega_\theta = \frac{r}{pq}\, dr. \tag{6}$$

However, in the case of the (E, r) distribution, one has to remember that E and r cannot vary independently inside the intervals $0 \leq r \leq p_0$, $1 \leq E \leq E_0$ with

$$p_0 = \sqrt{E_0^2 - 1}.$$

We find from (5), for a fixed value of r,

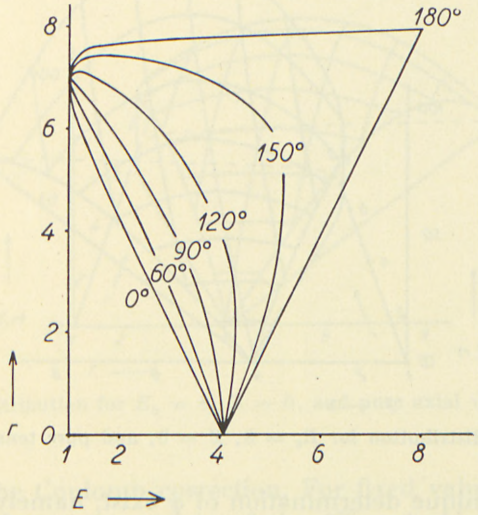

Fig. 1. Permitted area in the (E, r) plane for $E_0 = 8$. In the diagram, curves are drawn for constant values of φ referring to relation (10).

$$\frac{(E_0 - r)^2 + 1}{2(E_0 - r)} \leq E \leq \frac{(E_0 + r)^2 + 1}{2(E_0 + r)} \tag{8}$$

or, correspondingly, for a fixed value of E,

$$|p - q| \leq r \leq |p + q|, \tag{9}$$

where $p = \sqrt{E^2 - 1}$ and q is given by (4). Relations (8) and (9) are illustrated in Fig. 1 for a definite numerical example.

On the other hand, given values of E and r inside the permitted area determined by (8) and (9) give one and only one value for the angle φ inside the interval $0 \leq \varphi \leq \pi$. This follows

from the conservation of momentum, written in terms of E, r and φ,

$$\cos\varphi = \frac{q^2 - p^2 - r^2}{2\,pr}. \tag{10}$$

Relation (10) is also illustrated in Fig. 1. Two trivial excep-

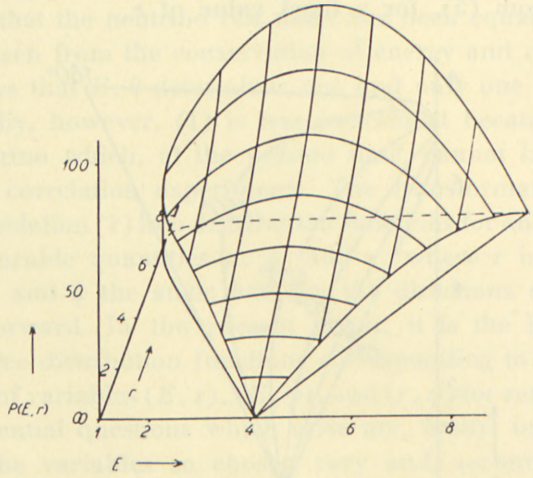

Fig. 2. (E, r) distribution for $E_0 = 8$, $Z = 0$, and pure tensor interaction.

tions to the unique determination of φ exist, namely, all φ values meet in the two points

$$(E, r) = \left(\frac{E_0^2 + 1}{2\,E_0}, 0\right), \tag{11}$$

corresponding to $p = q$, and

$$(E, r) = (0, E_0 - 1), \tag{12}$$

corresponding to $r = q^{\max} = E_0 - 1$. The limiting curves $r = p - q$, $(p > q)$ and $r = q + p$ correspond to $\varphi = \pi$, and the limiting curve $r = q - p$, $(p < q)$ corresponds to $\varphi = 0$.

The transformation from (1) to the (E, r) distribution can thus be carried out immediately, and one finds

$$\left.\begin{aligned} P(E, r)\,dE\,dr \\ = \frac{1}{2} F(Z, E)\left[rEq + brq + r\frac{a}{2}(r^2 - p^2 - q^2)\right] dE\,dr. \end{aligned}\right\} \tag{13}$$

This distribution function is illustrated in Fig. 2 for pure tensor interaction and $Z = 0$, and in Fig. 3 for pure axial vector interaction and $Z = 0$, in both cases for the same numerical example as chosen in Fig. 1.

In Figs. 2 and 3, and from formula (13), one sees that the (E, r) distribution is extremely simple, except for the complicated cut-off introduced by (8) or (9). For fixed values of r, the E distributions are parabolic and symmetric around $q = E = E_0/2$,

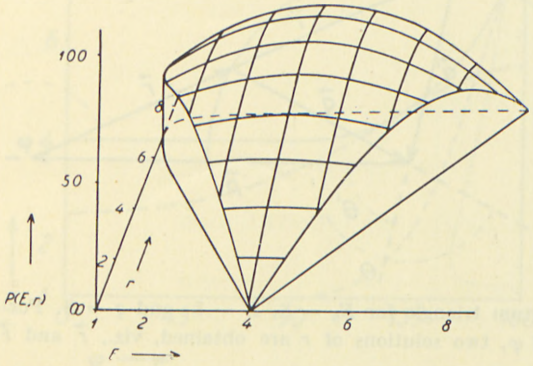

Fig. 3. (E, r) distribution for $E_0 = 8$, $Z = 0$, and pure axial vector interaction.

apart from the Coulomb correction. For fixed values of E, the r distributions are polynomials of the 3rd order in r, the entire expression consisting of r multiplied into a parabolic expression in r. The energy distribution for the kinetic energy R of the recoil is thus even simpler; we can write

$$R = r^2/(2M) \tag{14}$$

and get

$$\left.\begin{aligned} &P(E, R)\, dE\, dR \\ &= \frac{M}{2} F(Z, E) \left[Eq + bq + \frac{a}{2}(2MR - p^2 - q^2) \right] dE\, dR. \end{aligned}\right\} \tag{15}$$

For fixed R values, this equation shows the same general shapes of the β-energy distribution as (13) and give, for fixed E values, a linear dependence on R.

The (E, φ) Distribution[5].

While the (E, θ) distribution and, in some respects, the (E, r) distribution are very simple, this is not the case for the (E, φ) and (r, φ) distributions. This fact is of course entirely due to the conservation laws and is apparent from (10) and the corresponding curves for constant φ values in Fig. 1. It is immediately

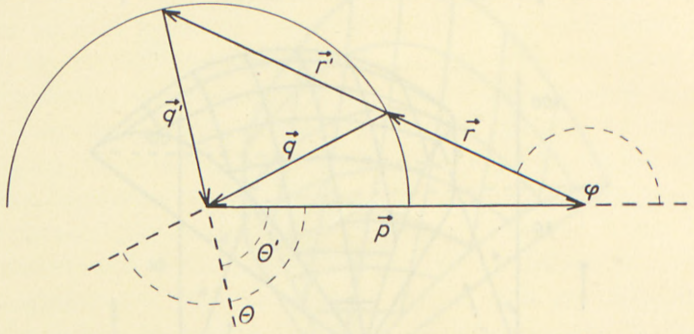

Fig. 4. Momentum triangle for $E_0 = 8$, $E = 5$, and $q = 3$. For a given value of φ, two solutions of r are obtained, viz., $\vec{r}$ and $\vec{r}'$.

seen that, when $p > q$, i. e., $E > \dfrac{E_0^2+1}{2E_0}$, and $\dfrac{\pi}{2} \leq \varphi \leq \pi$, given values of E and φ lead to two solutions for r. This finds of course also a simple geometrical interpretation, as illustrated by the momentum triangle shown in Fig. 4.

It is also evident that a given value of $E > \dfrac{E_0^2+1}{2E_0}$ only permits φ to vary between $\varphi' \leq \varphi \leq \pi$, where

$$\sin \varphi' = q/p, \quad \frac{\pi}{2} \leq \varphi' \leq \pi, \tag{16}$$

or, correspondingly, that a given value of $\varphi' \geq \dfrac{\pi}{2}$ permits as an interval for E

$$1 \leq E \leq E' = \left\{ \begin{array}{l} \dfrac{1}{\cos^2 \varphi'} (E_0 - \sin \varphi' \sqrt{E_0^2 - \cos^2 \varphi'}) \\ \rightarrow \dfrac{E_0^2+1}{2E_0} \quad \text{for} \quad \varphi' \rightarrow \dfrac{\pi}{2}. \end{array} \right\} \tag{17}$$

We have written down also the limiting value for $\varphi \rightarrow \dfrac{\pi}{2}$. The

formulas (16) and (17) constitute the limiting curves in the E, φ plane. These are illustrated in our specific numerical example in Fig. 5.

The description by our two variables is in this case not complete, as already mentioned. Curves for constant values of

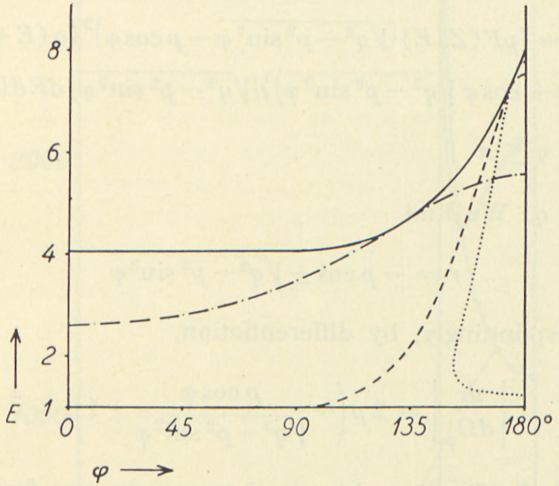

Fig. 5. Permitted area in the (E, φ) plane for $E_0 = 8$ and curves for constant r values.

..... $r = 7.5$
– – – – – – $r = 7$
– · – · – · $r = 3$.

r intersect in the area $\frac{\pi}{2} \leq \varphi \leq \pi$ and $E > \frac{E_0^2 + 1}{2 E_0}$, as illustrated in Fig. 5. For $r < E_0 - 1$, they extend from $\varphi = 0$ to $\varphi = \pi$, for $r = E_0 - 1$, they extend from $\varphi = \frac{\pi}{2}$ to $\varphi = \pi$, whereas for $r > E_0 - 1$, the curves begin at $\varphi = \pi$ and end at $\varphi = \pi$ again.

When performing the transformation from (13) to the (E, φ) distribution, we clearly have to distinguish between the two cases $p \geq q$ and $p \leq q$.

A: $p \leq q$. We find

$$r = -p \cos \varphi + \sqrt{q^2 - p^2 \sin^2 \varphi} \qquad (18\,a)$$

and, correspondingly, by differentiation with respect to φ,

$$\left|\frac{dr}{d\Omega_\varphi}\right| = 2p\left(1 - \frac{p\cos\varphi}{\sqrt{q^2 - p^2\sin^2\varphi}}\right). \tag{19 a}$$

Consequently, we find the (E, φ) distribution given by

$$\left.\begin{aligned} P(E,\varphi)\,dE\,d\Omega_\varphi = [pF(Z,E)(\sqrt{q^2 - p^2\sin^2\varphi} - p\cos\varphi)^2\{q(E+b) \\ - ap(p\sin^2\varphi + \cos\varphi\sqrt{q^2 - p^2\sin^2\varphi})\}/\sqrt{q^2 - p^2\sin^2\varphi}]\,dE\,d\Omega_\varphi, \end{aligned}\right\} \tag{20 a}$$

where $0 \leq \varphi \leq \pi$.

B: $p \geq q$. We find

$$r = -p\cos \pm \sqrt{q^2 - p^2\sin^2\varphi} \tag{18 b}$$

and, correspondingly, by differentiation,

$$\left|\frac{dr}{d\Omega_\varphi}\right| = 2p\left(-\frac{p\cos\varphi}{\sqrt{q^2 - p^2\sin^2\varphi}} \pm 1\right). \tag{19 b}$$

The (E, φ) distribution adds up of two parts, one from each of the solutions to (18 b), i. e.,

$$\left.\begin{aligned} P(E,\varphi)\,dE\,d\Omega_\varphi = (20,\text{a}) + [pF(Z,E)(\sqrt{q^2 - p^2\sin^2\varphi} + p\cos\varphi)^2 \\ \{q(E+b) - ap(p\sin^2\varphi - \cos\varphi\sqrt{q^2 - p^2\sin^2\varphi})\}/\sqrt{q^2 - p^2\sin^2\varphi}]\,dE\,d\Omega_\varphi \end{aligned}\right\} \tag{20 b}$$

$$\left.\begin{aligned} = 2pF(Z,E)[\{(q^2 + p^2\cos 2\varphi)/\sqrt{q^2 - p^2\sin^2\varphi}\}\{q(E+b) - ap^2\sin^2\varphi\} \\ + 2ap^2\cos^2\varphi\sqrt{q^2 - p^2\sin^2\varphi}]\,dE\,d\Omega_\varphi, \end{aligned}\right\} \tag{20 c}$$

where $\varphi' \leq \varphi \leq \pi$. An experimental cut-off of low velocity recoils would affect the two terms in (20 b) differently. This is the reason why this intermediate formula is given.

In the limit of $p = q$, both (20 a) and (20 b) give the following expression for $\frac{\pi}{2} \leq \varphi \leq \pi$:

$$P(\varphi)\,d\Omega_\varphi = 4p^3F|\cos\varphi|\{(E+b) + ap\cos 2\varphi\}\,d\Omega_\varphi; \tag{20 d}$$

for $0 \leq \varphi \leq \frac{\pi}{2}$, they give $P(\varphi) = 0$.

The distribution functions (20) are illustrated in Figs. 6 for fixed values of E. For $p < q$, the curves are quite regular and extend from $\varphi = 0$ to $\varphi = \pi$. As $p \to q$, the probability for $\varphi < \frac{\pi}{2}$ becomes smaller and smaller relative to the probability

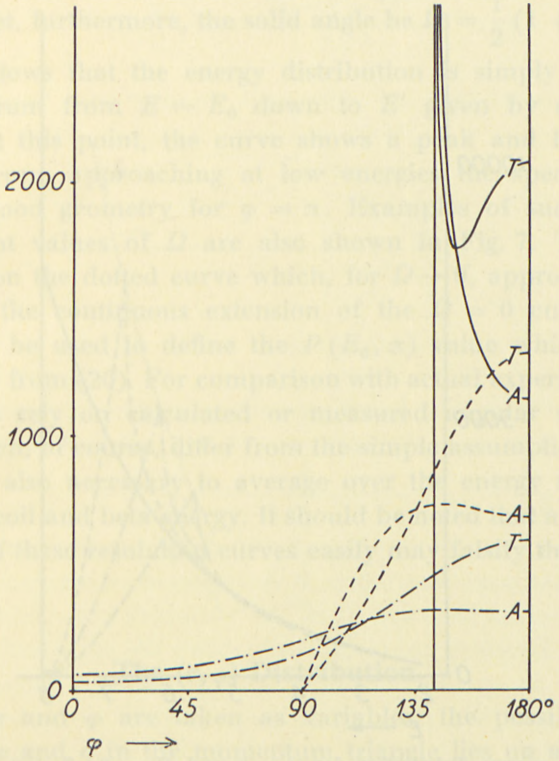

Fig. 6. Relative angular correlations for tensor and axial vector interactions per unit solid angle interval for $E_0 = 8$, $Z = 0$.

— · — · — · $E = 2.79$ $p/q = 0.5$
– – – – – – – – – $E = 4.063$ $p = q$
———————— $E = 5$.

for $\varphi > \frac{\pi}{2}$ until, for $p = q$, the probability for $\varphi \leq \frac{\pi}{2}$ is zero and the distribution then extends in a regular manner from $\varphi = \frac{\pi}{2}$ to $\varphi = \pi$. For $p > q$, the curves have an integrable singularity at $\varphi = \varphi'$ and extend from $\varphi = \varphi'$ to $\varphi = \pi$. For $p = p_0$, one sees that $\varphi = \pi$, and it should be noted that the limiting value

for $p = p_0$, $\varphi = \pi$, is undetermined from formula (20). For values of $p/q < 1$, the curves for T and A interaction intersect at $\operatorname{tg} \varphi'' = -q/p$; for $p/q = x < 1$, the curves for T and A interaction intersect at $\varphi'' > \frac{\pi}{2}$, given by $x^2 = \frac{1 + 3\cos 2\varphi''}{(1 + 2\cos 2\varphi'')(1 - \cos 2\varphi'')}$.

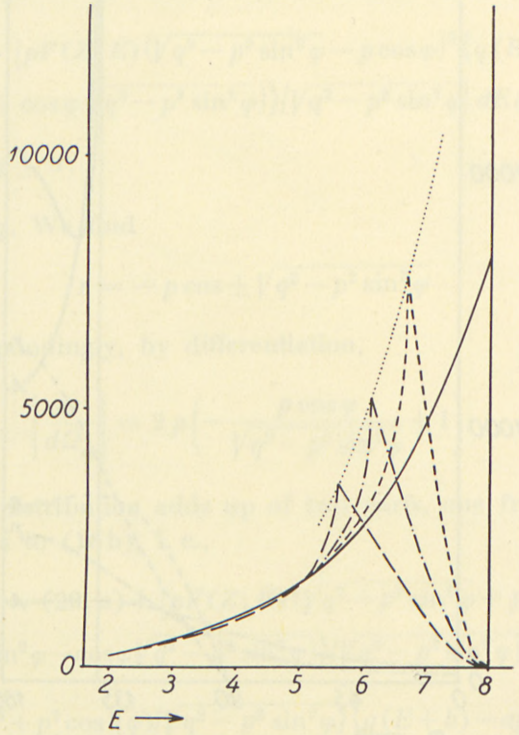

Fig. 7. Beta energy distribution for $\varphi = \pi$ and $E_0 = 8$, $Z = 0$, $b = a = 0$. The curves are normalized per unit interval $d\Omega = \Omega$

——————— infinitely good geometry
– – – – – – – – – $\Omega = 1/100$
— · — · — · $\Omega = 1/40$
— — — — — $\Omega = 1/20$.

This means that for small values of φ the A interaction gives a larger probability than the T interaction; the opposite is the case for large values of φ; the cross-over is at $\varphi = \varphi''$.

Similar considerations apply to the E distributions for fixed values of φ. Let us consider a special case, viz. $\varphi = \pi$. A numerical example is shown in Fig. 7. This curve, however, calls only for

physical interest if infinitely good geometry can be obtained. If the instrument permits only a certain resolution as regards the angle, the spectrum (20) has to be integrated over this angular interval. Let us consider an idealized geometry with reception of all particles for $\varphi^* < \varphi < \pi$ and of no particles outside this interval. Let, furthermore, the solid angle be $\Omega = \frac{1}{2}(1 + \cos \varphi^*)$. It then follows that the energy distribution is simply the total beta spectrum from $E = E_0$ down to E' given by (17) with $\varphi = \varphi^*$. At this point, the curve shows a peak and leaves the beta spectrum, approaching at low energies the spectrum for infinitely good geometry for $\varphi = \pi$. Examples of such curves for different values of Ω are also shown in Fig. 7. The peak values lie on the dotted curve which, for $\Omega \to 0$, approaches the double of the continuous extension of the $\Omega = 0$ curve. This value may be used to define the $P(E_0, \pi)$ value which is undetermined from (20). For comparison with actual experiments[6]*, one has to rely on calculated or measured angular resolution curves which, of course, differ from the simple assumptions made here. It is also necessary to average over the energy resolution for both recoil and beta energy. It should be noted that a misinterpretation of these resolution curves easily may falsify the picture.

The (r, φ) Distribution.

When r and φ are taken as variables, the possible intersection of p and q in the momentum triangle lies on an ellipse, as illustrated in Fig. 8. This figure shows the limiting case of $r = E_0 - 1$, where the ellipse goes through the endpoint of $\vec{r}$, and where angles $\frac{\pi}{2} \leq \varphi \leq \pi$ are permitted. When $r > E_0 - 1$, the ellipse cuts the vector $\vec{r}$, giving two solutions of E for each value of φ. For $r < E_0 - 1$, the vector $\vec{r}$ is entirely surrounded by the ellipse, thus giving only one solution for E for each value of φ. We clearly have to make a distinction between these two cases.

* The author is indebted to Drs. Rustad and Ruby for sending their unpublished experimental data.

The limiting area in the (r, φ) plane is given by

$$\left.\begin{aligned} 0 \leq \varphi \leq \pi \quad &\text{for} \quad r < E_0 - 1, \\ \varphi^+ \leq \varphi \leq \pi \quad &\text{for} \quad r < E_0 - 1, \end{aligned}\right\} \tag{21}$$

where

$$\sin \varphi^+ = \frac{E_0^2 - 1 - r^2}{2r} \quad \text{and} \quad \frac{\pi}{2} \leq \varphi^+ \leq \varphi. \tag{22}$$

Equations (21) and (22) lead, in our case, to the picture

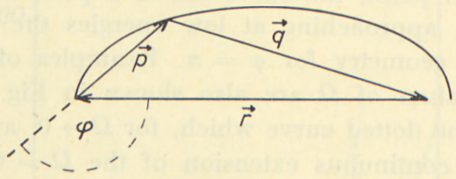

Fig. 8. Momentum triangle for $r = E_0 - 1 = 7$.

shown in Fig. 9. The figure also shows curves for constant values of E and illustrates the existence of two solutions to E for given values of $\varphi > \frac{\pi}{2}$ and $r > E_0 + 1$.

For the (E, r) and the (E, φ) distributions, the Coulomb correction enters quite naturally, since it is expressed as a function of E; this will not be the case when the variables are r and φ. In this case, it is therefore natural to write down the distribution function implicitly as a function of r and φ through expressions for $E = E(r, \varphi)$. We distinguish between the two cases

A: $r < E_0 - 1$. We then find

$$E = \frac{E_0 (E_0^2 + 1 - r^2) - r \cos \varphi \sqrt{(E_0^2 - 1 - r^2)^2 - 4 r^2 \sin^2 \varphi}}{2 (E_0^2 - r^2 \cos^2 \varphi)} \tag{23a}$$

and, correspondingly, by differentiation directly of (23a) or implicitly of (10),

$$\left.\begin{aligned} \left| \frac{dE}{d\Omega_\varphi} \right| &= \frac{2 p^2 r}{E_0 p + E r \cos \varphi} \\ &= \frac{4 p^2 r}{\sqrt{(E_0^2 - 1 - r^2)^2 - 4 r^2 \sin^2 \varphi}} \end{aligned}\right\} \tag{24a}$$

which, inserted in (13), gives the (r, φ) distribution

$$\left.\begin{aligned} & P(r,\varphi)\, dr\, d\Omega \\ &= F(Z,E)\,[rEq + brq + \frac{a}{2} r\,(r^2 - p^2 - q^2)]\,\frac{p^2 r}{E_0 p + Er\cos\varphi}\, dr\, d\Omega_\varphi. \end{aligned}\right\} \quad (25a)$$

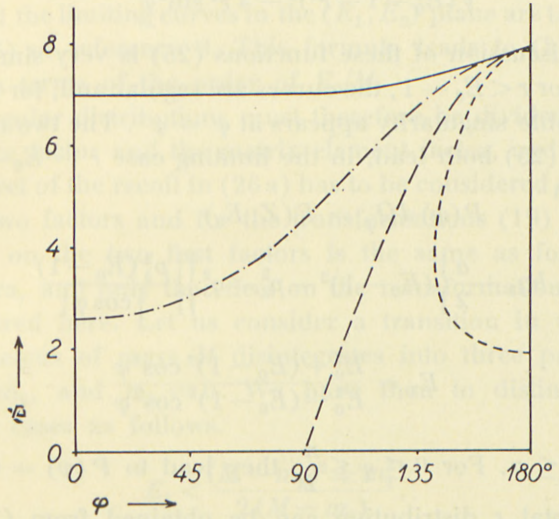

Fig. 9. Limiting area in the (r, φ) plane for $E_0 = 8$. Curves for constant values of E are given.

— · — · — · $E = 2.77$
— — — — — $E = (E_0^2 + 1)/(2\,E_0)$
- - - - - - - - - $E = 5.$

B: $r > E_0 - 1$. We then find

$$\left.\begin{matrix} E_1 \\ E_2 \end{matrix}\right\} = \frac{E_0\,(E_0^2 + 1 - r^2) \mp r\cos\varphi \sqrt{(E_0^2 - 1 - r^2)^2 - 4\,r^2\sin^2\varphi}}{2\,(E_0^2 - r^2\cos^2\varphi)} \quad (23b)$$

and, by differentiation,

$$\left.\begin{aligned} \left|\frac{dE}{d\Omega_\varphi}\right| &= \pm\frac{2\,p^2 r}{E_0 p + Er\cos\varphi} \\ &= \frac{4\,p^2 r}{\sqrt{(E_0^2 - 1 - r^2)^2 - 4\,r^2\sin^2\varphi}}. \end{aligned}\right\} \quad (24b)$$

Consequently, one obtains for the (r, φ) distribution

$$P(r,\varphi)\,dr\,d\Omega_\varphi = \left\{ F(Z,E_1)\left[rE_1q_1 + brq_1 + \frac{a}{2}r(r^2-p_1^2-q_1^2)\right]p_1^2 \right.$$
$$\left. + F(Z,E_2)\left[rE_2q_2 + brq_2 + \frac{a}{2}r(r^2-p_2^2-q_2^2)\right]p_2^2 \right\} \qquad (25\,b)$$
$$\frac{2\,r\,dr\,d\Omega_\varphi}{\sqrt{(E_0^2-1-r^2)^2-4\,r^2\sin^2\varphi}}.$$

The discussion of these functions (25) is very similar to that of (20). For $r < E_0 - 1$, the curves are regular and, for $r > E_0 - 1$, an integrable singularity appears at $\varphi = \varphi^+$. The two distribution functions (25) both lead, in the limiting case $r = E_0 - 1$, to the expression

$$P(\varphi)\,d\Omega_\varphi = F(Z,E_3)$$
$$\left[E_3q_3 + bq_3 + \frac{a}{2}\left\{(E_0-1)^2-p_3^2-q_3^2\right\}\right]\frac{p_3^2(E_0-1)}{|\cos\varphi|} \qquad (25\,c)$$

with

$$E_3 = \frac{E_0^2+(E_0-1)^2\cos^2\varphi}{E_0^2-(E_0-1)^2\cos^2\varphi} \qquad (23\,c)$$

for $\frac{\pi}{2} \leq \varphi \leq \pi$. For $0 \leq \varphi \leq \frac{\pi}{2}$, they lead to $P(\varphi) = 0$.

The total r distribution can be obtained from (13) by an integration over E between the limits (8). In the $Z = 0$, $b = 0$ approximation, the result has been given previously[7]. The total φ distribution can be found by integration of (20) or (25). This integration leads to very complicated integrals. Numerical calculations have been carried out for the neutron decay[5),8)].*

The Influence of the Recoil.

It seems of some interest to study the effects which occur when the kinetic energy of the recoil is not neglected in the conservation of energy (4). We then find

$$E_0 = E + q + r^2/2\,m_R, \qquad (26\,a)$$

where m_R is the recoil mass. If we include the rest mass in the recoil energy $\mathcal{R}$, we get the relativistic expression

* After the conclusion of this paper, the author has received an article by M. E. Rose (O.R.N.L. 1591, 1953) which deals with methods of calculation for the total distribution in certain limits.

$$M = E + \mathcal{R} + q = E_1 + E_2 + E_3, \tag{26b}$$

where M is the mass of the mother nucleus.

Here, it should be kept in mind that the phase space factor

$$rEq\, dE\, dr = E_1 E_2 E_3\, dE_1\, dE_2 \tag{27}$$

in (13) has the exact relativistic form when E_3 is inserted from (26b)[9] and the limiting curves in the (E_1, E_2) plane are taken from formula (6) of reference 9. This formula leads to (8) and (9) apart from terms of the order of E_0/M.

The angular distributions must therefore be divided into this phase space factor and the matrix element factor, and the effect of the neglect of the recoil in (26a) has to be considered separately for these two factors and for the transformations (19) and (24). The effect on the two first factors is the same as for allowed beta spectra, and only the effect on the transformations need to be considered here. Let us consider a transition in which the mother nucleus of mass M disintegrates into three particles of mass m_1, m_2, and $m_3 = 0$. We have then to distinguish between two cases as follows.

A: $$E_1 < \frac{(M-m_2)^2 + m_1^2}{2(M-m_1)}.$$

Here, the angle θ_{12} between the momenta p_1 and p_2 may vary independently between the limits $0 < \theta_{12} \leq \pi$.

B: $$E_1 > \frac{(M-m_2)^2 + m_1^2}{2(M-m_1)}.$$

In this case, we have $\theta'_{12} \leq \theta_{12} \leq \pi$, where θ'_{12} is given by

$$\sin\theta'_{12} = \frac{M(E_1^{\max} - E_1)}{m_2 p_1}; \quad \frac{\pi}{2} \leq \theta'_{12} \leq \pi. \tag{28}$$

When terms of the order of E_0/M are neglected, (28) leads to (16) and (22).

In case A, we find the upper sign and, in case B, both signs in

$$p_2 = \left.\frac{-p_1\cos\theta_{12}\,[M(E_1^{\max}-E_1)+m_2^2] \pm (M-E_1)\sqrt{M^2(E_1^{\max}-E_1)^2 - m_2^2 p_1^2 \sin^2\theta_{12}}}{[(M-E_1)^2 - p_1^2\cos^2\theta_{12}]}\right\} \tag{29}$$

and

$$E_2 = \frac{(M-E_1)\left[M(E_1^{\max}-E_1)+m_2^2\right] \mp p_1 \cos\theta_{12} \sqrt{M^2(E_1^{\max}-E_1)^2 - m_2^2 p_1^2 \sin^2\theta_{12}}}{\left[(M-E_1)^2 - p_1^2\cos^2\theta_{12}\right]} \cdot \quad (30)$$

(29) and (30) correspond to (18) and (23) in the usual limit. By squaring E_2 and p_2, and subtracting, it is easily seen that the result is m_2^2, provided the signs are kept in the order given in (29) and (30).

Formula (10) is also valid relativistically and may now be written in the form

$$2\, p_1 p_2 \cos\theta_{12} = p_3^2 - p_1^2 - p_2^2 \quad (31)$$

which, by differentiation, leads to

$$\frac{dp_2}{d\Omega_{\theta_{12}}} = \frac{2\, p_1 p_2 E_2}{(M-E_1)\, p_2 + E_2 p_1 \cos\theta_{12}} = \frac{2\, p_1 p_2 E_2}{\pm\sqrt{M^2(E_1^{\max}-E_1)^2 - m_2^2 p_1^2 \sin^2\theta_{12}}} \quad (32)$$

which shows the same general features as (19) and (24) and leads to singularities in the (E_1, θ_{12}) distributions when (28) is fulfilled. Formula (19) is obtained from (32) in the limit $m_2 = E_2 = M$, and (24) is obtained from (32) directly by inserting $2ME_1 = r^2$, i. e. in the limit $m_1 = M$. The effect of the recoil is therefore a small shift in the position of the singularities and then, if this shift is neglected, i. e., if (20) or (25) is compared with the true distribution function for fixed values of the square roots in (19), (24), and (32), the changes are of the order of magnitude E_0/M, only.

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Denmark.

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AN APPROPRIATE METHOD FOR INTEGRATION OF THE MOTION OF PERIODIC COMETS

BY

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With an Auxiliary Table

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When the method of numerical integration is used to calculate the perturbed motion of a planet or comet, a certain difficulty arises if it is desired to take the effect of the four inner planets into account.

First of all, the mass of Mercury is so small that, except for very rare cases, it can safely be included in that of the Sun. We shall assume that this procedure is adopted throughout this paper. Thus, no distinction is made between heliocentric co-ordinates and co-ordinates referred to the centre of mass of the Sun and Mercury. However, even the planets Venus, the Earth, and Mars cause difficulties. These planets revolve so rapidly around the Sun that the use of an interval exceeding 10 or 20 days is impossible if the differences are going to remain manageable.

When the objects are more than four or five astronomical units from the Sun this problem is solved by including the masses of the four inner planets into that of the Sun, and neglecting the attractions of the inner planets on the Sun. Though this does seem to be generally recognized, such a procedure assumes the use of a system of co-ordinates originating in the gravity centre of the Sun and the four inner planets.

At smaller distances from the Sun the attractions from Venus, the Earth and Mars on the object as well as on the Sun must be taken rigorously into account.

For objects moving in highly elliptical orbits the usual practice is to use both of these methods alternately. This, however, requires some care at the point where the transition from one method to the other is made, if discontinuities are not to be introduced in the co-ordinates.

The method proposed in this paper is designed to avoid this difficulty. The point of the method is to use a co-ordinate system with its origin in the centre of mass of the Sun + Mercury, Venus,

the Earth, and Mars in the summation schemes at all times, and to omit the attractions on the Sun from these planets. The particular advantages of this choice are the following two:

1. The differences in the summation schemes must always remain smooth, even at the point of changing over from one procedure to the other.
2. The distance of the Sun from the origin of the co-ordinate system will never exceed $5 \cdot 10^{-6}$ astronomical units. Thus, the correction necessary to reduce the co-ordinates, and the attractions on the Sun from the outer planets, to the chosen system from the heliocentric system, in which they generally are tabulated, are sufficiently small to be neglected in nearly all cases. This would not be the case if a system with its origin in the centre of mass of the whole planetary system had been chosen.

The use of a co-ordinate system such as the one proposed here requires the co-ordinates of the Sun in this system to be available. This paper gives the co-ordinates of the Sun with opposite signs for every 10 days during the years 1920—1960. The tabular values are denoted $-\bar{x}_0$, $-\bar{y}_0$, $-\bar{z}_0$ and computed by the equations:

$$-\bar{x}_0 = \frac{\sum_{i=2}^{4} m_i x_i}{1 + \sum_{i=1}^{4} m_i} \qquad -\bar{y}_0 = \frac{\sum_{i=2}^{4} m_i y_i}{1 + \sum_{i=1}^{4} m_i} \qquad -\bar{z}_0 = \frac{\sum_{i=2}^{4} m_i z_i}{1 + \sum_{i=1}^{4} m_i}$$

The masses of the inner planets, which are used in the equations, are in accordance with Astronomical Journal 1170 (G. M. Clemence):

1 = Mercury	1:6 000 000
2 = Venus	1: 408 000
3 = Earth + Moon	1: 329 406
4 = Mars	1:3 093 500

The co-ordinates of Venus, the Earth and Mars used in the computation are those given in the "Planetary Co-ordinates 1920—1960", published by H. M. Nautical Almanac Office.

Then the numerical integration of the perturbed rectangular co-ordinates of a body can be done, either by means of the equation:

$$\left.\begin{aligned} w^2 \frac{d^2 \bar{x}}{dt^2} &= -w^2 k^2 (1+m_1) \frac{\bar{x}-\bar{x}_0}{r^3} \\ &\quad - \sum_{i=2}^{8}{}' w^2 k^2 m_i \frac{x-x_i}{\Delta_i^3} \\ &\quad - \sum_{i=5}^{8}{}' w^2 k^2 m_i \frac{\bar{x}_i}{\bar{r}_i^3}. \end{aligned}\right\} 1.$$

Or, if the distance of the body from the Sun is more than 4—5 units, by means of the equation:

$$\left.\begin{aligned} w^2 \frac{d^2 \bar{x}}{dt^2} &= -w^2 k^2 \left(1 + \sum_{i=1}^{4}{}' m_i\right) \frac{\bar{x}}{\bar{r}^3} \\ &\quad - \sum_{i=5}^{8}{}' w^2 k^2 m_i \frac{\bar{x}-\bar{x}_i}{\Delta_i^3} \\ &\quad - \sum_{i=5}^{8}{}' w^2 k^2 m_i \frac{\bar{x}_i}{\bar{r}_i^3}. \end{aligned}\right\} 2.$$

In which:

$$\begin{aligned} x &= \bar{x}-\bar{x}_0 & r^2 &= (\bar{x}-\bar{x}_0)^2+(\bar{y}-\bar{y}_0)^2+(\bar{z}-\bar{z}_0)^2 \\ y &= \bar{y}-\bar{y}_0 & \Delta_i^2 &= (x-x_i)^2+(y-y_i)^2+(z-z_i)^2 \\ z &= \bar{z}-\bar{z}_0 & \Delta_i^2 &= (\bar{x}-\bar{x}_i)^2+(\bar{z}-\bar{z}_i)^2+(\bar{z}-\bar{z}_i)^2 \\ & & \bar{r}_i^2 &= \bar{x}_i^2+\bar{y}_i^2+\bar{z}_i^2 \end{aligned}$$

x, y, z are co-ordinates referred to the centre of mass of the Sun + Mercury.

$\bar{x}$, $\bar{y}$, $\bar{z}$ are co-ordinates referred to the centre of mass of the Sun + the four inner planets.

Instructions for the use of the method.

Given are the osculating elements: M, e, a, P_x, P_y, P_z, Q_x, Q_y, Q_z, $k\sqrt{1+m_1}$. We adopt the elements to be referred to the barycentre Sun + Mercury. With these elements we compute the rectangular co-ordinates and velocities for the osculating date: x, y, z, $w\frac{dx}{dt}$, $w\frac{dy}{dt}$, $w\frac{dz}{dt}$.

From these we then have to subtract: $-\bar{x}_0$, $-\bar{y}_0$, $-\bar{z}_0$, $-w\frac{dx_0}{dt}$, $-w\frac{dy_0}{dt}$, $-w\frac{dz_0}{dt}$ where the negative velocities of the Sun are deduced from the table by:

$$10\frac{df}{dt} = \frac{1}{60}(-f_{-3} + 9f_{-2} - 45f_{-1} + 45f_{+1} - 9f_{+2} + f_{+3}).$$

We then get $\bar{x}$, $\bar{y}$, $\bar{z}$, $w\frac{d\bar{x}}{dt}$, $w\frac{d\bar{y}}{dt}$, $w\frac{d\bar{z}}{dt}$ and from these values the integration schemes can be built up.

If the radius vector of the body is less than 4—5 units we use equation 1; if it is greater, equation 2.

If the interval is greater than 80 days, the heliocentric co-ordinates for Jupiter and Saturn given in "Planetary Co-ordinates 1920—1960" will not be sufficient for computing the third terms in the two equations accurately by 9 decimals.

In the rare cases when a comet passes very close to one of the outer planets, the second terms must also be computed by means of six or more decimals. So the co-ordinates for the five outer planets given in the "Astronomical Papers" Vol. XII will be well suited.

These co-ordinates are integrated by means of an equation similar to equation 2, thus they are also referred to the origin in the centre of mass of the Sun and the four inner planets. Consequently the co-ordinates of the comet in the second term of equation 1 must not be made heliocentric by adding the tabular values $-\bar{x}_0$, $-\bar{y}_0$, $-\bar{z}_0$.

If equation 2 is used both the co-ordinates of the outer planets and those of the comet are referred to the same origin.

When computing an accurate geocentric ephemeris, the tabular values $-\bar{x}_0$, $-\bar{y}_0$, $-\bar{z}_0$ must always be added to the

barycentric co-ordinates of the body $\bar{x}$, $\bar{y}$, $\bar{z}$. The resulting co-ordinates x, y, z cannot in any case be distinguished from true heliocentric co-ordinates by observation.

If osculating elements are to be derived for some date, the barycentric co-ordinates and velocities derived from the schemes have of course to be changed to the heliocentric system by adding

$$-\bar{x}_0,\ -\bar{y}_0,\ -\bar{z}_0,\ -w\frac{d\bar{x}_0}{dt},\ -w\frac{d\bar{y}_0}{dt},\ -w\frac{d\bar{z}_0}{dt}.$$

As an illustration of the use of the two equations, we compute the acceleration of the periodic comet Schwassmann-Wachmann (2) for 1940 June 20.0 (Interval 20 days).

This date is chosen because Venus and the Earth have then almost the same heliocentric longitude. Thus the disagreement between the results obtained will be particularly great, but all the same only a few units in the 9th decimal.

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Numerical examples.

Equation 1.

$''\overline{F}$	+ 3.95361694 5	— 0.39710974 7	— 0.37222441 1	r^2	15.92613559
$\overline{F}$: 12	— 61366	+ 6249	+ 5895		
$-\overline{F}''$: 240	+ 4	+ 9	+ 3	r	3.99075627
$-\bar{x}_0, -\bar{y}_0, -\bar{z}_0$	— 56	— 406	— 176	$\frac{w^2k^2(1+m_1)}{r^3}$	0.00186233298
x , y , z	+ 3.95300276	— 0.39705123	— 0.37716719		

Planet No.	2	3	4	5	6	7	8
$x - x_i$	+ 4.025	+ 3.977	+ 4.908	— 0.40914	— 3.4668	— 7.95	+ 34.05
$y - y_i$	+ 0.264	+ 0.535	— 1.603	— 2.59555	— 5.5935	— 14.70	— 2.63
$z - z_i$	+ 0.084	+ 0.027	— 0.956	— 1.21364	— 2.2054	— 6.48	— 2.05
Δ_i^2	16.277	16.103	27.572	8.37720	48.1697	321.28	1170.52
$10^8 w^2k^2m_i$	29.011	35.933	3.826	11301.31	3380.31	517.6	612.8
$10^8 w^2k^2m_i : \Delta_i^3$	0.442	0.556	0.026	466.102	10.111	0.0899	0.0153

1. Term	— 0.00736180 74	+ 0.00073944 16	+ 0.00070241 09
2. Term	+ 221 83	+ 1267 34	+ 588 57
3. Term	— 436 94	— 227 42	— 85 96
Acceleration	— 0.00736395 8	+ 0.00074984 1	+ 0.00070743 7
$-''\overline{F}$ (May 31.0)	— 3.98820410 0	+ 0.53999358 2	+ 0.43472291 1
$2''\overline{F}$ (June 20.0)	+ 7.90723389 0	— 0.79421949 4	— 0.75444882 2
$''\overline{F}$ (July 10.0)	+ 3.91166583 2	— 0.25347607 1	— 0.31901847 4

Equation 2.

$''\overline{F}$	+ 3.95361694 5	— 0.39710974 7	— 0.37722441 1	$\bar{r}^2$	15.92613546
$\overline{F}$: 12	— 61366	+ 6249	+ 5895	$\bar{r}$	3.99075625
$\overline{F}''$: 240	+ 4	+ 9	+ 3	$\frac{w^2k^2(1+\sum_{i=1}^{4} m_1)}{\bar{r}^3}$	0.00186234382
$\bar{x}$, $\bar{y}$, $\bar{z}$	+ 3.95300332	— 0.39704717	— 0.37716543		

Planet No.	5	6	7	8
$\bar{x} - \bar{x}_i$	— 0.40914	— 3.4668	— 7.95	+ 34.05
$\bar{y} - \bar{y}_i$	— 2.59555	— 5.5935	— 14.70	— 2.63
$\bar{z} - \bar{z}_i$	— 1.21364	— 2.2054	— 6.48	— 2.05
Δ_i^2	8.37720	48.1697	321.28	1170.52
$10^8 w^2k^2m_i$	11301.31	3380.31	517.6	612.8
$10^8 w^2k^2m_i : \Delta_i^3$	466.102	10.111	0 0899	0.0153

1. Term	— 0.00736185 13	+ 0.00073943 83	+ 0.00070241 17
2. Term	+ 225 95	+ 1267 71	+ 588 59
3. Term	— 436 94	— 227 42	— 85 96
Acceleration	— 0.00736396 1	+ 0.00074984 1	+ 0.00070743 8
$-''\overline{F}$ (May 31.0)	— 3.98820410 0	+ 0.53999358 2	+ 0.43472291 1
$2''\overline{F}$ (June 20.0)	+ 7.90723389 0	— 0.79421949 4	— 0.75444882 2
$''\overline{F}$ (July 10.0)	+ 3.91166582 9	— 0.25347607 1	— 0.31901847 3

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1919 Oct. 8.5.....	2422240.5	+ 415.20	+ 198.59	+ 79.78
18.5.....	250.5	+ 361.81	+ 275.41	+ 115.68
28.5.....	260.5	+ 291.05	+ 339.08	+ 146.30
Nov. 7.5.....	270.5	+ 206.80	+ 386.35	+ 170.02
17.5.....	280.5	+ 113.73	+ 415.00	+ 185.59
27.5.....	290.5	+ 16.94	+ 423.96	+ 192.32
Dec. 7.5.....	300.5	— 78.34	+ 413.36	+ 190.04
17.5.....	310.5	— 167.14	+ 384.56	+ 179.14
27.5.....	320.5	— 245.10	+ 339.94	+ 160.54
1920 Jan. 6.5.....	330.5	— 308.75	+ 282.76	+ 135.59
16.5.....	340.5	— 355.68	+ 216.83	+ 105.94
26.5.....	350.5	— 384.72	+ 146.27	+ 73.48
Feb. 5.5.....	360.5	— 395.84	+ 75.14	+ 40.10
15.5.....	370.5	— 390.18	+ 7.20	+ 7.62
25.5.....	380.5	— 369.80	— 54.33	— 22.32
Mar. 6.5.....	390.5	— 337.51	— 106.97	— 48.41
16.5.....	400.5	— 296.60	— 149.08	— 69.70
26.5.....	410.5	— 250.57	— 179.94	— 85.66
Apr. 5.5.....	420.5	— 202.86	— 199.67	— 96.14
15.5.....	430.5	— 156.60	— 209.22	— 101.43
25.5.....	440.5	— 114.42	— 210.20	— 102.14
May 5.5.....	450.5	— 78.21	— 204.73	— 99.18
15.5.....	460 5	— 49.07	— 195.21	— 93.64
25.5.....	470.5	— 27.22	— 184.09	— 86.71
June 4.5.....	480.5	— 12.03	— 173.67	— 79.54
14.5.....	490.5	— 2.09	— 165.81	— 73.17
24.5.....	500.5	+ 4.58	— 161.80	— 68.38
July 4.5.....	510.5	+ 10.34	— 162.20	— 65.68
14.5.....	520.5	+ 17.62	— 166.74	— 65.19
24.5.....	530.5	+ 28.66	— 174.39	— 66.66
Aug. 3.5.....	540.5	+ 45.23	— 183.45	— 69.52
13.5.....	550.5	+ 68.39	— 191.72	— 72.90
23.5.....	560.5	+ 98.35	— 196.71	— 75.74
Sept. 2.5.....	570.5	+ 134.40	— 195.96	— 76.93
12.5.....	580.5	+ 174.92	— 187.29	— 75.40
22.5.....	590.5	+ 217.52	— 168.99	— 70.25
Oct. 2.5.....	600.5	+ 259.25	— 140.10	— 60.83
12.5.....	610.5	+ 296.79	— 100.48	— 46.86
22.5.....	620.5	+ 326.75	— 50.87	— 28.40
Nov. 1.5.....	630.5	+ 345.95	+ 7.06	— 5.96
11.5.....	640.5	+ 351.67	+ 70.89	+ 19.63
21.5.....	650.5	+ 341.91	+ 137.48	+ 47.16
Dec. 1.5.....	660.5	+ 315.54	+ 203.24	+ 75.19
11.5.....	670.5	+ 272.46	+ 264.31	+ 102.07
21.5.....	680.5	+ 213.67	+ 316.81	+ 126.11
31.5.....	690.5	+ 141.28	+ 357.16	+ 145.66
1921 Jan. 10.5.....	700.5	+ 58.43	+ 382.27	+ 159.23
20.5.....	710.5	— 30.90	+ 389.84	+ 165.64

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1921 Jan. 20.5	2422710.5	— 30.90	+ 389.84	+ 165.64
30.5	720.5	— 122.03	+ 378.57	+ 164.07
Feb. 9.5	730.5	— 209.96	+ 348.24	+ 154.19
19.5	740.5	— 289.66	+ 299.86	+ 136.19
Mar. 1.5	750.5	— 356.42	+ 235.60	+ 110.77
11.5	760.5	— 406.22	+ 158.69	+ 79.15
21.5	770.5	— 435.99	+ 73.21	+ 42.98
31.5	780.5	— 443.90	— 16.18	+ 4.21
Apr. 10.5	790.5	— 429.42	— 104.57	— 34.99
20.5	800.5	— 393.38	— 187.09	— 72.45
30.5	810.5	— 337.93	— 259.30	— 106.09
May 10.5	820.5	— 266.31	— 317.43	— 134.07
20.5	830.5	— 182.67	— 358.57	— 154.93
30.5	840.5	— 91.77	— 380.90	— 167.61
June 9.5	850.5	+ 1.36	— 383.73	— 171.58
19.5	860.5	+ 91.68	— 367.47	— 166.81
29.5	870.5	+ 174.45	— 333.65	— 153.76
July 9.5	880.5	+ 245.54	— 284.70	— 133.38
19.5	890.5	+ 301.62	— 223.88	— 106.99
29.5	900.5	+ 340.35	— 154.98	— 76.23
Aug. 8.5	910.5	+ 360.52	— 82.12	— 42.95
18.5	920.5	+ 362.06	— 9.48	— 9.07
28.5	930.5	+ 346.06	+ 59.03	+ 23.52
Sept. 7.5	940.5	+ 314.61	+ 119.95	+ 53.11
17.5	950.5	+ 270.68	+ 170.54	+ 78.27
27.5	960.5	+ 217.81	+ 208.93	+ 97.94
Oct. 7.5	970.5	+ 159.86	+ 234.24	+ 111.49
17.5	980.5	+ 100.69	+ 246.56	+ 118.75
27.5	2422990.5	+ 43.83	+ 246.89	+ 120.03
Nov. 6.5	2423000.5	— 7.76	+ 236.99	+ 115.97
16.5	010.5	— 51.87	+ 219.16	+ 107.57
26.5	020.5	— 87.20	+ 196.01	+ 96.00
Dec. 6.5	030.5	— 113.34	+ 170.18	+ 82.53
16.5	040.5	— 130.73	+ 144.13	+ 68.41
26.5	050.5	— 140.56	+ 119.92	+ 54.74
1922 Jan. 5.5	060.5	— 144.54	+ 99.08	+ 42.42
15.5	070.5	— 144.75	+ 82.48	+ 32.09
25.5	080.5	— 143.40	+ 70.35	+ 24.05
Feb. 4.5	090.5	— 142.60	+ 62.23	+ 18.33
14.5	100.5	— 144.17	+ 57.10	+ 14.64
24.5	110.5	— 149.46	+ 53.44	+ 12.42
Mar. 6.5	120.5	— 159.19	+ 49.41	+ 10.90
16.5	130.5	— 173.41	+ 43.01	+ 9.21
26.5	140.5	— 191.38	+ 32.32	+ 6.42
Apr. 5.5	150.5	— 211.64	+ 15.65	+ 1.64
15.5	160.5	— 232.09	— 8.21	— 5.85
25.5	170.5	— 250.12	— 39.85	— 16.53
May 5.5	180.5	— 262.88	— 79.04	— 30.56

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1922 May 5.5.....	2423180.5	−262.88	− 79.04	− 30.56
15.5.....	190.5	−267.46	−124.75	− 47.74
25.5.....	200.5	−261.28	−175.08	− 67.46
June 4.5.....	210.5	−242.28	−227.41	− 88.76
14.5.....	220.5	−209.24	−278.55	−110.38
24.5.....	230.5	−161.91	−324.96	−130.84
July 4.5.....	240.5	−101.14	−363.07	−148.57
14.5.....	250.5	− 28.90	−389.52	−162.02
24.5.....	260.5	+ 51.88	−401.52	−169.82
Aug. 3.5.....	270.5	+137.42	−396.98	−170.88
13.5.....	280.5	+223.37	−374.80	−164.48
23.5.....	290.5	+305.05	−334.90	−150.34
Sept. 2.5.....	300.5	+377.81	−278.30	−128.68
12.5.....	310.5	+437.30	−207.06	−100.16
22.5.....	320.5	+479.76	−124.25	− 65.94
Oct. 2.5.....	330.5	+502.26	− 33.68	− 27.54
12.5.....	340.5	+502.93	+ 60.22	+ 13.20
22.5.....	350.5	+481.13	+152.67	+ 54.19
Nov. 1.5.....	360.5	+437.48	+238.88	+ 93.31
11.5.....	370.5	+373.90	+314.27	+128.44
21.5.....	380.5	+293.51	+374.82	+157.66
Dec. 1.5.....	390.5	+200.49	+417.34	+179.36
11.5.....	400.5	+ 99.79	+439.67	+192.36
21.5.....	410.5	− 3.15	+440.88	+196.00
31.5.....	420.5	−102.80	+421.31	+190.20
1923 Jan. 10.5.....	430.5	−193.89	+382.58	+175.44
20.5.....	440.5	−271.80	+327.42	+152.77
30.5.....	450.5	−332.84	+259.51	+123.69
Feb. 9.5.....	460.5	−374.53	+183.15	+ 90.06
19.5.....	470.5	−395.65	+102.97	+ 53.93
Mar. 1.5.....	480.5	−396.33	+ 23.58	+ 17.43
11.5.....	490.5	−377.93	− 50.74	− 17.42
21.5.....	500.5	−342.91	−116.30	− 48.79
31.5.....	510.5	−294.55	−170.21	− 75.20
Apr. 10.5.....	520.5	−236.74	−210.53	− 95.55
20.5.....	530.5	−173.62	−236.30	−109.21
30.5.....	540.5	−109.37	−247.60	−116.02
May 10.5.....	550.5	− 47.84	−245.41	−116.23
20.5.....	560.5	+ 7.63	−231.53	−110.55
30.5.....	570.5	+ 54.42	−208.39	− 99.98
June 9.5.....	580.5	+ 90.83	−178.85	− 85.78
19.5.....	590.5	+116.05	−145.93	− 69.39
29.5.....	600.5	+130.30	−112.58	− 52.22
July 9.5.....	610.5	+134.68	− 81.46	− 35.63
19.5.....	620.5	+131.04	− 54.66	− 20.76
29.5.....	630.5	+121.78	− 33.59	− 8.48
Aug. 8.5.....	640.5	+109.61	− 18.84	+ 0.72
18.5.....	650.5	+ 97.18	− 10.18	+ 6.70

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1923 Aug. 18.5.....	2423650.5	+ 97.18	— 10.18	+ 6.70
28.5.....	660.5	+ 86.90	— 6.59	+ 9.74
Sept. 7.5.....	670.5	+ 80.59	— 6.46	+ 10.42
17.5.....	680.5	+ 79.37	— 7.74	+ 9.58
27.5.....	690.5	+ 83.48	— 8.20	+ 8.22
Oct. 7.5.....	700.5	+ 92.34	— 5.70	+ 7.33
17.5.....	710.5	+ 104.58	+ 1.55	+ 7.87
27.5.....	720.5	+ 118.18	+ 14.82	+ 10.59
Nov. 6.5.....	730.5	+ 130.69	+ 34.72	+ 15.98
16.5.....	740.5	+ 139.49	+ 61.10	+ 24.20
26.5.....	750.5	+ 142.00	+ 93.01	+ 35.11
Dec. 6.5.....	760.5	+ 135.96	+ 128.80	+ 48.19
16.5.....	770.5	+ 119.62	+ 166.21	+ 62.64
26.5.....	780.5	+ 91.92	+ 202.50	+ 77.41
1924 Jan. 5.5.....	790.5	+ 52.66	+ 234.67	+ 91.24
15.5.....	800.5	+ 2.54	+ 259.64	+ 102.82
25.5.....	810.5	— 56.76	+ 274.51	+ 110.83
Feb. 4.5.....	820.5	— 122.73	+ 276.82	+ 114.05
14.5.....	830.5	— 192.03	+ 264.69	+ 111.52
24.5.....	840.5	— 260.72	+ 237.13	+ 102.56
Mar. 5.5.....	850.5	— 324.56	+ 194.06	+ 86.91
15.5.....	860.5	— 379.20	+ 136.52	+ 64.76
25.5.....	870.5	— 420.57	+ 66.56	+ 36.76
Apr. 4.5.....	880.5	— 445.19	— 12.78	+ 4.05
14.5.....	890.5	— 450.47	— 97.66	— 31.86
24.5.....	900.5	— 434.90	— 183.68	— 69.15
May 4.5.....	910.5	— 398.24	— 266.18	— 105.79
14.5.....	920.5	— 341.53	— 340.57	— 139.74
24.5.....	930.5	— 267.07	— 402.61	— 169.04
June 3.5.....	940.5	— 178.27	— 448.72	— 191.93
13.5.....	950.5	— 79.41	— 476.21	— 207.04
23.5.....	960.5	+ 24.62	— 483.43	— 213.42
July 3.5.....	970.5	+ 128.58	— 469.85	— 210.58
13.5.....	980.5	+ 227.30	— 436.14	— 198.59
23.5.....	2423990.5	+ 315.85	— 384.03	— 178.01
Aug. 2.5.....	2424000.5	+ 389.94	— 316.28	— 149.87
12.5.....	010.5	+ 446.10	— 236.47	— 115.64
22.5.....	020.5	+ 481.88	— 148.78	— 77.09
Sept. 1.5.....	030.5	+ 496.02	— 57.78	— 36.25
11.5.....	040.5	+ 488.49	+ 31.89	+ 4.79
21.5.....	050.5	+ 460.50	+ 115.80	+ 43.92
Oct. 1.5.....	060.5	+ 414.40	+ 189.97	+ 79.24
11.5.....	070.5	+ 353.51	+ 251.16	+ 109.09
21.5.....	080.5	+ 281.90	+ 297.07	+ 132.21
31.5.....	090.5	+ 204.09	+ 326.43	+ 147.82
Nov. 10.5.....	100.5	+ 124.66	+ 339.11	+ 155.64
20.5.....	110.5	+ 48.01	+ 336.04	+ 155.90
30.5.....	120.5	— 22.04	+ 319.08	+ 149.28

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1924 Nov. 30.5.....	2424120.5	— 22.04	+ 319.08	+ 149.28
Dec. 10.5.....	130.5	— 82.44	+ 290.83	+ 136.87
20.5.....	140.5	— 131.07	+ 254.38	+ 120.04
30.5.....	150.5	— 166.89	+ 213.01	+ 100.31
1925 Jan. 10.0.....	160.5	— 189.85	+ 169.98	+ 79.24
20	170.5	— 200.86	+ 128.22	+ 58.30
30	180.5	— 201.58	+ 90.17	+ 38.78
Feb. 9	190.5	— 194.26	+ 57.61	+ 21.71
19	200.5	— 181.48	+ 31.59	+ 7.76
Mar. 1	210.5	— 165.96	+ 12.40	— 2.72
11	220.5	— 150.24	— 0.39	— 9.74
21	230.5	— 136.53	— 7.86	— 13.65
31	240.5	— 126.52	— 11.60	— 15.08
Apr. 10	250.5	— 121.20	— 13.56	— 14.88
20	260.5	— 120.83	— 15.81	— 14.01
30	270.5	— 124.85	— 20.35	— 13.48
May 10	280.5	— 131.96	— 28.88	— 14.21
20	290.5	— 140.22	— 42.62	— 16.94
30	300.5	— 147.29	— 62.10	— 22.16
June 9	310.5	— 150.58	— 87.09	— 30.02
19	320.5	— 147.60	— 116.53	— 40.28
29	330.5	— 136.23	— 148.61	— 52.38
July 9	340.5	— 114.96	— 180.90	— 65.38
19	350.5	— 83.15	— 210.56	— 78.14
29	360.5	— 41.10	— 234.53	— 89.34
Aug. 8	370.5	+ 9.85	— 249.93	— 97.65
18	380.5	+ 67.45	— 254.22	— 101.86
28	390.5	+ 128.59	— 245.54	— 100.97
Sept. 7	400.5	+ 189.61	— 222.80	— 94.29
17	410.5	+ 246.49	— 185.91	— 81.53
27	420.5	+ 295.18	— 135.77	— 62.85
Oct. 7	430.5	+ 331.90	— 74.25	— 38.82
17	440.5	+ 353.36	— 4.12	— 10.46
27	450.5	+ 357.10	+ 71.07	+ 20.85
Nov. 6	460.5	+ 341.61	+ 147.27	+ 53.44
16	470.5	+ 306.55	+ 220.09	+ 85.44
26	480.5	+ 252.77	+ 285.13	+ 114.89
Dec. 6	490.5	+ 182.35	+ 338.23	+ 139.90
16	500.5	+ 98.51	+ 375.80	+ 158.74
26	510.5	+ 5.45	+ 395.03	+ 170.00
1926 Jan. 5	520.5	— 91.88	+ 394.18	+ 172.68
15	530.5	— 188.11	+ 372.71	+ 166.30
25	540.5	— 277.80	+ 331.32	+ 150.90
Feb. 4	550.5	— 355.82	+ 271.99	+ 127.16
14	560.5	— 417.71	+ 197.83	+ 96.23
24	570.5	— 459.98	+ 112.85	+ 59.76
Mar. 6	580.5	— 480.38	+ 21.72	+ 19.72
16	590.5	— 478.01	— 70.60	— 21.70

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1926 Mar. 16	2424590.5	−478.01	− 70.60	− 21.70
26	600.5	−453.34	−159.19	− 62.24
Apr. 5	610.5	−408.17	−239.46	− 99.79
15	620.5	−345.43	−307.49	−132.41
25	630.5	−268.94	−360.22	−158.55
May 5	640.5	−183.19	−395.61	−177.08
15	650.5	− 92.96	−412.68	−187.33
25	660.5	− 3.05	−411.57	−189.17
June 4	670.5	+ 82.04	−393.48	−182.92
14	680.5	+ 158.32	−360.47	−169.38
24	690.5	+ 222.58	−315.39	−149.70
July 4	700.5	+ 272.56	−261.58	−125.35
14	710.5	+ 307.03	−202.68	− 97.99
24	720.5	+ 325.84	−142.34	− 69.32
Aug. 3	730.5	+ 329.87	− 83.97	− 41.01
13	740.5	+ 320.94	− 30.50	− 14.57
23	750.5	+ 301.57	+ 15.83	+ 8.79
Sept. 2	760.5	+ 274.76	+ 53.60	+ 28.20
12	770.5	+ 243.70	+ 82.28	+ 43.20
22	780.5	+ 211.47	+ 102.20	+ 53.73
Oct. 2	790.5	+ 180.73	+ 114.49	+ 60.15
12	800.5	+ 153.55	+ 120.84	+ 63.11
22	810.5	+ 131.21	+ 123.30	+ 63.51
Nov. 1	820.5	+ 114.15	+ 124.08	+ 62.35
11	830.5	+ 102.01	+ 125.23	+ 60.66
21	840.5	+ 93.69	+ 128.46	+ 59.39
Dec. 1	850.5	+ 87.54	+ 134.99	+ 59.26
11	860.5	+ 81.56	+ 145.38	+ 60.76
21	870.5	+ 73.61	+ 159.56	+ 64.12
31	880.5	+ 61.64	+ 176.77	+ 69.20
1927 Jan. 10	890.5	+ 43.92	+ 195.63	+ 75.63
20	900.5	+ 19.19	+ 214.30	+ 82.72
30	910.5	− 13.18	+ 230.54	+ 89.61
Feb. 9	920.5	− 53.04	+ 241.98	+ 95.28
19	930.5	− 99.48	+ 246.24	+ 98.68
Mar. 1	940.5	−150.77	+ 241.21	+ 98.76
11	950.5	−204.45	+ 225.21	+ 94.64
21	960.5	−257.45	+ 197.19	+ 85.65
31	970.5	−306.31	+ 156.92	+ 71.46
Apr. 10	980.5	−347.40	+ 105.08	+ 52.10
20	2424990.5	−377.20	+ 43.29	+ 28.04
30	2425000.5	−392.67	− 25.94	+ 0.15
May 10	010.5	−391.44	− 99.32	− 30.31
20	020.5	−372.14	−172.99	− 61.77
30	030.5	−334.50	−242.78	− 92.46
June 9	040.5	−279.44	−304.54	−120.54
19	050.5	−209.09	−354.41	−144.24
29	060.5	−126.62	−389.10	−161.95

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1927 June 29	2425060.5	—126.62	—389.10	—161.95
July 9	070.5	— 36.09	—406.15	—172.42
19	080.5	+ 57.85	—404.10	—174.76
29	090.5	+150.19	—382.58	—168.57
Aug. 8	100.5	+235.94	—342.34	—153.96
18	110.5	+310.39	—285.27	—131.51
28	120.5	+369.41	—214.21	—102.30
Sept. 7	130.5	+409.68	—132.86	— 67.83
17	140.5	+428.96	— 45.54	— 29.89
27	150.5	+426.16	+ 43.05	+ 9.45
Oct. 7	160.5	+401.50	+128.12	+ 48.06
17	170.5	+356.47	+205.08	+ 83.78
27	180.5	+293.74	+269.82	+114.66
Nov. 6	190.5	+217.07	+319.00	+139.02
16	200.5	+130.98	+350.27	+155.59
26	210.5	+ 40.50	+362.40	+163.59
Dec. 6	220.5	— 49.17	+355.41	+162.81
16	230.5	—133.08	+330.47	+153.55
26	240.5	—206.83	+289.86	+136.68
1928 Jan. 5	250.5	—266.87	+236.70	+113.47
15	260.5	—310.71	+174.77	+ 85.57
25	270.5	—337.05	+108.10	+ 54.82
Feb. 4	280.5	—345.79	+ 40.77	+ 23.10
14	290.5	—337.96	— 23.45	— 7.75
24	300.5	—315.56	— 81.30	— 36.09
Mar. 5	310.5	—281.33	—130.25	— 60.58
15	320.5	—238.56	—168.59	— 80.24
25	330.5	—190.75	—195.51	— 94.50
Apr. 4	340.5	—141.38	—211.10	—103.19
14	350.5	— 93.65	—216.23	—106.56
24	360.5	— 50.26	—212.50	—105.21
May 4	370.5	— 13.21	—202.02	—100.04
14	380.5	+ 16.29	—187.19	— 92.13
24	390.5	+ 37.92	—170.55	— 82.69
June 3	400.5	+ 52.22	—154.44	— 72.92
13	410.5	+ 60.52	—140.86	— 63.87
23	420.5	+ 64.75	—131.19	— 56.41
July 3	430.5	+ 67.26	—126.14	— 51.09
13	440.5	+ 70.51	—125.58	— 48.09
23	450.5	+ 76.80	—128.63	— 47.24
Aug. 2	460.5	+ 88.01	—133.68	— 48.01
12	470.5	+105.33	—138.62	— 49.59
22	480.5	+129.10	—141.01	— 50.95
Sept. 1	490.5	+158.78	—138.42	— 51.01
11	500.5	+192.91	—128.62	— 48.69
21	510.5	+229.23	—109.90	— 43.06
Oct. 1	520.5	+264.90	— 81.18	— 33.48
11	530.5	+296.67	— 42.25	— 19.61

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1928 Oct. 11	2425530.5	+ 296.67	— 42.25	— 19.61
21	540.5	+ 321.23	+ 6.24	— 1.48
31	550.5	+ 335.41	+ 62.77	+ 20.46
Nov. 10	560.5	+ 336.48	+ 124.96	+ 45.41
20	570.5	+ 322.40	+ 189.78	+ 72.21
30	580.5	+ 291.98	+ 253.70	+ 99.44
Dec. 10	590.5	+ 245.07	+ 312.89	+ 125.48
20	600.5	+ 182.57	+ 363.50	+ 148.66
30	610.5	+ 106.52	+ 401.92	+ 167.32
1929 Jan. 9	620.5	+ 19.96	+ 425.02	+ 179.96
19	630.5	— 73.14	+ 430.45	+ 185.38
29	640.5	— 168.15	+ 416.80	+ 182.74
Feb. 8	650.5	— 260.08	+ 383.77	+ 171.65
18	660.5	— 343.84	+ 332.28	+ 152.26
28	670.5	— 414.65	+ 264.41	+ 125.24
Mar. 10	680.5	— 468.37	+ 183.31	+ 91.78
20	690.5	— 501.83	+ 93.02	+ 53.49
30	700.5	— 513.00	— 1.80	+ 12.34
Apr. 9	710.5	— 501.23	— 96.19	— 29.54
19	720.5	— 467.19	— 185.25	— 69.92
29	730.5	— 412.88	— 264.42	— 106.68
May 9	740.5	— 341.44	— 329.79	— 137.95
19	750.5	— 256.95	— 378.31	— 162.17
29	760.5	— 164.12	— 408.00	— 178.25
June 8	770.5	— 67.98	— 417.98	— 185.56
18	780.5	+ 26.36	— 408.52	— 183.99
28	790.5	+ 114.13	— 380.98	— 173.95
July 8	800.5	+ 191.07	— 337.68	— 156.31
18	810.5	+ 253.73	— 281.75	— 132.33
28	820.5	+ 299.62	— 216.90	— 103.62
Aug. 7	830.5	+ 327.35	— 147.21	— 71.99
17	840.5	+ 336.70	— 76.80	— 39.34
27	850.5	+ 328.57	— 9.64	— 7.57
Sept. 6	860.5	+ 304.93	+ 50.78	+ 21.61
16	870.5	+ 268.60	+ 101.65	+ 46.73
26	880.5	+ 223.01	+ 141.03	+ 66.70
Oct. 6	890.5	+ 171.96	+ 167.92	+ 80.84
16	900.5	+ 119.26	+ 182.32	+ 88.94
26	910.5	+ 68.43	+ 185.13	+ 91.25
Nov. 5	920.5	+ 22.45	+ 178.03	+ 88.39
15	930.5	— 16.44	+ 163.27	+ 81.31
25	940.5	— 46.89	+ 143.41	+ 71.16
Dec. 5	950.5	— 68.45	+ 121.08	+ 59.21
15	960.5	— 81.52	+ 98.74	+ 46.69
25	970.5	— 87.28	+ 78.49	+ 34.72
1930 Jan. 4	980.5	— 87.42	+ 61.86	+ 24.21
14	2425990.5	— 84.05	+ 49.77	+ 15.80
24	2426000.5	— 79.41	+ 42.47	+ 9.82

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1930 Jan. 24	2426000.5	— 79.41	+ 42.47	+ 9.82
Feb. 3	010.5	— 75.64	+ 39.54	+ 6.32
13	020.5	— 74.65	+ 39.95	+ 5.00
23	030.5	— 77.88	+ 42.18	+ 5.31
Mar. 5	040.5	— 86.12	+ 44.38	+ 6.50
15	050.5	— 99.52	+ 44.49	+ 7.67
25	060.5	—117.43	+ 40.50	+ 7.85
Apr. 4	070.5	—138.48	+ 30.65	+ 6.13
14	080.5	—160.61	+ 13.61	+ 1.74
24	090.5	—181.27	— 11.35	— 5.85
May 4	100.5	—197.58	— 44.16	— 16.88
14	110.5	—206.62	— 83.92	— 31.20
24	120.5	—205.70	—128.88	— 48.29
June 3	130.5	—192.68	—176.57	— 67.24
13	140.5	—166.16	—223.86	— 86.85
23	150.5	—125.75	—267.29	—105.67
July 3	160.5	— 72.12	—303.32	—122.15
13	170.5	— 7.04	—328.60	—134.75
23	180.5	+ 66.72	—340.25	—142.10
Aug. 2	190.5	+ 145.54	—336.14	—143.07
12	200.5	+ 225.17	—315.03	—136.90
22	210.5	+ 301.04	—276.73	—123.27
Sept. 1	220.5	+ 368.54	—222.13	—102.32
11	230.5	+ 423.34	—153.18	— 74.68
21	240.5	+ 461.65	— 72.78	— 41.43
Oct. 1	250.5	+ 480.49	+ 15.32	— 4.05
11	260.5	+ 477.93	+ 106.81	+ 35.65
21	270.5	+ 453.22	+ 196.96	+ 75.64
31	280.5	+ 406.86	+ 281.02	+ 113.81
Nov. 10	290.5	+ 340.67	+ 354.39	+ 148.04
20	300.5	+ 257.65	+ 413.04	+ 176.40
30	310.5	+ 161.90	+ 453.69	+ 197.26
Dec. 10	320.5	+ 58.30	+ 474.10	+ 209.41
20	330.5	— 47.74	+ 473.23	+ 212.13
30	340.5	—150.68	+ 451.31	+ 205.30
1931 Jan. 9	350.5	—245.19	+ 409.84	+ 189.36
19	360.5	—326.56	+ 351.48	+ 165.29
29	370.5	—390.99	+ 279.83	+ 134.57
Feb. 8	380.5	—435.84	+ 199.16	+ 99.04
18	390.5	—459.75	+ 114.11	+ 60.74
28	400.5	—462.70	+ 29.33	+ 21.81
Mar. 10	410.5	—445.92	— 50.83	— 15.68
20	420.5	—411.74	—122.57	— 49.88
30	430.5	—363.38	—182.87	— 79.25
Apr. 9	440.5	—304.66	—229.65	—102.62
19	450.5	—239.72	—261.83	—119.31
29	460.5	—172.75	—279.32	—129.06
May 9	470.5	—107.66	—282.98	—132.10

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1931 May 9	2426470.5	−107.66	−282.98	−132.10
19	480.5	− 47.86	−274.49	−129.05
29	490.5	+ 3.93	−256.20	−120.87
June 8	500.5	+ 45.85	−230.88	−108.80
18	510.5	+ 77.01	−201.53	− 94.22
28	520.5	+ 97.48	−171.08	− 78.56
July 8	530.5	+108.23	−142.20	− 63.16
18	540.5	+111.01	−117.02	− 49.18
28	550.5	+108.15	− 97.01	− 37.51
Aug. 7	560.5	+102.28	− 82.83	− 28.68
17	570.5	+ 96.03	− 74.30	− 22.84
27	580.5	+ 91.81	− 70.48	− 19.76
Sept. 6	590.5	+ 91.47	− 69.78	− 18.88
16	600.5	+ 96.14	− 70.18	− 19.36
26	610.5	+106.14	− 69.44	− 20.23
Oct. 6	620.5	+120.89	− 65.42	− 20.48
16	630.5	+139.03	− 56.25	− 19.13
26	640.5	+158.56	− 40.61	− 15.42
Nov. 5	650.5	+176.99	− 17.83	− 8.84
15	660.5	+191.65	+ 12.00	+ 0.84
25	670.5	+199.86	+ 48.00	+ 13.46
Dec. 5	680.5	+199.27	+ 88.57	+ 28.55
15	690.5	+187.98	+131.44	+ 45.30
25	700.5	+164.82	+173.86	+ 62.68
1932 Jan. 4	710.5	+129.42	+212.81	+ 79.42
14	720.5	+ 82.37	+245.13	+ 94.18
24	730.5	+ 25.16	+267.83	+105.60
Feb. 3	740.5	− 39.76	+278.30	+112.44
13	750.5	−109.18	+274.54	+113.63
23	760.5	−179.20	+255.35	+108.45
Mar. 4	770.5	−245.60	+220.49	+ 96.55
14	780.5	−304.00	+170.77	+ 78.03
24	790.5	−350.26	+108.08	+ 53.47
Apr. 3	800.5	−380.80	+ 35.28	+ 23.90
13	810.5	−392.84	− 43.92	− 9.21
23	820.5	−384.71	−125.25	− 44.12
May 3	830.5	−355.95	−204.10	− 78.83
13	840.5	−307.40	−275.90	−111.31
23	850.5	−241.16	−336.40	−139.61
June 2	860.5	−160.42	−381.96	−161.97
12	870.5	− 69.32	−409.80	−176.97
22	880.5	+ 27.38	−418.15	−183.61
July 2	890.5	+124.59	−406.38	−181.36
12	900.5	+217.15	−374.98	−170.23
22	910.5	+300.20	−325.58	−150.70
Aug. 1	920.5	+369.42	−260.79	−123.77
11	930.5	+421.31	−184.08	− 90.83
21	940.5	+453.36	− 99.52	− 53.61

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1932 Aug. 21	2426940.5	+ 453.36	— 99.52	— 53.61
31	950.5	+ 464.23	— 11.61	— 14.09
Sept. 10	960.5	+ 453.80	+ 75.06	+ 25.65
20	970.5	+ 423.16	+ 156.08	+ 63.54
30	980.5	+ 374.57	+ 227.46	+ 97.64
Oct. 10	2426990.5	+ 311.28	+ 285.92	+ 126.31
20	2427000.5	+ 237.30	+ 329.10	+ 148.27
30	010.5	+ 157.09	+ 355.67	+ 162.68
Nov. 9	020.5	+ 75.25	+ 365.42	+ 169.25
19	030.5	— 3.81	+ 359.20	+ 168.16
29	040.5	— 76.21	+ 338.82	+ 160.06
Dec. 9	050.5	— 138.80	+ 306.83	+ 146.02
19	060.5	— 189.40	+ 266.29	+ 127.39
29	070.5	— 226.86	+ 220.53	+ 105.69
1933 Jan. 8	080.5	— 251.05	+ 172.81	+ 82.49
18	090.5	— 262.76	+ 126.13	+ 59.29
28	100.5	— 263.63	+ 82.98	+ 37.41
Feb. 7	110.5	— 255.83	+ 45.25	+ 17.90
17	120.5	— 241.94	+ 14.09	+ 1.49
27	130.5	— 224.67	— 10.13	— 11.42
Mar. 9	140.5	— 206.59	— 27.75	— 20.81
19	150.5	— 189.96	— 39.76	— 26.98
29	160.5	— 176.53	— 47.69	— 30.52
Apr. 8	170.5	— 167.40	— 53.43	— 32.25
18	180.5	— 162.88	— 59.03	— 33.12
28	190.5	— 162.52	— 66.49	— 34.11
May 8	200.5	— 165.11	— 77.52	— 36.16
18	210.5	— 168.79	— 93.38	— 40.01
28	220.5	— 171.25	— 114.64	— 46.18
June 7	230.5	— 169.94	— 141.14	— 54.83
17	240.5	— 162.38	— 171.87	— 65.78
27	250.5	— 146.40	— 205.10	— 78.44
July 7	260.5	— 120.46	— 238.40	— 91.95
17	270.5	— 83.84	— 268.93	— 105.14
27	280.5	— 36.77	— 293.65	— 116.70
Aug. 6	290.5	+ 19.48	— 309.60	— 125.29
16	300.5	+ 82.68	— 314.18	— 129.67
26	310.5	+ 149.80	— 305.41	— 128.79
Sept. 5	320.5	+ 217.12	— 282.13	— 121.93
15	330.5	+ 280.63	— 244.10	— 108.75
25	340.5	+ 336.18	— 192.11	— 89.35
Oct. 5	350.5	+ 379.86	— 127.94	— 64.26
15	360.5	+ 408.28	— 54.30	— 34.46
25	370.5	+ 418.79	+ 25.36	— 1.30
Nov. 4	380.5	+ 409.69	+ 106.96	+ 33.57
14	390.5	+ 380.44	+ 186.12	+ 68.26
24	400.5	+ 331.69	+ 258.36	+ 100.82
Dec. 4	410.5	+ 265.33	+ 319.42	+ 129.30

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1933 Dec. 4	2427410.5	+ 265.33	+ 319.42	+ 129.30
14	420.5	+ 184.41	+ 365.56	+ 151.92
24	430.5	+ 92.99	+ 393.83	+ 167.20
1934 Jan. 3	440.5	— 4.08	+ 402.25	+ 174.05
13	450.5	— 101.49	+ 390.04	+ 171.88
23	460.5	— 193.78	+ 357.70	+ 160.66
Feb. 2	470.5	— 275.76	+ 306.98	+ 140.92
12	480.5	— 342.86	+ 240.77	+ 113.75
22	490.5	— 391.45	+ 162.91	+ 80.70
Mar. 4	500.5	— 419.06	+ 77.93	+ 43.67
14	510.5	— 424.58	— 9.31	+ 4.80
24	520.5	— 408.28	— 93.90	— 33.69
Apr. 3	530.5	— 371.72	— 171.30	— 69.67
13	540.5	— 317.62	— 237.53	— 101.22
23	550.5	— 249.66	— 289.45	— 126.76
May 3	560.5	— 172.14	— 324.93	— 145.11
13	570.5	— 89.73	— 342.90	— 155.57
23	580.5	— 7.15	— 343.38	— 157.93
June 2	590.5	+ 71.14	— 327.41	— 152.49
12	600.5	+ 141.18	— 296.97	— 139.97
22	610.5	+ 199.76	— 254.79	— 121.48
July 2	620.5	+ 244.57	— 204.12	— 98.45
12	630.5	+ 274.34	— 148.54	— 72.48
22	640.5	+ 288.86	— 91.65	— 45.28
Aug. 1	650.5	+ 288.97	— 36.83	— 18.48
11	660.5	+ 276.42	+ 12.97	+ 6.41
21	670.5	+ 253.69	+ 55.52	+ 28.17
31	680.5	+ 223.77	+ 89.36	+ 45.92
Sept. 10	690.5	+ 189.86	+ 113.93	+ 59.17
20	700.5	+ 155.03	+ 129.54	+ 67.88
30	710.5	+ 122.02	+ 137.29	+ 72.37
Oct. 10	720.5	+ 92.93	+ 138.88	+ 73.28
20	730.5	+ 69.13	+ 136.39	+ 71.52
30	740.5	+ 51.12	+ 132.04	+ 68.11
Nov. 9	750.5	+ 38.57	+ 127.97	+ 64.11
19	760.5	+ 30.43	+ 125.96	+ 60.46
29	770.5	+ 25.08	+ 127.30	+ 57.94
Dec. 9	780.5	+ 20.49	+ 132.67	+ 57.10
19	790.5	+ 14.50	+ 142.05	+ 58.16
29	800.5	+ 5.02	+ 154.76	+ 61.07
1935 Jan. 8	810.5	— 9.77	+ 169.51	+ 65.44
18	820.5	— 31.19	+ 184.48	+ 70.66
28	830.5	— 59.97	+ 197.48	+ 75.85
Feb. 7	840.5	— 96.06	+ 206.13	+ 80.02
17	850.5	— 138.65	+ 208.06	+ 82.12
27	860.5	— 186.09	+ 201.10	+ 81.10
Mar. 9	870.5	— 236.00	+ 183.53	+ 76.04
19	880.5	— 285.39	+ 154.23	+ 66.28

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1935 Mar. 19	2427880.5	−285.39	+154.23	+ 66.28
29	890.5	−330.79	+112.88	+ 51.43
Apr. 8	900.5	−368.61	+ 60.07	+ 31.49
18	910.5	−395.30	− 2.66	+ 6.89
28	920.5	−407.76	− 72.88	− 21.53
May 8	930.5	−403.53	−147.34	− 52.54
18	940.5	−381.14	−222.22	− 84.59
28	950.5	−340.22	−293.35	−115.92
June 7	960.5	−281.56	−356.52	−144.66
17	970.5	−207.20	−407.81	−169.02
27	980.5	−120.24	−443.82	−187.37
July 7	2427990.5	− 24.67	−461.98	−198.39
17	2428000.5	+ 74.84	−460.68	−201.15
27	010.5	+173.30	−439.41	−195.17
Aug. 6	020.5	+265.62	−398.79	−180.50
16	030.5	+346.99	−340.57	−157.67
26	040.5	+413.14	−267.48	−127.69
Sept. 5	050.5	+460.60	−183.16	− 92.02
15	060.5	+486.92	− 91.86	− 52.45
25	070.5	+490.83	+ 1.72	− 11.00
Oct. 5	080.5	+472.30	+ 92.76	+ 30.16
15	090.5	+432.63	+176.60	+ 68.88
25	100.5	+374.30	+249.00	+103.15
Nov. 4	110.5	+300.89	+306.50	+131.22
14	120.5	+216.81	+346.53	+151.74
24	130.5	+126.99	+367.69	+163.86
Dec. 4	140.5	+ 36.60	+369.74	+167.24
14	150.5	− 49.41	+353.66	+162.10
24	160.5	−126.54	+321.52	+149.20
1936 Jan. 3	170.5	−191.14	+276.26	+129.75
13	180.5	−240.59	+221.50	+105.30
23	190.5	−273.40	+161.18	+ 77.62
Feb. 2	200.5	−289.31	+ 99.28	+ 48.58
12	210.5	−289.13	+ 39.55	+ 19.97
22	220.5	−274.71	− 14.77	− 6.57
Mar. 3	230.5	−248.64	− 61.10	− 29.68
13	240.5	−214.04	− 97.71	− 48.38
23	250.5	−174.31	−123.72	− 62.06
Apr. 2	260.5	−132.85	−139.12	− 70.52
12	270.5	− 92.77	−144.74	− 73.98
22	280.5	− 56.74	−142.09	− 72.99
May 2	290.5	− 26.75	−133.21	− 68.43
12	300.5	− 3.99	−120.46	− 61.35
22	310.5	+ 11.20	−106.32	− 52.94
June 1	320.5	+ 19.37	− 93.12	− 44.36
11	330.5	+ 21.82	− 82.84	− 36.69
21	340.5	+ 20.50	− 76.89	− 30.77
July 1	350.5	+ 17.77	− 75.98	− 27.17

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1936 July 1	2428350.5	+ 17.77	— 75.98	— 27.17
11	360.5	+ 16.14	— 80.01	— 26.08
21	370.5	+ 17.98	— 88.09	— 27.34
31	380.5	+ 25.21	— 98.61	— 30.42
Aug. 10	390.5	+ 39.14	—109.43	— 34.50
20	400.5	+ 60.21	—118.08	— 38.55
30	410.5	+ 87.95	—122.03	— 41.43
Sept. 9	420.5	+120.97	—118.96	— 42.05
19	430.5	+157.08	—107.02	— 39.44
29	440.5	+193.43	— 85.02	— 32.88
Oct. 9	450.5	+226.80	— 52.59	— 21.96
19	460.5	+253.80	— 10.26	— 6.69
29	470.5	+271.19	+ 40.57	+ 12.57
Nov. 8	480.5	+276.14	+ 97.65	+ 35.06
18	490.5	+266.47	+158.02	+ 59.66
28	500.5	+240.85	+218.19	+ 84.98
Dec. 8	510.5	+198.95	+274.37	+109.42
18	520.5	+141.55	+322.68	+131.32
28	530.5	+ 70.50	+359.48	+149.00
1937 Jan. 7	540.5	— 11.28	+381.57	+160.95
17	550.5	— 99.92	+386.47	+165.91
27	560.5	—190.89	+372.67	+162.99
Feb. 6	570.5	—279.22	+339.73	+151.76
16	580.5	—359.82	+288.41	+132.29
26	590.5	—427.86	+220.68	+105.22
Mar. 8	600.5	—479.14	+139.57	+ 71.66
18	610.5	—510.33	+ 49.05	+ 33.19
28	620.5	—519.30	— 46.27	— 8.25
Apr. 7	630.5	—505.23	—141.45	— 50.53
17	640.5	—468.66	—231.53	— 91.41
27	650.5	—411.46	—311.88	—128.77
May 7	660.5	—336.67	—378.50	—160.66
17	670.5	—248.29	—428.22	—185.51
27	680.5	—151.00	—458.91	—202.13
June 6	690.5	— 49.86	—469.55	—209.85
16	700.5	+ 50.02	—460.26	—208.48
26	710.5	+143.75	—432.26	—198.38
July 6	720.5	+226.96	—387.78	—180.35
16	730.5	+296.06	—329.85	—155.63
26	740.5	+348.39	—262.13	—125.76
Aug. 5	750.5	+382.37	—188.66	— 92.56
15	760.5	+397.60	—113.62	— 57.93
25	770.5	+394.80	— 40.98	— 23.77
Sept. 4	780.5	+375.76	+ 25.67	+ 8.16
14	790.5	+343.14	+ 83.39	+ 36.36
24	800.5	+300.29	+130.09	+ 59.65
Oct. 4	810.5	+250.91	+164.63	+ 77.31
14	820.5	+198.78	+186.82	+ 89.03

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1937 Oct. 14	2428820.5	+ 198.78	+ 186.82	+ 89.03
24	830.5	+ 147.41	+ 197.41	+ 94.99
Nov. 3	840.5	+ 99.84	+ 197.93	+ 95.73
13	850.5	+ 58.37	+ 190.49	+ 92.13
23	860.5	+ 24.43	+ 177.54	+ 85.30
Dec. 3	870.5	— 1.42	+ 161.63	+ 76.45
13	880.5	— 19.48	+ 145.16	+ 66.77
23	890.5	— 30.79	+ 130.18	+ 57.37
1938 Jan. 2	900.5	— 36.98	+ 118.21	+ 49.13
12	910.5	— 40.05	+ 110.15	+ 42.70
22	920.5	— 42.15	+ 106.27	+ 38.41
Feb. 1	930.5	— 45.38	+ 106.14	+ 36.29
11	940.5	— 51.56	+ 108.75	+ 36.08
21	950.5	— 62.08	+ 112.60	+ 37.22
Mar. 3	960.5	— 77.75	+ 115.82	+ 38.98
13	970.5	— 98.64	+ 116.40	+ 40.44
23	980.5	— 124.09	+ 112.29	+ 40.66
Apr. 2	2428990.5	— 152.71	+ 101.71	+ 38.72
12	2429000.5	— 182.40	+ 83.31	+ 33.84
22	010.5	— 210.58	+ 56.34	+ 25.47
May 2	020.5	— 234.29	+ 20.81	+ 13.36
12	030.5	— 250.53	— 22.40	— 2.35
22	040.5	— 256.53	— 71.57	— 21.16
June 1	050.5	— 249.99	— 124.21	— 42.17
11	060.5	— 229.41	— 177.21	— 64.16
21	070.5	— 194.24	— 227.04	— 85.69
July 1	080.5	— 145.02	— 270.08	— 105.16
11	090.5	— 83.37	— 302.85	— 121.01
21	100.5	— 11.98	— 322.37	— 131.79
31	110.5	+ 65.59	— 326.32	— 136.32
Aug. 10	120.5	+ 145.14	— 313.29	— 133.77
20	130.5	+ 222.10	— 282.92	— 123.73
30	140.5	+ 291.80	— 235.93	— 106.26
Sept. 9	150.5	+ 349.83	— 174.09	— 81.93
19	160.5	+ 392.28	— 100.17	— 51.74
29	170.5	+ 416.04	— 17.79	— 17.11
Oct. 9	180.5	+ 419.00	+ 68.80	+ 20.19
19	190.5	+ 400.19	+ 154.96	+ 58.17
29	200.5	+ 359.96	+ 235.92	+ 94.70
Nov. 8	210.5	+ 299.92	+ 307.07	+ 127.70
18	220.5	+ 222.89	+ 364.30	+ 155.20
28	230.5	+ 132.82	+ 404.23	+ 175.53
Dec. 8	240.5	+ 34.47	+ 424.50	+ 187.43
18	250.5	— 66.82	+ 423.92	+ 190.12
28	260.5	— 165.53	+ 402.56	+ 183.39
1939 Jan. 7	270.5	— 256.34	+ 361.78	+ 167.63
17	280.5	— 334.45	+ 304.09	+ 143.76
27	290.5	— 395.99	+ 232.98	+ 113.20

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1939 Jan. 27	2429290.5	— 395.99	+ 232.98	+ 113.20
Feb. 6	300.5	— 438.17	+ 152.67	+ 77.74
16	310.5	— 459.51	+ 67.76	+ 39.43
26	320.5	— 459.84	— 17.11	+ 0.38
Mar. 8	330.5	— 440.26	— 97.52	— 37.32
18	340.5	— 403.01	— 169.60	— 71.78
28	350.5	— 351.21	— 230.24	— 101.41
Apr. 7	360.5	— 288.66	— 277.25	— 125.02
17	370.5	— 219.49	— 309.41	— 141.84
27	380.5	— 147.90	— 326.53	— 151.59
May 7	390.5	— 77.87	— 329.35	— 154.42
17	400.5	— 12.92	— 319.46	— 150.90
27	410.5	+ 44.13	— 299.13	— 141.97
June 6	420.5	+ 91.28	— 271.10	— 128.82
16	430.5	+ 127.48	— 238.34	— 112.84
26	440.5	+ 152.66	— 203.83	— 95.44
July 6	450.5	+ 167.66	— 170.26	— 78.00
16	460.5	+ 174.09	— 139.86	— 61.70
26	470.5	+ 174.18	— 114.20	— 47.47
Aug. 5	480.5	+ 170.48	— 94.04	— 35.90
15	490.5	+ 165.62	— 79.36	— 27.19
25	500.5	+ 161.98	— 69.32	— 21.18
Sept. 4	510.5	+ 161.44	— 62.44	— 17.35
14	520.5	+ 165.19	— 56.80	— 14.93
24	530.5	+ 173.60	— 50.23	— 12.98
Oct. 4	540.5	+ 186.20	— 40.63	— 10.50
14	550.5	+ 201.67	— 26.15	— 6.56
24	560.5	+ 218.08	— 5.49	— 0.40
Nov. 3	570.5	+ 232.99	+ 22.03	+ 8.51
13	580.5	+ 243.75	+ 56.32	+ 20.36
23	590.5	+ 247.73	+ 96.54	+ 35.02
Dec. 3	600.5	+ 242.54	+ 141.04	+ 52.00
13	610.5	+ 226.30	+ 187.60	+ 70.53
23	620.5	+ 197.81	+ 233.45	+ 89.54
1940 Jan. 2	630.5	+ 156.68	+ 275.50	+ 107.77
12	640.5	+ 103.48	+ 310.58	+ 123.85
22	650.5	+ 39.69	+ 335.64	+ 136.41
Feb. 1	660.5	— 32.24	+ 347.98	+ 144.18
11	670.5	— 109.07	+ 345.52	+ 146.06
21	680.5	— 186.89	+ 326.96	+ 141.27
Mar. 2	690.5	— 261.38	+ 291.97	+ 129.45
12	700.5	— 328.07	+ 241.28	+ 110.64
22	710.5	— 382.71	+ 176.69	+ 85.38
Apr. 1	720.5	— 421.54	+ 101.01	+ 54.70
11	730.5	— 441.63	+ 17.92	+ 20.03
21	740.5	— 441.10	— 68.28	— 16.89
May 1	750.5	— 419.30	— 152.95	— 54.04
11	760.5	— 376.90	— 231.42	— 89.36

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1940 Apr. 1	2429720.5	−421.54	+ 101.01	+ 54.70
11	730.5	−441.63	+ 17.92	+ 20.03
21	740.5	−441.10	− 68.28	− 16.89
May 1	750.5	−419.30	−152.95	− 54.04
11	760.5	−376.90	−231.42	− 89.36
21	770.5	−315.83	−299.31	−120.86
31	780.5	−239.17	−352.84	−146.70
June 10	790.5	−150.93	−389.04	−165.40
20	800.5	− 55.83	−405.97	−175.86
30	810.5	+ 41.04	−402.78	−177.50
July 10	820.5	+ 134.50	−379.80	−170.21
20	830.5	+ 219.62	−338.46	−154.40
30	840.5	+ 291.99	−281.22	−130.99
Aug. 9	850.5	+ 347.97	−211.40	−101.31
19	860.5	+ 384.89	−133.00	− 67.03
29	870.5	+ 401.24	− 50.42	− 30.10
Sept. 8	880.5	+ 396.70	+ 31.79	+ 7.43
18	890.5	+ 372.20	+ 109.18	+ 43.50
28	900.5	+ 329.82	+ 177.76	+ 76.16
Oct. 8	910.5	+ 272.67	+ 234.18	+ 103.73
18	920.5	+ 204.62	+ 275.96	+ 124.91
28	930.5	+ 130.05	+ 301.69	+ 138.81
Nov. 7	940.5	+ 53.51	+ 311.03	+ 145.07
17	950.5	− 20.60	+ 304.70	+ 143.82
27	960.5	− 88.36	+ 284.41	+ 135.67
Dec. 7	970.5	−146.59	+ 252.64	+ 121.64
17	980.5	−193.03	+ 212.40	+ 103.05
27	2429990.5	−226.44	+ 166.97	+ 81.41
1941 Jan. 6	2430000.5	−246.60	+ 119.62	+ 58.27
16	010.5	−254.26	+ 73.36	+ 35.14
26	020.5	−250.97	+ 30.76	+ 13.36
Feb. 5	030.5	−238.89	− 6.26	− 6.00
15	040.5	−220.57	− 36.49	− 22.16
25	050.5	−198.73	− 59.45	− 34.69
Mar. 7	060.5	−175.97	− 75.44	− 43.55
17	070.5	−154.63	− 85.38	− 48.99
27	080.5	−136.53	− 90.77	− 51.58
Apr. 6	090.5	−122.86	− 93.45	− 52.12
16	100.5	−114.04	− 95.47	− 51.54
26	110.5	−109.74	− 98.85	− 50.85
May 6	120.5	−108.85	−105.37	− 50.96
16	130.5	−109.60	−116.32	− 52.68
26	140.5	−109.76	−132.41	− 56.54
June 5	150.5	−106.82	−153.55	− 62.76
15	160.5	− 98.32	−178.86	− 71.20
25	170.5	− 82.07	−206.70	− 81.33
July 5	180.5	− 56.50	−234.75	− 92.33
15	190.5	− 20.80	−260.24	−103.08

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1941 July 15	2430190.5	— 20.80	—260.24	—103.08
25	200.5	+ 24.86	—280.19	—112.30
Aug. 4	210.5	+ 79.32	—291.65	—118.67
14	220.5	+140.44	—292.04	—120.95
24	230.5	+205.25	—279.34	—118.09
Sept. 3	240.5	+270.10	—252.35	—109.37
13	250.5	+331.00	—210.78	— 94.42
23	260.5	+383.81	—155.38	— 73.31
Oct. 3	270.5	+424.61	— 87.87	— 46.57
13	280.5	+449.98	— 10.90	— 15.16
23	290.5	+457.19	+ 72.06	+ 19.61
Nov. 2	300.5	+444.48	+156.96	+ 56.05
12	310.5	+411.25	+239.39	+ 92.31
22	320.5	+358.05	+314.84	+126.39
Dec. 2	330.5	+286.71	+378.96	+156.33
12	340.5	+200.23	+427.93	+180.32
22	350.5	+102.65	+458.64	+196.82
1942 Jan. 1	360.5	— 1.20	+469.03	+204.67
11	370.5	—105.93	+458.16	+203.24
21	380.5	—206.03	+426.38	+192.43
31	390.5	—296.17	+375.33	+172.72
Feb. 10	400.5	—371.62	+307.81	+145.13
20	410.5	—428.54	+227.60	+111.20
Mar. 2	420.5	—464.28	+139.20	+ 72.81
12	430.5	—477.52	+ 47.52	+ 32.11
22	440.5	—468.29	— 42.50	— 8.67
Apr. 1	450.5	—438.00	—126.18	— 47.36
11	460.5	—389.21	—199.41	— 81.98
21	470.5	—325.47	—258.89	—110.86
May 1	480.5	—251.01	—302.30	—132.77
11	490.5	—170.47	—328.39	—146.91
21	500.5	— 88.58	—337.00	—153.00
31	510.5	— 9.84	—329.00	—151.24
June 10	520.5	+ 61.71	—306.23	—142.28
20	530.5	+122.74	—271.26	—127.19
30	540.5	+170.83	—227.28	—107.32
July 10	550.5	+204.55	—177.75	— 84.24
20	560.5	+223.56	—126.28	— 59.64
30	570.5	+228.54	— 76.22	— 35.14
Aug. 9	580.5	+221.09	— 30.53	— 12.26
19	590.5	+203.62	+ 8.48	+ 7.78
29	600.5	+179.02	+ 39.30	+ 24.06
Sept. 8	610.5	+150.41	+ 61.30	+ 36.07
18	620.5	+120.88	+ 74.73	+ 43.70
28	630.5	+ 93.14	+ 80.61	+ 47.27
Oct. 8	640.5	+ 69.35	+ 80.60	+ 47.41
18	650.5	+ 50.89	+ 76.78	+ 44.98
28	660.5	+ 38.31	+ 71.36	+ 41.01

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1942 Oct. 28	2430660.5	+ 38.31	+ 71.36	+ 41.01
Nov. 7	670.5	+ 31.32	+ 66.49	+ 36.55
17	680.5	+ 28.90	+ 64.01	+ 32.58
27	690.5	+ 29.41	+ 65.26	+ 29.88
Dec. 7	700.5	+ 30.84	+ 70.97	+ 29.03
17	710.5	+ 30.98	+ 81.19	+ 30.29
27	720.5	+ 27.67	+ 95.30	+ 33.61
1943 Jan. 6	730.5	+ 19.03	+ 112.03	+ 38.66
16	740.5	+ 3.63	+ 129.60	+ 44.81
26	750.5	— 19.35	+ 145.83	+ 51.22
Feb. 5	760.5	— 49.99	+ 158.33	+ 56.90
15	770.5	— 87.56	+ 164.68	+ 60.77
25	780.5	— 130.55	+ 162.68	+ 61.78
Mar. 7	790.5	— 176.66	+ 150.50	+ 58.99
17	800.5	— 222.98	+ 126.94	+ 51.68
27	810.5	— 266.12	+ 91.56	+ 39.44
Apr. 6	820.5	— 302.48	+ 44.83	+ 22.20
16	830.5	— 328.54	— 11.86	+ 0.33
26	840.5	— 341.15	— 76.19	— 25.39
May 6	850.5	— 337.78	— 145.03	— 53.77
16	860.5	— 316.87	— 214.63	— 83.31
26	870.5	— 277.92	— 280.89	— 112.28
June 5	880.5	— 221.64	— 339.63	— 138.84
15	890.5	— 149.92	— 386.93	— 161.22
25	900.5	— 65.75	— 419.36	— 177.76
July 5	910.5	+ 26.94	— 434.30	— 187.14
15	920.5	+ 123.58	— 430.07	— 188.39
25	930.5	+ 219.17	— 406.10	— 181.02
Aug. 4	940.5	+ 308.68	— 362.93	— 165.04
14	950.5	+ 387.26	— 302.23	— 140.95
24	960.5	+ 450.59	— 226.69	— 109.74
Sept. 3	970.5	+ 495.16	— 139.88	— 72.83
13	980.5	+ 518.44	— 46.05	— 32.00
23	2430990.5	+ 519.06	+ 50.12	+ 10.72
Oct. 3	2431000.5	+ 496.92	+ 143.77	+ 53.16
13	010.5	+ 453.22	+ 230.19	+ 93.16
23	020.5	+ 390.39	+ 305.07	+ 128.66
Nov. 2	030.5	+ 311.95	+ 364.83	+ 157.89
12	040.5	+ 222.28	+ 406.80	+ 179.44
22	050.5	+ 126.34	+ 429.43	+ 192.38
Dec. 2	060.5	+ 29.35	+ 432.37	+ 196.34
12	070.5	— 63.64	+ 416.49	+ 191.49
22	080.5	— 148.01	+ 383.74	+ 178.50
1944 Jan. 1	090.5	— 219.97	+ 337.02	+ 158.58
11	100.5	— 276.69	+ 279.89	+ 133.23
21	110.5	— 316.54	+ 216.31	+ 104.24
31	120.5	— 339.04	+ 150.30	+ 73.48
Feb. 10	130.5	— 344.89	+ 85.70	+ 42.78

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1944 Feb. 10	2431130.5	−344.89	+ 85.70	+ 42.78
20	140.5	−335.77	+ 25.84	+ 13.81
Mar. 1	150.5	−314.17	− 26.58	− 12.02
11	160.5	−283.14	− 69.67	− 33.65
21	170.5	−246.05	−102.41	− 50.41
31	180.5	−206.27	−124.67	− 62.04
Apr. 10	190.5	−166.98	−137.12	− 68.68
20	200.5	−130.87	−141.15	− 70.84
30	210.5	−100.02	−138.70	− 69.31
May 10	220.5	− 75.74	−132.07	− 65.15
20	230.5	− 58.44	−123.68	− 59.48
30	240.5	− 47.70	−115.86	− 53.47
June 9	250.5	− 42.30	−110.58	− 48.18
19	260.5	− 40.39	−109.28	− 44.47
29	270.5	− 39.66	−112.71	− 42.92
July 9	280.5	− 37.63	−120.82	− 43.75
19	290.5	− 31.94	−132.78	− 46.82
29	300.5	− 20.61	−147.00	− 51.62
Aug. 8	310.5	− 2.33	−161.38	− 57.34
18	320.5	+ 23.41	−173.43	− 62.96
28	330.5	+ 56.22	−180.59	− 67.34
Sept. 7	340.5	+ 94.74	−180.48	− 69.35
17	350.5	+136.82	−171.18	− 68.00
27	360.5	+179.65	−151.38	− 62.54
Oct. 7	370.5	+219.94	−120.63	− 52.51
17	380.5	+254.28	− 79.32	− 37.86
27	390.5	+279.33	− 28.78	− 18.91
Nov. 6	400.5	+292.16	+ 28.84	+ 3.60
16	410.5	+290.44	+ 90.64	+ 28.62
26	420.5	+272.68	+153.17	+ 54.75
Dec. 6	430.5	+238.39	+212.63	+ 80.43
16	440.5	+188.16	+265.14	+103.96
26	450.5	+123.70	+306.95	+123.67
1945 Jan. 5	460.5	+ 47.76	+334.79	+138.00
15	470.5	− 35.94	+346.06	+145.64
25	480.5	−122.96	+339.08	+145.65
Feb. 4	490.5	−208.37	+313.26	+137.52
14	500.5	−287.13	+269.19	+121.26
24	510.5	−354.38	+208.64	+ 97.41
Mar. 6	520.5	−405.85	+134.50	+ 67.02
16	530.5	−438.13	+ 50.61	+ 31.61
26	540.5	−448.94	− 38.54	− 6.94
Apr. 5	550.5	−437.34	−128.06	− 46.54
15	560.5	−403.71	−213.04	− 84.98
25	570.5	−349.78	−288.85	−120.12
May 5	580.5	−278.48	−351.44	−150.04
15	590.5	−193.70	−397.61	−173.12
25	600.5	−100.03	−425.14	−188.16

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1945 May 25	2431600.5	−100.03	−425.14	− 188.16
June 4	610.5	− 2.50	−432.95	−194.46
14	620.5	+ 93.81	−421.10	−191.80
24	630.5	+184.00	−390.72	−180.50
July 4	640.5	+263.69	−343.97	−161.32
14	650.5	+329.22	−283.86	−135.49
24	660.5	+377.88	−214.03	−104.55
Aug. 3	670.5	+408.03	−138.50	− 70.30
13	680.5	+419.19	− 61.46	− 34.64
23	690.5	+412.03	+ 13.03	+ 0.51
Sept. 2	700.5	+388.29	+ 81.35	+ 33.36
12	710.5	+350.62	+140.47	+ 62.36
22	720.5	+302.36	+188.22	+ 86.33
Oct. 2	730.5	+247.24	+223.34	+104.46
12	740.5	+189.09	+245.59	+116.43
22	750.5	+131.55	+255.63	+122.36
Nov. 1	760.5	+ 77.74	+254.93	+122.77
11	770.5	+ 30.09	+245.59	+118.52
21	780.5	− 9.81	+230.07	+110.72
Dec. 1	790.5	− 41.30	+210.95	+100.58
11	800.5	− 64.56	+190.68	+ 89.33
21	810.5	− 80.54	+171.41	+ 78.08
31	820.5	− 90.79	+154.75	+ 67.78
1946 Jan. 10	830.5	− 97.26	+141.70	+ 59.10
20	840.5	−102.08	+132.63	+ 52.44
30	850.5	−107.36	+127.22	+ 47.87
Feb. 9	860.5	−114.96	+124.54	+ 45.17
19	870.5	−126.31	+123.18	+ 43.84
Mar. 1	880.5	−142.26	+121.34	+ 43.15
11	890.5	−162.98	+117.01	+ 42.25
21	900.5	−187.90	+108.20	+ 40.20
31	910.5	−215.67	+ 93.12	+ 36.07
Apr. 10	920.5	−244.29	+ 70.39	+ 29.10
20	930.5	−271.20	+ 39.22	+ 18.71
30	940.5	−293.50	− 0.42	+ 4.64
May 10	950.5	−308.18	− 47.74	− 13.01
20	960.5	−312.42	−101.05	− 33.74
30	970.5	−303.91	−157.91	− 56.69
June 9	980.5	−281.04	−215.22	− 80.66
19	2431990.5	−243.17	−269.48	−104.20
29	2432000.5	−190.74	−317.00	−125.72
July 9	010.5	−125.29	−354.26	−143.62
19	020.5	− 49.43	−378.16	−156.43
29	030.5	+ 33.34	−386.27	−162.91
Aug. 8	040.5	+118.84	−377.06	−162.17
18	050.5	+202.48	−350.02	−153.76
28	060.5	+279.56	−305.70	−137.65
Sept. 7	070.5	+345.54	−245.77	−114.36

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1946 Sept. 7	2432070.5	+ 345.54	— 245.77	— 114.36
17	080.5	+ 396.42	— 172.86	— 84.83
27	090.5	+ 428.92	— 90.50	— 50.43
Oct. 7	100.5	+ 440.75	— 2.86	— 12.90
17	110.5	+ 430.76	+ 85.44	+ 25.80
27	120.5	+ 399.07	+ 169.62	+ 63.56
Nov. 6	130.5	+ 347.08	+ 245.03	+ 98.25
16	140.5	+ 277.46	+ 307.48	+ 127.90
26	150.5	+ 193.94	+ 353.47	+ 150.79
Dec. 6	160.5	+ 101.17	+ 380.48	+ 165.57
16	170.5	+ 4.38	+ 387.15	+ 171.43
26	180.5	— 90.98	+ 373.37	+ 168.05
1947 Jan. 5	190.5	— 179.57	+ 340.29	+ 155.74
15	200.5	— 256.58	+ 290.27	+ 135.34
25	210.5	— 318.04	+ 226.65	+ 108.20
Feb. 4	220.5	— 361.08	+ 153.50	+ 76.03
14	230.5	— 384.06	+ 75.32	+ 40.85
24	240.5	— 386.71	— 3.27	+ 4.73
Mar. 6	250.5	— 369.97	— 77.91	— 30.26
16	260.5	— 335.98	— 144.72	— 62.22
26	270.5	— 287.76	— 200.56	— 89.58
Apr. 5	280.5	— 229.00	— 243.20	— 111.10
15	290.5	— 163.82	— 271.37	— 126.02
25	300.5	— 96.36	— 284.82	— 134.02
May 5	310.5	— 30.60	— 284.27	— 135.22
15	320.5	+ 29.93	— 271.27	— 130.20
25	330.5	+ 82.38	— 248.06	— 119.86
June 4	340.5	+ 124.72	— 217.38	— 105.41
14	350.5	+ 155.87	— 182.23	— 88.21
24	360.5	+ 175.72	— 145.60	— 69.71
July 4	370.5	+ 185.07	— 110.26	— 51.30
14	380.5	+ 185.56	— 78.50	— 34.19
24	390.5	+ 179.43	— 51.97	— 19.36
Aug. 3	400.5	+ 169.28	— 31.52	— 7.41
13	410.5	+ 157.83	— 17.16	+ 1.40
23	420.5	+ 147.54	— 8.14	+ 7.22
Sept. 2	430.5	+ 140.44	— 3.00	+ 10.52
12	440.5	+ 137.86	+ .19	+ 12.10
22	450.5	+ 140.31	+ 3.63	+ 12.88
Oct. 2	460.5	+ 147.42	+ 9.48	+ 13.89
12	470.5	+ 158.02	+ 19.68	+ 16.08
22	480.5	+ 170.24	+ 35.67	+ 20.29
Nov. 1	490.5	+ 181.70	+ 58.24	+ 27.06
11	500.5	+ 189.78	+ 87.43	+ 36.66
21	510.5	+ 191.80	+ 122.51	+ 49.02
Dec. 1	520.5	+ 185.35	+ 162.00	+ 63.70
11	530.5	+ 168.47	+ 203.72	+ 79.98
21	540.5	+ 139.86	+ 245.01	+ 96.82

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1947 Dec. 21	2432540.5	+ 139.86	+ 245.01	+ 96.82
31	550.5	+ 99.02	+ 282.82	+ 112.99
1948 Jan. 10	560.5	+ 46.42	+ 314.00	+ 127.16
20	570.5	— 16.59	+ 335.48	+ 137.95
30	580.5	— 87.67	+ 344.54	+ 144.06
Feb. 9	590.5	— 163.66	+ 339.03	+ 144.40
19	600.5	— 240.70	+ 317.58	+ 138.17
29	610.5	— 314.53	+ 279.74	+ 124.94
Mar. 10	620.5	— 380.67	+ 226.16	+ 104.72
20	630.5	— 434.83	+ 158.53	+ 78.02
30	640.5	— 473.18	+ 79.59	+ 45.80
Apr. 9	650.5	— 492.68	— 7.06	+ 9.47
19	660.5	— 491.34	— 97.15	— 29.22
29	670.5	— 468.36	— 186.02	— 68.30
May 9	680.5	— 424.29	— 268.97	— 105.67
19	690.5	— 360.90	— 341.54	— 139.32
29	700.5	— 281.20	— 399.84	— 167.37
June 8	710.5	— 189.12	— 440.77	— 188.28
18	720.5	— 89.32	— 462.24	— 200.89
28	730.5	+ 13.08	— 463.24	— 204.55
July 8	740.5	+ 112.90	— 443.95	— 199.07
18	750.5	+ 205.12	— 405.64	— 184.81
28	760.5	+ 285.22	— 350.65	— 162.61
Aug. 7	770.5	+ 349.40	— 282.18	— 133.75
17	780.5	+ 394.86	— 204.14	— 99.86
27	790.5	+ 419.88	— 120.88	— 62.84
Sept. 6	800.5	+ 423.96	— 36.96	— 24.75
16	810.5	+ 407.84	+ 43.20	+ 12.36
26	820.5	+ 373.42	+ 115.52	+ 46.53
Oct. 6	830.5	+ 323.61	+ 176.56	+ 76.04
16	840.5	+ 262.17	+ 223.76	+ 99.54
26	850.5	+ 193.36	+ 255.53	+ 116.08
Nov. 5	860.5	+ 121.66	+ 271.40	+ 125.24
15	870.5	+ 51.44	+ 271.96	+ 127.07
25	880.5	— 13.40	+ 258.76	+ 122.14
Dec. 5	890.5	— 69.63	+ 234.15	+ 111.37
15	900.5	— 114.94	+ 201.04	+ 96.05
25	910.5	— 148.04	+ 162.64	+ 77.63
1949 Jan. 4	920.5	— 168.61	+ 122.13	+ 57.64
14	930.5	— 177.35	+ 82.51	+ 37.57
24	940.5	— 175.72	+ 46.30	+ 18.74
Feb. 3	950.5	— 165.85	+ 15.44	+ 2.22
13	960.5	— 150.25	— 8.89	— 11.21
23	970.5	— 131.59	— 26.20	— 21.15
Mar. 5	980.5	— 112.48	— 36.79	— 27.52
15	2432990.5	— 95.26	— 41.60	— 30.58
25	2433000.5	— 81.76	— 42.12	— 30.92
Apr. 4	010.5	— 73.16	— 40.26	— 29.33

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1949 Apr. 4	2433010.5	— 73.16	— 40.26	— 29.33
14	020.5	— 69.92	— 38.09	— 26.77
24	030.5	— 71.70	— 37.70	— 24.25
May 4	040.5	— 77.40	— 40.93	— 22.74
14	050.5	— 85.22	— 49.19	— 23.07
24	060.5	— 92.88	— 63.24	— 25.80
June 3	070.5	— 97.81	— 83.11	— 31.21
13	080.5	— 97.40	—108.00	— 39.19
23	090.5	— 89.37	—136.32	— 49.27
July 3	100.5	— 71.96	—165.80	— 60.62
13	110.5	— 44.20	—193.68	— 72.15
23	120.5	— 6.06	—216.93	— 82.57
Aug. 2	130.5	+ 41.45	—232.54	— 90.54
12	140.5	+ 96.37	—237.80	— 94.77
22	150.5	+ 155.83	—230.58	— 94.18
Sept. 1	160.5	+ 216.30	—209.50	— 87.95
11	170.5	+ 273.81	—174.12	— 75.67
21	180.5	+ 324.26	—124.98	— 57.31
Oct. 1	190.5	+ 363.68	— 63.66	— 33.34
11	200.5	+ 388.56	+ 7.36	— 4.62
21	210.5	+ 396.07	+ 84.74	+ 27.56
31	220.5	+ 384.35	+ 164.54	+ 61.62
Nov. 10	230.5	+ 352.58	+ 242.41	+ 95.70
20	240.5	+ 301.19	+ 313.86	+ 127.84
30	250.5	+ 231.82	+ 374.55	+ 156.08
Dec. 10	260.5	+ 147.32	+ 420.60	+ 178.60
20	270.5	+ 51.59	+ 448.83	+ 193.82
30	280.5	— 50.64	+ 457.05	+ 200.56
1950 Jan. 9	290.5	—154.04	+ 444.19	+ 198.11
19	300.5	—253.11	+ 410.49	+ 186.33
29	310.5	—342.51	+ 357.42	+ 165.62
Feb. 8	320.5	—417.42	+ 287.67	+ 136.97
18	330.5	—473.90	+ 204.93	+ 101.84
28	340.5	—509.16	+ 113.65	+ 62.10
Mar. 10	350.5	—521.73	+ 18.70	+ 19.88
20	360.5	—511.49	— 74.94	— 22.59
30	370.5	—479.70	—162.54	— 63.11
Apr. 9	380.5	—428.81	—239.89	— 99.67
19	390.5	—362.27	—303.58	—130.55
29	400.5	—284.27	—351.14	—154.46
May 9	410.5	—199.42	—381.18	—170.53
19	420.5	—112.45	—393.39	—178.42
29	430.5	— 27.94	—388.52	—178.26
June 8	440.5	+ 49.99	—368.25	—170.64
18	450.5	+ 117.92	—335.09	—156.58
28	460.5	+ 173.26	—292.12	—137.38
July 8	470.5	+ 214.45	—242.77	—114.60
18	480.5	+ 240.98	—190.61	— 89.88

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1950 July 18	2433480.5	+ 240.98	— 190.61	— 89.88
28	490.5	+ 253.37	— 139.02	— 64.86
Aug. 7	500.5	+ 253.11	— 91.01	— 41.07
17	510.5	+ 242.44	— 48.93	— 19.77
27	520.5	+ 224.17	— 14.39	— 1.90
Sept. 6	530.5	+ 201.34	+ 11.89	+ 11.99
16	540.5	+ 176.99	+ 30.06	+ 21.74
26	550.5	+ 153.84	+ 41.04	+ 27.62
Oct. 6	560.5	+ 134.01	+ 46.43	+ 30.21
16	570.5	+ 118.93	+ 48.23	+ 30.36
26	580.5	+ 109.16	+ 48.62	+ 29.06
Nov. 5	590.5	+ 104.45	+ 49.72	+ 27.35
15	600.5	+ 103.77	+ 53.35	+ 26.18
25	610.5	+ 105.50	+ 60.86	+ 26.35
Dec. 5	620.5	+ 107.61	+ 72.96	+ 28.42
15	630.5	+ 107.86	+ 89.71	+ 32.65
25	640.5	+ 104.05	+ 110.46	+ 38.98
1951 Jan. 4	650.5	+ 94.26	+ 133.92	+ 47.08
14	660.5	+ 76.99	+ 158.27	+ 56.29
24	670.5	+ 51.38	+ 181.26	+ 65.77
Feb. 3	680.5	+ 17.33	+ 200.42	+ 74.48
13	690.5	— 24.58	+ 213.26	+ 81.32
23	700.5	— 72.75	+ 217.43	+ 85.19
Mar. 5	710.5	— 124.93	+ 211.00	+ 85.09
15	720.5	— 178.18	+ 192.62	+ 80.25
25	730.5	— 229.06	+ 161.71	+ 70.19
Apr. 4	740.5	— 273.88	+ 118.61	+ 54.79
14	750.5	— 309.00	+ 64.57	+ 34.34
24	760.5	— 331.08	+ 1.81	+ 9.61
May 4	770.5	— 337.44	— 66.62	— 18.30
14	780.5	— 326.29	— 137.01	— 47.90
24	790.5	— 296.90	— 205.27	— 77.47
June 3	800.5	— 249.75	— 267.16	— 105.16
13	810.5	— 186.53	— 318.68	— 129.16
23	820.5	— 110.06	— 356.28	— 147.78
July 3	830.5	— 24.08	— 377.17	— 159.62
13	840.5	+ 66.94	— 379.50	— 163.64
23	850.5	+ 158.07	— 362.49	— 159.28
Aug. 2	860.5	+ 244.29	— 326.48	— 146.44
12	870.5	+ 320.74	— 272.96	— 125.53
22	880.5	+ 383.06	— 204.43	— 97.48
Sept. 1	890.5	+ 427.60	— 124.30	— 63.62
11	900.5	+ 451.71	— 36.69	— 25.66
21	910.5	+ 453.86	+ 53.80	+ 14.42
Oct. 1	920.5	+ 433.80	+ 142.40	+ 54.48
11	930.5	+ 392.52	+ 224.38	+ 92.36
21	940.5	+ 332.30	+ 295.46	+ 126.03
31	950.5	+ 256.49	+ 351.96	+ 153.67

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1951 Oct. 31	2433950.5	+ 256.49	+ 351.96	+ 153.67
Nov. 10	960.5	+ 169.36	+ 391.15	+ 173.86
20	970.5	+ 75.79	+ 411.35	+ 185.62
30	980.5	— 19.09	+ 412.10	+ 188.51
Dec. 10	2433990.5	— 110.21	+ 394.14	+ 182.64
20	2434000.5	— 192.92	+ 359.30	+ 168.66
30	010.5	— 263.36	+ 310.39	+ 147.69
1952 Jan. 9	020.5	— 318.62	+ 250.91	+ 121.22
19	030.5	— 356.93	+ 184.77	+ 91.02
29	040.5	— 377.72	+ 116.01	+ 58.93
Feb. 8	050.5	— 381.56	+ 48.47	+ 26.81
18	060.5	— 370.04	— 14.43	— 3.65
28	070.5	— 345.58	— 69.92	— 31.01
Mar. 9	080.5	— 311.19	— 116.01	— 54.16
19	090.5	— 270.19	— 151.59	— 72.38
29	100.5	— 225.99	— 176.40	— 85.37
Apr. 8	110.5	— 181.80	— 191.03	— 93.21
18	120.5	— 140.37	— 196.78	— 96.37
28	130.5	— 103.89	— 195.51	— 95.62
May 8	140.5	— 73.73	— 189.47	— 91.94
18	150.5	— 50.46	— 181.07	— 86.47
28	160.5	— 33.75	— 172.62	— 80.37
June 7	170.5	— 22.50	— 166.13	— 74.70
17	180.5	— 14.95	— 163.09	— 70.33
27	190.5	— 8.86	— 164.29	— 67.88
July 7	200.5	— 1.80	— 169.79	— 67.59
17	210.5	+ 8.56	— 178.80	— 69.36
27	220.5	+ 24.21	— 189.82	— 72.71
Aug. 6	230.5	+ 46.50	— 200.79	— 76.87
16	240.5	+ 75.97	— 209.24	— 80.82
26	250.5	+ 112.26	— 212.63	— 83.44
Sept. 5	260.5	+ 154.07	— 208.55	— 83.60
15	270.5	+ 199.26	— 195.04	— 80.31
25	280.5	+ 245.00	— 170.77	— 72.77
Oct. 5	290.5	+ 287.99	— 135.20	— 60.54
15	300.5	+ 324.76	— 88.70	— 43.52
25	310.5	+ 351.90	— 32.54	— 22.03
Nov. 4	320.5	+ 366.37	+ 31.13	+ 3.22
14	330.5	+ 365.72	+ 99.41	+ 31.15
24	340.5	+ 348.34	+ 168.82	+ 60.38
Dec. 4	350.5	+ 313.62	+ 235.50	+ 89.31
14	360.5	+ 262.01	+ 295.46	+ 116.22
24	370.5	+ 195.12	+ 344.84	+ 139.36
1953 Jan. 3	380.5	+ 115.63	+ 380.21	+ 157.14
13	390.5	+ 27.21	+ 398.79	+ 168.17
23	400.5	— 65.71	+ 398.71	+ 171.41
Feb. 2	410.5	— 158.17	+ 379.17	+ 166.26
12	420.5	— 245.04	+ 340.56	+ 152.66

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1953 Feb. 12	2434420.5	−245.04	+340.56	+152.66
22	430.5	−321.31	+284.48	+131.05
Mar. 4	440.5	−382.52	+213.67	+102.42
14	450.5	−425.06	+131.84	+68.21
24	460.5	−446.40	+43.41	+30.27
Apr. 3	470.5	−445.32	−46.74	−9.32
13	480.5	−421.96	−133.69	−48.36
23	490.5	−377.81	−212.70	−84.68
May 3	500.5	−315.57	−279.63	−116.31
13	510.5	−238.96	−331.10	−141.57
23	520.5	−152.42	−364.73	−159.20
June 2	530.5	−60.89	−379.23	−168.38
12	540.5	+30.59	−374.47	−168.82
22	550.5	+117.12	−351.39	−160.75
July 2	560.5	+194.29	−311.97	−144.86
12	570.5	+258.36	−259.04	−122.28
22	580.5	+306.50	−196.11	−94.50
Aug. 1	590.5	+336.97	−127.12	−63.24
11	600.5	+349.14	−56.17	−30.39
21	610.5	+343.51	+12.73	+2.16
31	620.5	+321.70	+75.93	+32.63
Sept. 10	630.5	+286.23	+130.41	+59.48
20	640.5	+240.33	+173.92	+81.47
30	650.5	+187.67	+205.16	+97.80
Oct. 10	660.5	+132.03	+223.78	+108.10
20	670.5	+77.03	+230.40	+112.45
30	680.5	+25.82	+226.42	+111.35
Nov. 9	690.5	−19.13	+213.89	+105.63
19	700.5	−56.21	+195.26	+96.38
29	710.5	−84.67	+173.10	+84.80
Dec. 9	720.5	−104.64	+149.90	+72.13
19	730.5	−117.05	+127.82	+59.50
29	740.5	−123.41	+108.53	+47.87
1954 Jan. 8	750.5	−125.67	+93.10	+37.95
18	760.5	−125.97	+81.96	+30.15
28	770.5	−126.44	+74.84	+24.60
Feb. 7	780.5	−128.99	+70.87	+21.07
17	790.5	−135.12	+68.69	+19.11
27	800.5	−145.76	+66.50	+18.01
Mar. 9	810.5	−161.15	+62.34	+16.93
19	820.5	−180.82	+54.20	+14.92
29	830.5	−203.49	+40.24	+11.08
Apr. 8	840.5	−227.27	+19.06	+4.61
18	850.5	−249.63	−10.20	−5.08
28	860.5	−267.72	−47.65	−18.30
May 8	870.5	−278.57	−92.56	−34.95
18	880.5	−279.32	−143.35	−54.60
28	890.5	−267.64	−197.64	−76.42

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1954 May 28	2434890.5	−267.64	−197.64	− 76.42
June 7	900.5	−241.84	−252.37	− 99.22
17	910.5	−201.23	−304.06	−121.57
27	920.5	−146.15	−349.04	−141.90
July 7	930.5	− 78.08	−383.75	−158.59
17	940.5	+ 0.46	−405.03	−170.16
27	950.5	+ 85.95	−410.39	−175.35
Aug. 6	960.5	+174.28	−398.19	−173.22
16	970.5	+260.80	−367.83	−163.27
26	980.5	+340.76	−319.79	−145.46
Sept. 5	2434990.5	+409.55	−255.64	−120.26
15	2435000.5	+463.04	−177.98	− 88.58
25	010.5	+497.81	− 90.28	− 51.79
Oct. 5	020.5	+511.42	+ 3.27	− 11.59
15	030.5	+502.56	+ 98.02	+ 30.02
25	040.5	+471.19	+189.12	+ 70.91
Nov. 4	050.5	+418.57	+271.82	+108.93
14	060.5	+347.22	+341.78	+141.98
24	070.5	+260.79	+395.31	+168.40
Dec. 4	080.5	+163.87	+429.72	+186.64
14	090.5	+ 61.68	+443.40	+195.82
24	100.5	− 40.26	+436.05	+195.54
1955 Jan. 3	110.5	−136.53	+408.61	+185.99
13	120.5	−222.16	+363.24	+167.93
23	130.5	−292.98	+303.15	+142.64
Feb. 2	140.5	−345.88	+232.30	+111.79
12	150.5	−379.00	+155.14	+ 77.33
22	160.5	−391.78	+ 76.26	+ 41.35
Mar. 4	170.5	−384.96	+ 0.11	+ 5.92
14	180.5	−360.43	− 69.38	− 27.04
24	190.5	−321.02	−128.94	− 55.88
Apr. 3	200.5	−270.30	−176.18	− 79.34
13	210.5	−212.24	−209.69	− 96.55
23	220.5	−150.93	−229.05	−107.13
May 3	230.5	− 90.34	−234.78	−111.12
13	240.5	− 33.98	−228.30	−109.02
23	250.5	+ 15.24	−211.71	−101.68
June 2	260.5	+ 55.22	−187.61	− 90.22
12	270.5	+ 84.78	−158.93	− 75.97
22	280.5	+103.73	−128.59	− 60.35
July 2	290.5	+112.76	− 99.32	− 44.71
12	300.5	+113.42	− 73.42	− 30.28
22	310.5	+107.87	− 52.53	− 18.01
Aug. 1	320.5	+ 98.67	− 37.55	− 8.55
11	330.5	+ 88.48	− 28.53	− 2.14
21	340.5	+ 79.81	− 24.74	+ 1.32
31	350.5	+ 74.68	− 24.75	+ 2.32
Sept. 10	360.5	+ 74.47	− 26.62	+ 1.61

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1955 Sept. 10	2435360.5	+ 74.47	— 26.62	+ 1.61
20	370.5	+ 79.73	— 28.16	+ 0.15
30	380.5	+ 90.15	— 27.12	— 1.05
Oct. 10	390.5	+ 104.59	— 21.54	— 0.99
20	400.5	+ 121.19	— 9.90	+ 1.18
30	410.5	+ 137.58	+ 8.70	+ 6.05
Nov. 9	420.5	+ 151.09	+ 34.39	+ 13.93
19	430.5	+ 159.00	+ 66.52	+ 24.78
29	440.5	+ 158.81	+ 103.69	+ 38.20
Dec. 9	450.5	+ 148.44	+ 143.78	+ 53.50
19	460.5	+ 126.49	+ 184.15	+ 69.67
29	470.5	+ 92.34	+ 221.80	+ 85.49
1956 Jan. 8	480.5	+ 46.28	+ 253.52	+ 99.63
18	490.5	— 10.42	+ 276.25	+ 110.70
28	500.5	— 75.55	+ 287.17	+ 117.39
Feb. 7	510.5	— 146.06	+ 284.06	+ 118.57
17	520.5	— 218.18	+ 265.42	+ 113.40
27	530.5	— 287.67	+ 230.72	+ 101.39
Mar. 8	540.5	— 350.08	+ 180.44	+ 82.52
18	550.5	— 401.08	+ 116.17	+ 57.23
28	560.5	— 436.81	+ 40.54	+ 26.44
Apr. 7	570.5	— 454.12	— 42.94	— 8.49
17	580.5	— 450.93	— 130.04	— 45.85
27	590.5	— 426.32	— 216.16	— 83.67
May 7	600.5	— 380.70	— 296.56	— 119.87
17	610.5	— 315.76	— 366.76	— 152.40
27	620.5	— 234.40	— 422.78	— 179.39
June 6	630.5	— 140.49	— 461.44	— 199.24
16	640.5	— 38.69	— 480.52	— 210.76
26	650.5	+ 65.91	— 478.94	— 213.23
July 6	660.5	+ 168.06	— 456.74	— 206.44
16	670.5	+ 262.67	— 415.12	— 190.69
26	680.5	+ 345.11	— 356.34	— 166.78
Aug. 5	690.5	+ 411.46	— 283.56	— 135.98
15	700.5	+ 458.77	— 200.65	— 99.89
25	710.5	+ 485.16	— 112.00	— 60.42
Sept. 4	720.5	+ 489.99	— 22.19	— 19.65
14	730.5	+ 473.84	+ 64.23	+ 20.34
24	740.5	+ 438.49	+ 143.09	+ 57.55
Oct. 4	750.5	+ 386.75	+ 210.81	+ 90.19
14	760.5	+ 322.32	+ 264.64	+ 116.82
24	770.5	+ 249.44	+ 302.83	+ 136.44
Nov. 3	780.5	+ 172.63	+ 324.70	+ 148.52
13	790.5	+ 96.31	+ 330.68	+ 153.04
23	800.5	+ 24.52	+ 322.16	+ 150.47
Dec. 3	810.5	— 39.37	+ 301.37	+ 141.70
13	820.5	— 92.88	+ 271.13	+ 127.93
23	830.5	— 134.51	+ 234.57	+ 110.60

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1956 Dec. 23	2435830.5	−134.51	+234.57	+110.60
1957 Jan. 2	5840.5	−163.77	+194.88	+ 91.23
12	850.5	−181.09	+155.08	+ 71.30
22	860.5	−187.96	+117.73	+ 52.14
Feb. 1	870.5	−186.17	+ 84.85	+ 34.87
11	880.5	−178.17	+ 57.73	+ 20.29
21	890.5	−166.55	+ 36.95	+ 8.88
Mar. 3	900.5	−153.86	+ 22.35	+ 0.75
13	910.5	−142.38	+ 13.08	− 4.30
23	920.5	−133.95	+ 7.75	− 6.80
Apr. 2	930.5	−129.80	+ 4.58	− 7.50
12	940.5	−130.40	+ 1.53	− 7.32
22	950.5	−135.46	− 3.41	− 7.24
May 2	960.5	−143.94	− 12.06	− 8.20
12	970.5	−154.09	− 25.80	− 11.02
22	980.5	−163.66	− 45.43	− 16.28
June 1	2435990.5	−170.11	− 70.98	− 24.25
11	2436000.5	−170.82	−101.69	− 34.84
21	010.5	−163.48	−135.99	− 47.58
July 1	020.5	−146.25	−171.63	− 61.67
11	030.5	−118.12	−205.82	− 76.00
21	040.5	− 78.97	−235.50	− 89.29
31	050.5	− 29.72	−257.60	−100.18
Aug. 10	060.5	+ 27.72	−269.34	−107.33
20	070.5	+ 90.56	−268.45	−109.62
30	080.5	+155.26	−253.43	−106.19
Sept. 9	090.5	+217.85	−223.71	− 96.56
19	100.5	+274.18	−179.68	− 80.64
29	110.5	+320.21	−122.80	− 58.84
Oct. 9	120.5	+352.31	− 55.43	− 31.97
19	130.5	+367.52	+ 19.19	− 1.26
29	140.5	+363.78	+ 97.18	+ 31.72
Nov. 8	150.5	+340.13	+174.22	+ 65.16
18	160.5	+296.80	+245.81	+ 97.09
28	170.5	+235.24	+307.59	+125.56
Dec. 8	180.5	+158.14	+355.59	+148.69
18	190.5	+ 69.23	+386.53	+164.89
28	200.5	− 26.83	+398.08	+172.92
1958 Jan. 7	210.5	−124.84	+389.02	+172.00
17	220.5	−219.32	+359.41	+161.92
27	230.5	−304.89	+310.58	+143.00
Feb. 6	240.5	−376.71	+245.06	+116.17
16	250.5	−430.73	+166.44	+ 82.84
26	260.5	−464.05	+ 79.06	+ 44.83
Mar. 8	270.5	−475.05	− 12.25	+ 4.24
18	280.5	−463.50	−102.51	− 36.72
28	290.5	−430.51	−186.97	− 75.82
Apr. 7	300.5	−378.44	−261.36	−111.05

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1958 Apr. 7	2436300.5	— 378.44	— 261.36	— 111.05
17	310.5	— 310.66	— 322.19	— 140.65
27	320.5	— 231.32	— 366.90	— 163.28
May 7	330.5	— 145.01	— 393.99	— 178.05
17	340.5	— 56.51	— 403.06	— 184.54
27	350.5	+ 29.56	— 394.76	— 182.86
June 6	360.5	+ 108.99	— 370.74	— 173.58
16	370.5	+ 178.23	— 333.42	— 157.65
26	380.5	+ 234.59	— 285.89	— 136.40
July 6	390.5	+ 276.37	— 231.61	— 111.37
16	400.5	+ 302.94	— 174.17	— 84.24
26	410.5	+ 314.70	— 117.06	— 56.66
Aug. 5	420.5	+ 313.03	— 63.38	— 30.20
15	430.5	+ 300.12	— 15.62	— 6.19
25	440.5	+ 278.75	+ 24.48	+ 14.36
Sept. 4	450.5	+ 251.98	+ 56.03	+ 30.84
14	460.5	+ 222.88	+ 79.04	+ 43.03
24	470.5	+ 194.27	+ 94.32	+ 51.11
Oct. 4	480.5	+ 168.39	+ 103.35	+ 55.63
14	490.5	+ 146.80	+ 108.06	+ 57.38
24	500.5	+ 130.22	+ 110.59	+ 57.33
Nov. 3	510.5	+ 118.53	+ 113.07	+ 56.50
13	520.5	+ 110.86	+ 117.32	+ 55.86
23	530.5	+ 105.71	+ 124.75	+ 56.21
Dec. 3	540.5	+ 101.14	+ 136.14	+ 58.13
13	550.5	+ 94.99	+ 151.60	+ 61.92
23	560.5	+ 85.12	+ 170.59	+ 67.56
1959 Jan. 2	570.5	+ 69.62	+ 191.89	+ 74.75
12	580.5	+ 47.01	+ 213.74	+ 82.88
22	590.5	+ 16.42	+ 233.96	+ 91.11
Feb. 1	600.5	— 22.34	+ 250.14	+ 98.46
11	610.5	— 68.59	+ 259.79	+ 103.83
21	620.5	— 120.90	+ 260.60	+ 106.14
Mar. 3	630.5	— 177.01	+ 250.61	+ 104.41
13	640.5	— 234.02	+ 228.45	+ 97.84
23	650.5	— 288.48	+ 193.50	+ 85.94
Apr. 2	660.5	— 336.72	+ 146.04	+ 68.57
12	670.5	— 375.03	+ 87.29	+ 46.02
22	680.5	— 400.04	+ 19.41	+ 19.00
May 2	690.5	— 408.96	— 54.58	— 11.38
12	700.5	— 399.89	— 130.98	— 43.64
22	710.5	— 371.98	— 205.67	— 76.06
June 1	720.5	— 325.60	— 274.37	— 106.78
11	730.5	— 262.32	— 332.98	— 133.94
21	740.5	— 184.84	— 377.84	— 155.84
July 1	750.5	— 96.86	— 406.02	— 170.99
11	760.5	— 2.80	— 415.52	— 178.30
21	770.5	+ 92.42	— 405.38	— 177.13

Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1959 July 21	2436770.5	+ 92.42	— 405.38	— 177.13
31	780.5	+ 183.74	— 375.81	— 167.32
Aug. 10	790.5	+ 266.22	— 328.12	— 149.20
20	800.5	+ 335.39	— 264.67	— 123.63
30	810.5	+ 387.48	— 188.77	— 91.89
Sept. 9	820.5	+ 419.65	— 104.43	— 55.64
19	830.5	+ 430.23	— 16.19	— 16.82
29	840.5	+ 418.72	+ 71.19	+ 22.44
Oct. 9	850.5	+ 385.98	+ 153.00	+ 60.00
19	860.5	+ 334.05	+ 224.87	+ 93.79
29	870.5	+ 266.14	+ 283.07	+ 121.99
Nov. 8	880.5	+ 186.38	+ 324.72	+ 143.10
18	890.5	+ 99.52	+ 348.05	+ 156.10
28	900.5	+ 10.66	+ 352.42	+ 160.49
Dec. 8	910.5	— 75.17	+ 338.43	+ 156.30
18	920.5	— 153.31	+ 307.78	+ 144.11
28	930.5	— 219.84	+ 263.16	+ 125.01
1960 Jan. 7	940.5	— 271.75	+ 207.96	+ 100.43
17	950.5	— 307.21	+ 146.06	+ 72.08
27	960.5	— 325.54	+ 81.45	+ 41.82
Feb. 6	970.5	— 327.22	+ 17.98	+ 11.49
16	980.5	— 313.74	— 40.92	— 17.21
26	2436990.5	— 287.48	— 92.41	— 42.82
Mar. 7	2437000.5	— 251.41	— 134.47	— 64.21
17	010.5	— 208.86	— 165.89	— 80.63
27	020.5	— 163.26	— 186.39	— 91.74
Apr. 6	030.5	— 117.86	— 196.48	— 97.61
16	040.5	— 75.50	— 197.44	— 98.68
26	050.5	— 38.44	— 191.13	— 95.68
May 6	060.5	— 8.16	— 179.80	— 89.64
16	070.5	+ 14.68	— 165.90	— 81.69
26	080.5	+ 30.30	— 151.81	— 73.01
June 5	090.5	+ 39.74	— 139.64	— 64.70
15	100.5	+ 44.71	— 130.99	— 57.69
25	110.5	+ 47.42	— 126.80	— 52.64
July 5	120.5	+ 50.31	— 127.25	— 49.87
15	130.5	+ 55.79	— 131.69	— 49.34
25	140.5	+ 65.93	— 138.74	— 50.63
Aug. 4	150.5	+ 82.20	— 146.43	— 53.02
14	160.5	+ 105.31	— 152.36	— 55.53
24	170.5	+ 135.04	— 154.00	— 57.05
Sept. 3	180.5	+ 170.26	— 148.95	— 56.48
13	190.5	+ 208.96	— 135.20	— 52.78
23	200.5	+ 248.45	— 111.34	— 45.18
Oct. 3	210.5	+ 285.54	— 76.73	— 33.15
13	220.5	+ 316.82	— 31.65	— 16.59
23	230.5	+ 338.90	+ 22.74	+ 4.25
Nov. 2	240.5	+ 348.75	+ 84.41	+ 28.71

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MESON PRODUCTION IN MESON-NUCLEON COLLISIONS

BY

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CONTENTS

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The production of one meson in meson-nucleon collisions is investigated, with the purpose of expressing the differential and total cross section for the process in terms of the known parameters of elastic meson-nucleon scattering. As little use as possible is made of the questionable approximations of field theory. Particular attention is paid to the reaction in which two positive mesons emerge.

A comparison is made with nuclear reactions. Although the two processes are very different, such a comparison still makes it likely that the l, J contribution to meson production (l, relative orbital, J, total angular momentum) is in some rough way proportional to the l, J contribution to elastic scattering. The same conclusion is reached independently through a field theoretical approach; it is therefore believed to be true, although neither of the two methods is entirely cogent by itself.

Formulae are derived for the total cross section for production and the angular distribution of the emergent mesons. The first is considered to give only an order of magnitude, and agrees roughly with what a purely statistical approach to the problem would give. The angular distribution result is further supported by an argument of invariance under rotation of the coordinate system and is therefore considered to have good reliability as long as the l, J expansion remains workable.

I. Introduction.

In the last few years, the elastic scattering of π mesons by nucleons has been considered as providing one of the most direct experimental approaches to the problem of π meson-nucleon coupling and as such has been, and is, extensively studied from the experimental side (Anderson, Fermi, Long, Martin, and Nagle, 1952; Anderson, Fermi, Long, and Nagle, 1952; Anderson, Fermi, Nagle, and Yodh, 1952; Anderson, Fermi, Martin, and Nagle, 1953; Chedester, Isaacs, Sachs, and Steinberger, 1951; Fowler, Fowler, Shutt, Thorndyke, and Whittemore, 1952; Isaacs, Sachs, and Steinberger, 1952).

The first conclusion manifest from these experiments was that a perturbation treatment of the meson-nucleon scattering fails entirely to give agreement with the observations. Several kinds of approximations were then proposed, some based on strong coupling methods (Brueckner, 1952; Brueckner and

Watson, 1952 Wentzel, 1953), some on a Tamm-Dancoff approach (Chew, 1953; Bethe and Dyson, 1953; Fubini, 1953; Lévy and Marshak, 1954), and others (Drell and Henley, 1952). These theories showed on the whole a great improvement over the Born approximation (weak coupling), though the agreement with empirical data still remains partly qualitative. Also, from a theoretical point of view, none of these theories seems to be entirely justified, except perhaps when rather unnatural assumptions (e. g., cut-off momentum $\ll M$) are made at the start.

In the present paper, we are dealing with another effect which could provide useful information, viz. reactions of the type

$$\pi + \mathfrak{N} \rightarrow \mathfrak{N} + \pi + \pi, \tag{1}$$

where $\mathfrak{N}$ means nucleon. Such reactions can now be studied experimentally with the help of the big accelerating machines, and it might be useful to have some rough theoretical estimate of their relative frequencies and principal aspects*.

One could, in principle, make an attempt at developing an independent theory of such processes, following, for example, a Tamm-Dancoff approach. In view of the theoretical difficulties that make the Tamm-Dancoff approach not really consistent even in the elastic scattering problem, this would, however, be a rather ambitious programme. It might be more rewarding to look, first, for some formula connecting the cross sections for these effects to the phase shifts of elastic π meson scattering and, subsequently, to make use of the experimental values of these phase shifts.

In Chapter II, a general theory of processes such as (1) is given and a comparison is made with ordinary nuclear reactions. In Chapters III and IV, the special case

$$\pi^+ + P \rightarrow N + \pi^+ + \pi^+ \tag{2}$$

is examined, this time from the point of view of conventional meson field theory. In Chapter V, results concerning the total cross section and the angular distribution of the emitted mesons are given. It is shown, moreover, that the characteristic features of the latter are independent of any field theoretical approximation.

* Note added in proof: See also Bethe and Nelkin, Bull. Am. Phys. Soc. **29**, 30 (1954).

II. General theory.

1) Mathematical formalism.

To present a convenient formalism for both elastic meson scattering and mesoproduction of mesons, let us define $\psi_{(a)}^{(+)}$ and $\psi_{(a)}^{(1)}$ by means of the integral equations (LIPPMAN and SCHWINGER, 1950)

$$\psi_{(a)}^{(+)} = \varphi_{(a)} + \frac{1}{E_{(a)} + i\eta - H_0} H' \psi_{(a)}^{(+)}, \tag{3}$$

$$\psi_{(a)}^{(1)} = \varphi_{(a)} + P \frac{1}{E_{(a)} - H_0} H' \psi_{(a)}^{(1)}. \tag{4}$$

Here, H_0 is the free Hamiltonian, H' the interaction between meson and nucleon fields, $\varphi_{(a)}$ is an eigenvector of H_0 pertaining to state (a), P means Cauchy principal value, $\eta \to 0^+$.

Then, one can show (GOLDBERGER, 1951; BELINFANTE and MØLLER, 1954)

$$\psi_{(a)}^{(+)} = \psi_{(a)}^{(1)} - i\pi \sum_{(c)} \psi_{(c)}^{(1)} \delta(E_{(a)} - E_{(c)}) \boldsymbol{T}_{ca} \tag{5}$$

with the definition

$$\boldsymbol{T}_{ca} = (\varphi_{(c)}, H' \psi_{(a)}^{(+)}), \tag{6}$$

when $E_{(c)} = E_{(a)}$. $\boldsymbol{T}_{ca}$ is thus defined only on the energy shell. The matrix element $\boldsymbol{T}_{ca}$ is connected to the S or $R = S - 1$ matrix (MØLLER, 1945) by means of the relation

$$R_{ba} = -2\pi i \delta (E_{(a)} - E_{(b)}) \boldsymbol{T}_{ba}, \tag{7}$$

its most important property being that the rate of transition from state (a) to state (b) is given by (LIPPMAN and SCHWINGER, 1950)

$$w_{ba} = 2\pi \hbar^{-1} |\boldsymbol{T}_{ba}|^2 \varrho(E_{(a)}), \tag{8}$$

$\varrho(E_{(a)})$ being the density of states. It thus plays the same role in an exact theory as the lowest possible order perturbation matrix element in a perturbation theory.

As shown by GOLDBERGER (1951), instead of trying to compute $\boldsymbol{T}_{ba}$ directly, it is convenient to introduce the so-called reaction matrix

$$K_{ba} = (\varphi_{(b)}, H' \psi^{(1)}_{(a)}) \tag{9}$$

which is thus defined both on and out of the energy shell. Of course, states (a) and (b) can differ not only in the momenta and spin of the particles involved, but also in the *number* of these particles. (4) gives

$$K_{ba} = H'_{ba} + P \sum_{(c)}' \frac{H'_{bc} K_{ca}}{E_{(a)} - E_{(c)}}, \tag{10}$$

where the sum is to be taken over all possible states and (5) gives

$$\boldsymbol{T}_{ba} = \boldsymbol{K}_{ba} - i\pi \sum_{(c)} \boldsymbol{K}_{bc} \delta (E_{(a)} - E_{(c)}) \boldsymbol{T}_{ca}, \tag{11}$$

where $\boldsymbol{K}_{ba}$ means just K_{ba} when $E_{(a)} = E_{(b)}$; the sum $\sum_{(c)}$ is here to be taken only over states (c) which have the same energy as the initial and final ones. (11) is usually known as the Heitler integral equation. It can be shown (GOLDBERGER, 1951) that $\boldsymbol{K}_{ba}$ is a Hermitian matrix (on the energy shell only).

The problem of solving eq. (10) is very complicated, involving self-energies, elimination of divergences, etc. We shall thus focus our attention, in this section, on equation (11), which is much simpler because of the energy shell condition, and we shall attempt to derive as many results as possible from this equation only.

2) Solution of Heitler equation for energies lying below the threshold for meson production.

Equation (11) is especially simple in this case because the intermediate states (c) can contain only one meson.

We work in the system where the total momentum of the colliding particles is zero. States (a) (or (b) or (c)) are then specified by the momentum $\boldsymbol{k}_a = k_a \boldsymbol{n}_a$ (or $\boldsymbol{k}_b$ or $\boldsymbol{k}_c$) of, say, the meson and the value m_a (or m_b or m_c) of the z-component of the nucleon spin.

We first look for the most general form for $\boldsymbol{K}_{ba}$, and to that effect we consider the eigenvalue problem

$$\sum_a \boldsymbol{K}_{ba} f_a^A = K'^A f_b^A. \tag{12}$$

$\boldsymbol{K}_{ba}$ is, of course, an invariant with respect to rotations of the coordinate system. Applying a rotation $\mathfrak{R}$ to both sides, we thus get

$$\sum_a \boldsymbol{K}_{ba} (\mathfrak{R} f_a^A) = K'^A (\mathfrak{R} f_b^A); \tag{13}$$

therefore, if the eigenvalue K'^A is p–fold degenerate, with the eigenfunctions $f_a^{A,1} \cdots f_a^{A,q} \cdots f_a^{A,p}$, we have

$$\mathfrak{R} f_a^{A,q} = \sum_{q'=1}^{p} c_{qq'}^A f_a^{A,q'}. \tag{14}$$

Applying successively two rotations, we get, by a well-known argument, that the matrix $c_{qq'}^A$ is just the irreducible representation of dimension p of the rotation group. Setting $p = 2J + 1$, $q = M + J + 1$, and using J instead of A as a label for the corresponding eigenvalue, we thus have

$$\mathfrak{R} f_a^{JM} = \sum_{M'} \mathfrak{D}_{MM'}^J f_a^{JM'}, \tag{15}$$

which means that f_a^{JM} can be written as

$$f_a^{JM} = \sum_{l=J-1/2}^{J+1/2} d_{Jl}(k_a)\, \mathcal{Y}_{Jl1/2}^M(\boldsymbol{n}_a, m_a) \tag{16}$$

with (notations of [Blatt and Weisskopf, 1952] for the Clebsch-Gordan coefficients)

$$\mathcal{Y}_{Jl1/2}^M(\boldsymbol{n}_a\, m_a) = \sum_{\nu\nu'} C_{l1/2}(J, M, \nu, \nu')\, Y_l^\nu(\boldsymbol{n}_a)\, Y_{1/2}^{\nu'}(m_a); \tag{17}$$

here, $Y_{1/2}^{\nu'}(m_a)$ is the amplitude for finding the nucleon of state (a) with a spin whose projection on the z-axis is ν'. m_a having been chosen as the projection of spin of the now fixed z-axis, one has simply

$$Y^{\nu'}_{1/2}(m_a) = \delta_{\nu', m_a}. \tag{18}$$

Returning now to eq. (12), multiplying both sides by f_a^{A*} and summing over $A \equiv J, M$, one has, owing to the closure relation ($\boldsymbol{K}_{ba}$ Hermitian):

$$\boldsymbol{K}_{ba} = \sum_{lJ} K_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}^{M}_{Jl^{1/2}}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a, m_a). \tag{19}$$

This is thus the most general form for $\boldsymbol{K}_{ba}$. We write $K_{lJ}(k_a, k_a)$ since $k_b = k_a$ (energy shell). J can take both values $l + {}^1/_2$ and $l - {}^1/_2$. The reason that l is a good quantum number is the conservation of parity which forbids $\Delta l = \pm 1$.

So far, we have not introduced the charge coordinates. In fact, $\boldsymbol{K}_{ba}$ depends also on the values t_3, T_3 of the third component of the isotopic spin of nucleon and meson. If isotopic spin I is a good quantum number it is easy to see that $\boldsymbol{T}_{ba}$ and $\boldsymbol{K}_{ba}$ both can be expanded by

$$\left.\begin{aligned} &(\boldsymbol{K} \text{ or } \boldsymbol{T})_{t_{3b}, T_{3b}, t_{3a} T_{3a}} \\ &= \sum_{I=1/2}^{3/2} C_{1^{1/2}}(I, N, t_{3b}, T_{3b})\, (\boldsymbol{K}^I_{ba} \text{ or } \boldsymbol{T}^I_{ba})\, C_{1^{1/2}}(I, N, t_{3a}\, T_{3a}) \\ &\text{with } N = t_{3b} + T_{3b} = t_{3a} + T_{3a}; \end{aligned}\right\} \tag{20}$$

when carried into (11) these expansions give readily an equation for $\boldsymbol{T}^I_{ba}$ in terms of $\boldsymbol{K}^I_{ba}$, which is just equation (11) with $\boldsymbol{K}_{ba}$ replaced by $\boldsymbol{K}^I_{ba}$, and $\boldsymbol{T}_{ba}$ by $\boldsymbol{T}^I_{ba}$. The separation into total isotopic spin eigenvalues offers thus no difficulty. We can rewrite (19) as:

$$\boldsymbol{K}^I_{ba} = \sum_{lJ} K^I_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}^{M}_{Jl^{1/2}}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a, m_a). \tag{21}$$

Eq. (11) can now easily be solved by assuming for $\boldsymbol{T}^I_{ba}$ an expansion

$$\boldsymbol{T}^I_{ba} = \sum_{lJ} T^I_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}_{Jl^{1/2}}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a, m_a) \tag{22}$$

of the same form as (21). Carrying these expansions into (11), and making use of the orthonormality relations

$$\sum_{m_c} \int d\omega_c \, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_c, m_c) \, \mathcal{Y}^{M'}_{J'l'^{1/2}}(\boldsymbol{n}_c, m_c) = \delta_{J,J'} \delta_{M,M'} \delta_{l,l'}, \quad (23)$$

we get

$$T^I_{lJ}(k_a, k_a) = K^I_{lJ}(k_a, k_a) - i\pi\varrho \, K^I_{lJ}(k_a, k_a) \, T^I_{lJ}(k_a, k_a), \quad (24)$$

where $\varrho \, d\omega$ is the density in phase space. As is well known, relation (24) can be expressed parametrically by

$$\left.\begin{aligned} K^I_{lJ}(k_a, k_a) &= -(\pi\varrho)^{-1} \, tg \, \delta^I_{lJ}(k_a); \\ T^I_{lJ}(k_a, k_a) &= -(\pi\varrho)^{-1} \, e^{i\delta^I_{lJ}(k_a)} \sin \delta^I_{lJ}(k_a). \end{aligned}\right\} \quad (25)$$

Use of (8) and (25) readily shows that the δ are just the usual phase shifts.

Of course, one can express also the S-matrix in terms of the $\mathcal{Y}$-functions. Using (7) together with

$$\delta_{ba} = \sum_{lJM} \mathcal{Y}^M_{Jl^{1/2}}(\boldsymbol{n}_b, m_b) \, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a, m_a), \quad (26)$$

we get

$$\left.\begin{aligned} S_{ba} = \delta_{ba} + R_{ba} &= \sum_{lJ} [1 - 2\, i\pi\varrho \, T^I_{lJ}(k_a, k_a)] \\ &\times \sum_M \mathcal{Y}^M_{Jl^{1/2}}(\boldsymbol{n}_b, m_b) \, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a, m_a) \end{aligned}\right\} \quad (27)$$

$$= \sum_{lJ} S^I_{lJ}(k_a, k_a) \sum_M \mathcal{Y}^M_{Jl^{1/2}}(\boldsymbol{n}_b, m_b) \, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_a \, m_a) \quad (28)$$

with

$$S^I_{lJ} = 1 - 2\, i\pi\varrho \, T^I_{lJ} = \frac{1 - i\pi\varrho \, K^I_{lJ}}{1 + i\pi\varrho \, K^I_{lJ}} = e^{2i\delta^I_{lJ}}. \quad (29)$$

It may happen that $K^I_{lJ}(k_a, k_a)$ goes to infinity for a special value k_s of k_a; then, with the general notation

$$\varepsilon_a = (\mu^2 c^4 + k_a^2)^{1/2}, \quad (30)$$

one can set, in the neighbourhood of ε_s,

$$\frac{1}{K^I_{lJ}(k_a, k_a)} = \lambda(\varepsilon_a - \varepsilon_s) + \cdots; \tag{31}$$

λ being a constant, this gives

$$S^I_{lJ}(k_a, k_a) = 1 - 2i\frac{\pi\varrho\lambda^{-1}}{\varepsilon_a - \varepsilon_s + i\pi\varrho\lambda^{-1}}. \tag{32}$$

(32) corresponds to the Breit-Wigner one-level formula when there is only one channel open (elastic scattering). The resonance width is $\Gamma = 2\pi\varrho\lambda^{-1}$. Factors "$\omega$" accounting in the Breit-Wigner formula for the finite nuclear radii are, of course, not present in (32). Although one could introduce them formally by a slight modification of ε_s and Γ, such a formal modification of (29) and (32) seems to have no physical significance in our case and we therefore keep (32) as it is. Consequently, we have here no "potential" or "hard sphere" scattering. One can also interpret the resonance as being due to the formation of an excited nucleon of binding energy ε_s and mean lifetime $\hbar\,\Gamma^{-1}$ (MØLLER, 1946). It should perhaps be mentioned that one has a resonance in the usual sense only if δ crosses the value $\pi/2$ (K^I_{lJ} infinite).

3) Solution of Heitler equation for energies lying somewhat higher than the threshold for one-meson production.

(11) is now a set of two coupled integral equations. It will be convenient to use Greek indices for states containing two mesons and to reserve Latin indices for states containing one meson; then,

$$\boldsymbol{T}_{ba} = \boldsymbol{K}_{ba} - i\pi\sum_c \boldsymbol{K}_{bc}\,\delta(E_{(a)} - E_{(c)})\,\boldsymbol{T}_{ca} - i\pi\sum_\gamma \boldsymbol{K}_{b\gamma}\,\delta(E_{(a)} - E_{(\gamma)})\,\boldsymbol{T}_{\gamma a} \tag{33}$$

$$\boldsymbol{T}_{\beta a} = \boldsymbol{K}_{\beta a} - i\pi\sum_c \boldsymbol{K}_{\beta c}\,\delta(E_{(a)} - E_{(c)})\,\boldsymbol{T}_{ca} - i\pi\sum_\gamma \boldsymbol{K}_{\beta\gamma}\,\delta(E_{(a)} - E_{(\gamma)})\,\boldsymbol{T}_{\gamma a}. \tag{34}$$

Let us restrict our investigations to the domain of energies where $|\boldsymbol{T}_{\beta a}| \ll |\boldsymbol{T}_{ba}|$, i. e., where the elastic scattering is much more important than the meson production (apart from the difference in phase space densities). It seems probable that this domain

extends to rather high energies. The third terms in (33) and (34) are then comparatively very small. In the system of equations (33), (34), $\sum_c$ represents a summation which, owing to the energy shell condition, can extend only over the angles defining the direction of the intermediate meson; $\sum_\gamma$ is a summation both over the directions of the two mesons contained in state γ and over the magnitude of their momenta, this summation being restricted, however, to values satisfying the energy shell condition. In actual calculations, the δ-functions should of course be replaced by the densities in phase-space.

We first give an approximate solution of (33), (34) by neglecting the third term in the right-hand side of (34) and introducing (34) into (33), thus neglecting a term of order $|\boldsymbol{T}_{\gamma a}|^2$. One easily sees that it is possible to write

$$\boldsymbol{T}_{ba} = \mathfrak{K}_{ba} - i\pi \sum_c \mathfrak{K}_{bc}\, \delta(E_{(a)} - E_{(c)})\, \boldsymbol{T}_{ca} \tag{35}$$

with

$$\mathfrak{K}_{ba} = \boldsymbol{K}_{ba} - i\pi \sum_\gamma \boldsymbol{K}_{b\gamma}\, \delta(E_{(a)} - E_{(\gamma)})\, \boldsymbol{K}_{\gamma a}. \tag{36}$$

(36) is expected to be a good approximation as long as the imaginary part is smaller than the real part.

The case where $\boldsymbol{K}_{ba}$ becomes large ("resonance") deserves special consideration. A-priori we know nothing about the behaviour of $\boldsymbol{K}_{b\gamma}$, $\boldsymbol{K}_{\gamma a}$, so we cannot exclude the possibility that these quantities become large for the same energy as $\boldsymbol{K}_{ba}$. A simplified picture of this case is obtained by writing the $\boldsymbol{K}$'s as products:

$$\boldsymbol{K}_{ba} = f_b \cdot f_a; \qquad \boldsymbol{K}_{b\gamma} = f_b \cdot \varphi_\gamma, \tag{37}$$

where we assume that f_b, f_a, φ_γ all become very large in the neighbourhood of an energy ε_s. Then, (36) does not hold, but we can easily get a better approximation by iterating (34) any number of time before we introduce the result into (33). Thus, one gets (35) again, now with

$$\left.\begin{aligned} \mathfrak{K}_{ba} = \boldsymbol{K}_{ba} - i\pi \sum_\gamma \boldsymbol{K}_{b\gamma}\, \delta(\cdots)\, \boldsymbol{K}_{\gamma a} \\ + (-i\pi)^2 \sum_{\gamma\varepsilon} \boldsymbol{K}_{b\gamma}\, \delta(\cdots)\, \boldsymbol{K}_{\gamma\varepsilon}\, \delta(\cdots)\, \boldsymbol{K}_{\varepsilon a} + \cdots. \end{aligned}\right\} \tag{38}$$

With (37), this gives

$$\mathfrak{K}_{ba} = \frac{\boldsymbol{K}_{ba}}{1 + i\pi \sum_\gamma \varphi_\gamma \delta(\cdots) \varphi_\gamma}. \tag{39}$$

It seems therefore a reasonable approximation to write, in a general way,

$$\mathfrak{K}_{ba} = \frac{\boldsymbol{K}_{ba}}{1 + i a} \tag{40}$$

instead of (36), a being a real function of the incident energy which we do not specify any further. (36) is a particular case of (40) when a is small.

(35) can now be treated in the same way as (11) (§§ 1 and 3) and we get:

$$T^I_{lJ}(k_a, k_a) = \frac{\mathfrak{K}^I_{lJ}(k_a, k_a)}{1 + i\pi\varrho\, \mathfrak{K}^I_{lJ}(k_a, k_a)} \tag{41}$$

with

$$\mathfrak{K}^I_{lJ}(k_a, k_a) = \frac{K^I_{lJ}(k_a, k_a)}{1 + i a}. \tag{42}$$

In fact, the assumption (37) is somewhat oversimplified for our purpose. A more accurate treatment can be made if we anticipate over a general result proved in § 4 below and which states that expansions of the form

$$\boldsymbol{K}_{ba} = \sum_{lJ} K_{lJ} \sum_M \mathfrak{Y}^M_{Jl_{1/2}}(\boldsymbol{n}_b, m_b)\, \mathfrak{Y}^{M*}_{Jl_{1/2}}(\boldsymbol{n}_a, m_a)$$

$$\boldsymbol{K}_{\gamma a} = \sum_{lJl'l''} K_{lJl'l''} \sum_M \Phi^M_{l'l''(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)\, \mathfrak{Y}^{M*}_{Jl_{1/2}}(\boldsymbol{n}_a, m_a)$$

$$\boldsymbol{K}_{\beta\gamma} = \sum_{LJl_1l_2l'l''} K_{l_1l_2LJl'l''} \sum_M \Phi^M_{l_1l_2(L)J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_\beta)\, \Phi^{M*}_{l'l''(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)$$

are always possible, with the orthonormality property (use eq. (60) below)

$$\sum_{m\gamma} \int d\omega' \int d\omega'\, \Phi^M_{l_1l_2(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)\, \Phi^{M'*}_{l'l''(L')J'}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma) = \delta_{l_1l'}\, \delta_{l_2l''}\, \delta_{L,L'}\, \delta_{J,J'}\, \delta_{M,M'}.$$

K_{lJ} is a function of k_a and it is convenient to write it in the form $K_{lJ} = [f_{lJ}(k_a)]^2$, $K_{lJl'l''}$ is a function of k_a, k', k''. We replace assumption (37) by

$$K_{lJl'l''} = f_{lJ}(k_a) \cdot \varphi_{lJl'l''}(k',k'') \tag{37a}$$

and, similarly,

$$K_{l_1l_2LJl'l''} = \varphi_{LJl_1l_2}(k_1, k_2) \cdot \varphi_{LJl'l''}(k',k''). \tag{37b}$$

Equations (33) and (34) then give, by substitution,

$$\left.\begin{aligned} T_{lJ}(k_a, k_a) = f_{lJ}(k_a) \Big[& f_{lJ}(k_a) - i\pi\varrho\, f_{lJ}(k_a)\, T_{lJ}(k_a, k_a) \\ & - i\pi \sum_{k'k''l'l''} \varphi_{lJl'l''}(k', k'')\, T_{lJl'l''}(k', k'', k_a)\, \varrho(k', k'') \Big] \end{aligned}\right\} \tag{33a}$$

$$T_{lJl_1l_2}(k_1, k_2, k_a) = \varphi_{lJl_1l_2}(k_1, k_2) \text{ [same bracket]}; \tag{34a}$$

therefore,

$$T_{lJl_1l_2}(k_1, k_2, k_a) = \frac{\varphi_{lJl_1l_2}(k_1, k_2)}{f_{lJ}(k_a)}\, T_{lJ}(k_a, k_a). \tag{34b}$$

Substitution in the right-hand side of (33a) and solution with respect to $T_{lJ}(k_a, k_a)$ give again eq. (41), now however with

$$\mathfrak{K}^I_{lJ}(k_a, k_a) = \frac{K^I_{lJ}(k_a, k_a)}{1 + i a^I_{lJ}} \tag{42a}$$

$$a^I_{lJ} = \pi \sum_{k'k''l'l''} \varphi_{lJl'l''}(k', k'')\, \varphi_{lJl'l''}(k', k'')\, \varrho(k', k''). \tag{42b}$$

We get rid of the factors involving $\pi\varrho$ by defining new quantities:

$$L = -\pi\varrho K; \quad \mathfrak{L} = -\pi\varrho\mathfrak{K}; \quad R = -2\pi i\varrho T, \tag{43a, b, c}$$

(43c) being just a rewriting of (7). (41) is then

$$R^I_{lJ}(k_a, k_a) = 2i \frac{\mathfrak{L}^I_{lJ}(k_a, k_a)}{1 - i\mathfrak{L}^I_{lJ}(k_a, k_a)}, \tag{44}$$

and one has:

$$S^I_{lJ}(k_a, k_a) = 1 + R^I_{lJ}(k_a, k_a) = \frac{1 + i\,\mathfrak{Q}^I_{lJ}(k_a, k_a)}{1 - i\,\mathfrak{Q}^I_{lJ}(k_a, k_a)}. \qquad (45)$$

Let us define

$$f^I_{lJ}(k_a) = \frac{1}{\mathfrak{Q}^I_{lJ}(k_a, k_a)} = \frac{1}{L^I_{lJ}(k_a, k_a)} + i\,\frac{a^I_{lJ}}{L^I_{lJ}(k_a, k_a)}. \qquad (46)$$

(45) can be written:

$$S^I_{lJ}(k_a, k_a) = \frac{f^I_{lJ}(k_a) + i}{f^I_{lJ}(k_a) - i}, \qquad (47)$$

where the notations are somewhat similar to those used in nuclear reactions*. Here, as in the theory of nuclear reactions, the fact that $f^I_{lJ}(k_a)$ has now a (negative) imaginary part makes $|S^I_{lJ}|^2 < 1$, thus accounting for the possibility of processes ("reactions") different from elastic scattering. Here, these processes are the meson production. The total elastic cross section for l, J, I waves is

$$\sigma^I_{lJ\ \mathrm{el.}} \sim |S^I_{lJ} - 1|^2 = \frac{4}{[\Re\mathrm{e}\, f^I_{lJ}]^2 + [1 - \Im\mathrm{m}\, f^I_{lJ}]^2}, \qquad (48)$$

the total reaction cross section being (unitarity of S-matrix):

$$\sigma^I_{lJ\ \mathrm{prod.}} \sim 1 - |S^I_{lJ}|^2 = \frac{-4\,\Im\mathrm{m}\, f^I_{lJ}}{[\Re\mathrm{e}\, f^I_{lJ}]^2 + [1 - \Im\mathrm{m}\, f^I_{lJ}]^2}. \qquad (49)$$

(48), (49) are still rather formal because the ratio of the two cross sections depends (eq. (46)) on the quantity a^I_{lJ} which we have so far no means of evaluating.

In order to test the analogy with nuclear reactions it will be useful to study the case where a resonance should occur. In the theory of nuclear reactions, a resonance is said to take place for a certain energy $\varepsilon_s = (\mu^2 c^4 + k_s^2)^{1/2}$ if $\Re\mathrm{e}\, f^I_{lJ}(k_a) \to o$ when $k_a \to k_s$. Another important condition, however, should be fulfilled if

* Our f_{lJ} are, in fact, equivalent to $\frac{f_l - \Delta_l}{s_l}$ in the notations of Blatt and Weisskopf. Similarly, our ε_s are their ε'_s ("actual resonance energies"), see below.

we want the scattering and reaction cross section to obey the usual Breit-Wigner formulae for resonance processes. This condition is that $\Im \mathfrak{m}\, f^I_{lJ}(k_a)$ should be a finite, smoothly varying function in the neighbourhood of k_s (see formulae below). Therefore, as an inspection of (46) shows, we shall have a real analogy with nuclear reaction processes if, and only if, a^I_{lJ} tends to infinity for the same energy as L^I_{lJ}. This in turn implies that the reaction matrix elements, such as $\boldsymbol{K}_{\gamma a}$, for production processes increase proportionally to elastic ones, $\boldsymbol{K}_{ba}$, when the latter approach a resonance. Only if this condition is fulfilled will the analogy with nuclear reactions be somewhat more than a purely superficial one.

Then, defining $\Gamma^I_{lJ\ \mathrm{el.}}(k_a)$, $\Gamma^I_{lJ\ \mathrm{prod.}}(k_a)$ by

$$\frac{1}{L^I_{lJ}(k_a)} = \Re \mathfrak{e}\, f^I_{lJ}(k_a) = \frac{-2}{\Gamma^I_{lJ\ \mathrm{el.}}(k_a)}(\varepsilon_a - \varepsilon_s) + \cdots \tag{50}$$

$$\Im \mathfrak{m}\, f^I_{lJ}(k_a) = -\frac{\Gamma^I_{lJ\ \mathrm{prod.}}(k_a)}{\Gamma^I_{lJ\ \mathrm{el.}}(k_a)}, \tag{51}$$

and replacing in (48), (49), one gets the usual Breit-Wigner formulae

$$\sigma^I_{lJ\ \mathrm{el.}} \sim \frac{[\Gamma^I_{lJ\ \mathrm{el.}}(k_a)]^2}{(\varepsilon_a - \varepsilon_s)^2 + \frac{1}{4}[\Gamma^I_{lJ\ \mathrm{el.}}(k_a) + \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)]^2}, \tag{52}$$

$$\sigma^I_{lJ\ \mathrm{prod.}} \sim \frac{\Gamma^I_{lJ\ \mathrm{el.}}(k_a) \cdot \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)}{(\varepsilon_a - \varepsilon_s)^2 + \frac{1}{4}[\Gamma^I_{lJ\ \mathrm{el.}}(k_a) + \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)]^2}, \tag{53}$$

which, as is well known, can be written (Blatt and Weisskopf, chapter X) in the compact from

$$\sigma^I_{lJ\ \mathrm{el.}}(k_a) = \sigma^I_{lJ\,(c)}(k_a) \cdot \frac{\Gamma^I_{lJ\ \mathrm{el.}}}{\Gamma^I_{lJ}}, \tag{54 a}$$

$$\sigma^I_{lJ\ \mathrm{prod.}}(k_a) = \sigma^I_{lJ\,(c)}(k_a) \cdot \frac{\Gamma^I_{lJ\ \mathrm{prod.}}}{\Gamma^I_{lJ}}, \tag{54 b}$$

$$\sigma^I_{lJ\,(c)}(k_a) \propto \frac{\Gamma^I_{lJ} \cdot \Gamma^I_{lJ\ \mathrm{el.}}}{(\varepsilon_a - \varepsilon_s)^2 + \frac{1}{4}(\Gamma^I_{lJ})^2}\ ;\ \Gamma^I_{lJ} = \Gamma^I_{lJ\ \mathrm{el.}} + \Gamma^I_{lJ\ \mathrm{prod.}}\cdot \quad (54\,\mathrm{c\,d})$$

The second factors in (54a), (54b) are the branching ratios for the two possible phenomena, elastic scattering and meson production, (54d) is the total width, (54c) may be interpreted as the cross section for the formation of an excited nucleon.

These analogies with resonance nuclear reactions are not quite general: one can see that they depend on the possibility of making the approximation (42a) for $\mathfrak{K}^I_{lJ}$ and that they would break down if, for example, the correct approximation had been $\mathfrak{K}^I_{lJ} = K^I_{lJ}(1 - i\,a^I_{lJ})$, *whatever energy dependence we should then choose for* a^I_{lJ}. They can only be partial analogies, however, for, quite apart from the different nature of the involved particles, there are important differences between the two phenomena: the main difference is that, in mesoproduction, three particles emerge at the same time as compared to only two in nuclear reactions. This particular feature, which makes very non-physical the consideration of the inverse processes, prevents us, for example, from giving to (54a, b, c, d) the simple interpretation that it proves Bohr's hypothesis at resonances: it is true that (54a, b, c, d) show that formation and decay of the excited nucleon take place as independent processes, but we must keep in mind that here we have only one way of forming the excited nucleon; to say that the excited nucleon does not remember how it was formed has therefore no great meaning.

As mentioned before, any analogy with nuclear reactions would break down if the matrix elements $\boldsymbol{K}_{\gamma a}$ for production of a meson happened not to go to infinity precisely for the same energies ε_s as the elastic matrix elements $\boldsymbol{K}_{ba}$ if, of course, these energies exist. Equations (52), (53) would still be valid in that case, but $\Gamma^I_{lJ\ \mathrm{prod.}}$ would go to zero at ε_s, thus making $\sigma^I_{lJ\ \mathrm{prod.}}(\varepsilon_s) \to 0$ instead of going through a maximum.

In this chapter, we have not made any attempt to calculate explicitly the matrix elements of the K-matrix, i. e. to solve eq. (10); solution of eq. (10) always requires some more or less drastic assumptions in order to restrict the number of significant graphs (intermediate states). Although such assumptions are

customary in field theory (Born approximation, Tamm-Dancoff method) one should be very cautious in using them and not neglect any possible check of their results. Here we make also an assumption, though of a quite different kind: we assume that an analogy with nuclear reactions does exist. This leads us to the conclusion that some kind of proportionality between the production and elastic matrix elements of the K-matrix should exist. In Chapter IV, we shall find the same kind of proportionality by means of some plausible, but not altogether cogent, arguments concerning which of the graphs are most significant. In the author's opinion, the plausibility of both assumptions is somewhat reinforced by the fact that, though very different, they lead to the same conclusion in that respect.

As regards the scattering cross section, it is clear that, for energies above the meson production threshold, the conventional analysis in phase shifts cannot be rigorously valid any more. We quote the following relevant formulae:

$$S^I_{lJ}(k_a, k_a) = e^{-2q^I_{lJ}(k_a)}\, e^{2i\delta^I_{lJ}(k_a)} \tag{55 a}$$

with

$$e^{-2q^I_{lJ}} = \frac{\cos\,\delta^{(+)}}{\cos\,\delta^{(-)}};\quad 2\,\delta^I_{lJ} = \delta^{(+)} + \delta^{(-)} \tag{55 b, c}$$

and

$$\operatorname{tg}\delta^{(\pm)} = -\pi\varrho\, K^I_{lJ}\left(1 \pm \frac{\sigma^I_{lJ\ \text{prod.}}}{\sigma^I_{lJ\ \text{el.}}}\right). \tag{55 d}$$

4. Remarks on the matrix element T for production.

We first neglect the third term in the right hand side not only of (34), but also of (33): the phaseshift analysis of the scattering process is then possible as in § 2 (this is also apparent from formulae (55) with σ^I_{lJ} prod. $\ll \sigma^I_{lJ}$ el.).

In order to perform the integration (34) we look, as in § 2, for the most general form for the production matrix element $\boldsymbol{K}_{\beta a}$. This can be done easily by a slight extension of the argument in § 2. It will be convenient here to change somewhat our notations: states $a, b, c \ldots$ with one meson only will be referred

to by suffixes N_a, a with $N_a = 1$, states $\alpha, \beta, \gamma \ldots$ which contain two mesons by suffixes N_a, a with $N_a = 2, \ldots$ We then ask for the solution of the eigenvalue problem:

$$\sum_{N_a} \sum_{a} \boldsymbol{K}_{N_b, b; N_a, a} f^{A}_{N_a, a} = K'^{A} f_{N_b, b}. \tag{56}$$

The same argument as in § 2 shows that $f^{A}_{N_a, a} = f^{J, M}_{N_a, a}$ must satisfy

$$\Re f^{J, M}_{N_a, a} = \sum_{M'} \mathfrak{D}^{J}_{MM'} f^{J, M'}_{N_a, a}. \tag{57}$$

For the particular case $N_a = 1$, (57) reduces to (15) so that $f^{J, M}_{1, a}$ has the form (16) with (17). For the case $N_b = 2$, one has

$$\left.\begin{aligned} f^{JM}_{2, b} = \sum_{l_1 l_2 L} g_{l_1 l_2}(k_1, k_2 : k_a) \sum_{m_1 m_2 \mu m_s} C_{L^1/_2}(J, M, \mu, m_s) \\ \times C_{l_1 l_2}(L, \mu, m_1, m_2)\, Y^{m_1}_{l_1}(\boldsymbol{n}_1)\, Y^{m_2}_{l_2}(\boldsymbol{n}_2)\, Y^{m_s}_{1/_2}(m_b), \end{aligned}\right\} \tag{58}$$

where $\boldsymbol{k}_1 = k_1 \boldsymbol{n}_1$, $\boldsymbol{k}_2 = k_2 \boldsymbol{n}_2$ are the momenta of the emerging mesons in state b. Therefore,

$$\boldsymbol{K}_{2b; 1a} = \sum_{lJl_1 l_2} \boldsymbol{K}_{lJl_1 l_2}(k_1, k_2; k_a) \sum_{M} \Phi^{M}_{l_1 l_2 (L) J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_b\, \mathfrak{D}^{M*}_{JI^1/_2}(\boldsymbol{n}_a, m_a) \tag{59}$$

with

$$\Phi^{M}_{l_1 l_2 (L) J} = \sum_{m_1 m_2 \mu\, m_s} C_{L^1/_2}(J, M, \mu, m_s) C_{l_1 l_2}(L, \mu, m_1 m_2)\, Y^{m_1}_{l_1}(\boldsymbol{n}_1)\, Y^{m_2}_{l_2}(\boldsymbol{n}_2)\, Y^{m_s}_{1/_2}(m_b) \tag{60}$$

The summation in (59) should, from general symmetry requiments, be restricted to $l_1 + l_2 - l$ odd. For two emergent π^+ this gives:

$$L = l + 1 \text{ if } J = l + \frac{1}{2};\ L = l - 1 \text{ if } J = l - \frac{1}{2}. \tag{61}$$

when one keeps only the terms of (59) whose radial and angular dependences are both symmetrical. The remaining terms vanish for $k_1 = k_2$, and will be neglected.

No problem arises here with regard to isotopic spin indices since we are primarily interested in reaction (2): $P + \pi^+$ is a

pure $I = {}^3/_2$ state so that in (33) and (34) only $\boldsymbol{T}^{3/2}_{ba}$, $\boldsymbol{K}^{3/2}_{ba}$ can enter. We introduce (59) and (22) into (34) and make use of the orthonormality relation of the $\mathcal{Y}$'s. This gives (we note $\boldsymbol{T}^{++}$, the reaction where two π^+ come out)

$$\boldsymbol{T}^{++}_{2b1a} = \sum_{lJl_1l_2} T^{++}_{lJl_1l_2}(k_1, k_2; k_a) \sum_M \Phi^M_{l_1l_2(L)J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_b)\, \mathcal{Y}^{M*}_{Jl^1/_2}(\boldsymbol{n}_a, \boldsymbol{n}_a) \quad (62)$$

with

$$T^{++}_{lJl_1l_2} = K^{++}_{lJl_1l_2}(k_1, k_2; k_a) - i\pi\varrho\, K^{++}_{lJl_1l_2}(k_1, k_2; k_a)\, T^{3/2}_{lJ}(k_a, k_a) \quad (64)$$

$$= K^{++}_{lJl_1l_2}(k_1, k_2; k_a)\,[1 - i\pi\varrho\, T^{3/2}_{lJ}(k_a, k_a)], \quad (64\,a)$$

$$T^{++}_{lJl_1l_2}(k_1, k_2; k_a) = \frac{K^{++}_{lJl_1l_2}(k_1, k_2; k_a)}{K^{3/2}_{lJ}(k_a, k_a)}\, T^{3/2}_{lJ}(k_a, k_a) \quad (65)$$

(65) is here derived for $\boldsymbol{T}_{\gamma a}$ small; under the assumptions (37a, b) it is, however, valid for arbitrary $\boldsymbol{T}_{\gamma a}$ (cf. (34b)).

Relation (65) is a first step toward a solution of the problem stated in Chapter I, which is to express the matrix element of the T-matrix for production, and therefore the production cross section, in terms of the matrix elements of the T-matrix for scattering, i. e. in terms of the scattering phase shifts. Relation (65) is especially significant if we cling to the conclusion arrived at in § 3 from a comparison with nuclear reactions, that the K-matrix elements for production should show some proportionality with the K-matrix elements for scattering. We then expect that, if a particular $K^{3/2}_{lJ}$ becomes much larger than the others in some domain of energies, the corresponding K-matrix elements for production will also become large for these energies, thus making the ratio of the two K's in (65) roughly a constant. $T^{3/2}_{lJ}$ being large, we then expect $T^{++}_{lJl_1l_2}$ to be large. Therefore it is probable that the main contribution to the production cross section will be given by the l, J waves in this domain of energies.

In that case, one can use (62) together with (60) to make some predictions on the angular distribution of the emitted mesons. Of course, in order to get definite results, it will be necessary to limit oneself to low values of l_1 and l_2: it is, how-

ever, interesting to note that such predictions are possible without any approximation of a field theoretical nature, e. g., limiting the number of graphs. We defer the detailed study of angular distribution to Chapter V, where some possible experimental tests of the theory will be given.

III. The reaction $\pi^+ + P \to N + \pi^+ + \pi^+$ (2). Field theoretical approach.

While the general considerations of Chapter II have the great advantage of not being dependent on any simplification of the kind used in field theory, i. e. on any reduction of the number of graphs, it is clear that they are too formal to give any answer to many important questions; for example, they can lead to no conclusion concerning the relative magnitude of the production cross section. We now attack the problem from a different angle and look for a way of approach more connected to the ordinary methods of elementary quantum field theory.

The method that first suggests itself is, of course, the Feynman-Dyson covariant approach. Let us therefore write the two Feynman graphs (A) and $(\tilde{A})$ (Fig. 1) that represent the reaction to lowest order in the coupling constant G.

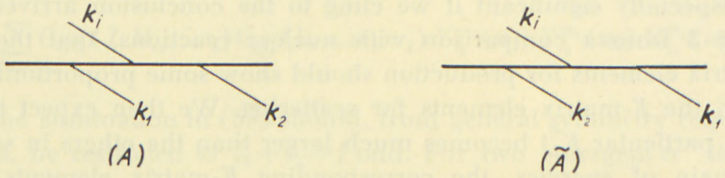

Fig. 1. The two covariant Feynman graphs of lowest order for $\pi+ + P \to N + \pi+ + \pi+$.

These graphs are symmetric in $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, the momenta of the emerging mesons, so that it is sufficient to calculate only one of them. The calculation of the matrix elements pertaining to these graphs follows the well-known pattern and does not give rise to any difficulty but, of course, in view of the failure of lowest order calculations in the elastic scattering problem, one cannot have much confidence in the result.

Instead of using the covariant method one can just as well use the old non-covariant lowest order approach. This method gives, of course, the same end-result as the covariant one, but it is instructive to see how the matrix elements pertaining to each of the non-covariant graphs combine with each other.

The six non-covariant graphs corresponding to the covariant graph (A) are (α), (β), (γ), (δ), (ε), (ζ) (Fig. 2).

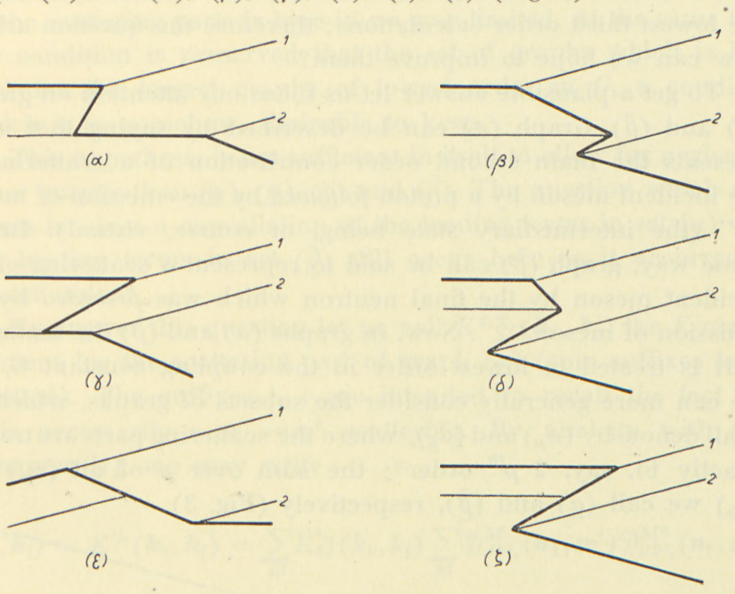

Fig. 2. The six non-covariant graphs corresponding to the covariant graph (A).

The six graphs corresponding to ($\tilde{A}$) are just the symmetrics of those with respect to interchange of k_1 and k_2 (1 and 2 on Fig. 2) and will be denoted by ($\tilde{\alpha}$), ($\tilde{\beta}$), ($\tilde{\gamma}$), ($\tilde{\delta}$), ($\tilde{\varepsilon}$), ($\tilde{\zeta}$).

Let us now choose a not too large incident energy, so that k_i, k_1, k_2, $\varepsilon_i = (\mu^2 c^4 + k_i^2)^{1/2}$, $\varepsilon_1 = (\mu^2 c^4 + k_1^2)^{1/2}$, $\varepsilon_2 = (\mu^2 c^4 + k_2^2)^{1/2}$ are all of order μ compared to M (μ = meson mass, M = nucleon mass); roughly speaking, this is satisfied when the incident energy of the π meson in laboratory system is significantly smaller than 1 Bev. Then a quite straightforward calculation of the matrix elements (we assume γ_5 coupling throughout) gives the following results:

(i) both graphs (α) and (β) (and of course ($\tilde{\alpha}$) and ($\tilde{\beta}$)) give contributions larger than the contributions of the various other

graphs by a factor of order M/μ ; (ii) however, the contributions of (α) and $(\tilde{\beta})$ (and of $(\tilde{\alpha})$ and (β)) cancel to leading order in M/μ.

As a consequence of this latter fact, all 12 graphs contribute significantly (in spite of (i)) to the total matrix element of lowest order. Of course, this cancellation occurs so-to-speak automatically in the covariant method.

As mentioned before, there are reasons for not believing in the lowest third order calculations; therefore the question arises: how can we hope to improve them?

To get a plausible answer let us focus our attention on graphs (α) and $(\tilde{\beta})$. Graph (α) can be described by saying that it represents the main second order contribution of a scattering of the incident meson by a proton *followed* by the emission of meson "2" (the intermediary state being, of course, virtual). In the same way, graph $(\tilde{\beta})$ can be said to represent a scattering of the incident meson by the final neutron which was *preceded* by the emission of meson "2". Now, in graphs (α) and $(\tilde{\beta})$ the scattering part is treated to lowest order in the coupling constant G, but we can more generally consider the subsets of graphs, which we shall denote by (α_p) and $(\tilde{\beta}_p)$, where the scattering parts are treated exactly to, say, 2 p^{th} order*; the sum over p of all (α_p) and $(\tilde{\beta}_p)$ we call $(\underline{\alpha})$ and $(\underline{\tilde{\beta}})$, respectively (Fig. 3).

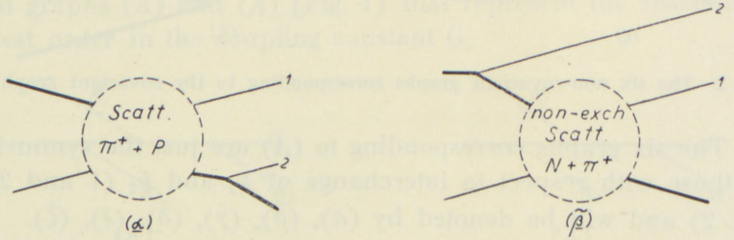

Fig. 3. The "improved" graphs. (α) and (β) are sums over all the graphs one would get by drawing the "scattering parts" (hidden by circles on the figure) to any order.

For $p = 1$, what has been said before shows that the matrix elements of (α_p), $(\tilde{\alpha}_p)$, (β_p), $(\tilde{\beta}_p)$, taken separately, are larger than all other matrix elements of lowest order in G. Our hypothesis will be that this is still true more generally for $(\underline{\alpha})$, $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$, $(\underline{\tilde{\beta}})$

* Note that some of the graphs entering in (α_p) may be identical with some entering in (β_p), as would be the case for (ε); then they should be taken only once. This, however, will not modify the general argument.

compared separately to the sum of all other possible graphs. This hypothesis, which constitutes the whole of our approximation, is of course largely arbitrary, as is the case in all approximations where one selects particular sets of a finite or infinite number of graphs and neglects others. It seems, however, a less drastic simplification than that introduced, for example, by the Tamm-Dancoff method, because the number of simultaneous meson lines in the scattering parts is here in no way limited. At the same time the condition is preserved that the set of graphs which is kept includes the largest graphs of lowest order in G, a condition which it is, somehow, desirable to keep.

This hypothesis is not sufficient in itself to allow for neglecting other graphs than $(\underline{\alpha})$, $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$ and $(\underline{\tilde{\beta}})$. The question which now arises is: does a cancellation of the leading terms in set $(\underline{\alpha})$ with the leading terms in set $(\underline{\tilde{\beta}})$ still occur here as it occurred in lowest order?

To answer this question let us call $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_1)$ the K-matrix element for the scattering part of graph $(\underline{\alpha})$ (spin suffixes being omitted). The suffixes $+ +$ are intended to recall the fact that it is necessarily a $\pi^+ \to \pi^+$ scattering. By analogy with (19) (Chapter II) one may write

$$K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i) \equiv K^{3/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \sum_{lJ} K^{3/2}_{lJ}(k_1, k_i) \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_1, m') \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_i, m_i). \quad (66)$$

The matrix element of the scattering part of graph $(\underline{\tilde{\beta}})$ is not taken exactly in the c. m. system; however, its main properties will be the same as if it were and, to alleviate notations, we call it $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$. The suffixes $- -$ recall the fact that this matrix element pertains also to the experimentally studied $P + \pi^-$ non-exchange scattering. Expansion (20) in total isotopic spin eigenvalues, with use of the appropriate Clebsch-Gordan coefficients, gives

$$K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \frac{1}{3}\left[K^{3/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) + 2\,K^{1/2}(\boldsymbol{k}_1, \boldsymbol{k}_i)\right] \quad (67)$$

with

$$K^{1/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \sum_{lJ} K^{1/2}_{lJ}(k_1, k_i) \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_1, m_l) \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_i, m'). \quad (68)$$

Now, the quantities $K^I_{lJ}(k_1, k_i)$ appearing in (66) are not taken on the energy shell ($k_1 \neq k_i$). In order to compare them with their values $K^I_{lJ}(k_i, k_i)$ on the energy shell, we must make use of a very rough and qualitative result of the Tamm-Dancoff method applied to the scattering K-matrix (see Appendix I for a formulation of this method). This will be the only place where we make use of a result derived by the Tamm-Dancoff method and, even if the Tamm-Dancoff method is not quantitatively reliable for scattering, it is felt that the qualitative and rough result of it used here has a high degree of probability. Let us call $B^I_{lJ}(k_1, k_i)$ the lowest order approximation of $K^I_{lJ}(k_1, k_i)$ in an expansion in the coupling constant: i. e. the l, J, I coefficient in an expansion, similar to (66), of $B(\boldsymbol{k}_1, \boldsymbol{k}_i)$, $B(\boldsymbol{k}_1, \boldsymbol{k}_i)$ being the Born approximation matrix element. The result just referred to is that the ratios

$$y^I_{lJ} = \frac{K^I_{lJ}(k_1, k_i)}{B^I_{lJ}(k_1, k_i)} \tag{69}$$

should not be extremely different off the energy shell from what they are on the energy shell ($k_1 = k_i$). This means that

$$K^I_{lJ}(k_1, k_i) \approx K^I_{lJ}(k_i, k_i) \cdot F_{lJ}(k_1, k_i) \text{ with } F_{lJ}(k_1, k_i) = \frac{B^I_{lJ}(k_1, k_i)}{B^I_{lJ}(k_i, k_i)} \tag{70}$$

as the dependence of $B^I_{lJ}(k_1, k_i)$ on k_1 is independent of I. $F_{lJ}(k_1, k_i)$ is the same for $I = {}^3/_2$ and $I = {}^1/_2$. Therefore,

$$\frac{K^{3/2}_{lJ}(k_1, k_i)}{K^{1/2}_{lJ}(k_1, k_i)} \approx \frac{K^{3/2}_{lJ}(k_i, k_i)}{K^{1/2}_{lJ}(k_i, k_i)}. \tag{71}$$

(71) means that in order to compare $K^{3/2}_{lJ}$ and $K^{1/2}_{lJ}$ we can take their values on the energy shell, which are experimentally known through the corresponding phase shifts δ (eq. (25)).

In order to compare $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ and $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ we carry (66) and (68) into (67), thus obtaining an expansion of $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ of the same form as (66), $K^{3/2}_{lJ}$ being replaced by $\frac{1}{3}(K^{3/2}_{lJ} + 2\,K^{1/2}_{lJ})$. Now, for energies studied so far it happens

(ANDERSON, FERMI, MARTIN, and NAGLE, 1953) that all $\delta_{lJ}^{1/2}$ are of opposite sign to the corresponding $\delta_{lJ}^{3/2}$ and of such relative values as to make

$$\frac{1}{3}(K_{lJ}^{3/2} + 2K_{lJ}^{1/2}) \ll K_{lJ}^{3/2}. \tag{72}$$

Therefore experimental evidence leads us to the conclusion that $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i) \gg K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$, at least up to k_i values corresponding to energies of 135 Mev. Although these energies are still somewhat below the threshold for meson production (176 Mev), it seems rather likely that the extrapolation to this energy and beyond is correct. Also, detailed inspection of the phase shift values given by ANDERSON et. al. shows that $y_{lJ}^{I}(k_1, k_i)$ given by (69) should be very different indeed from its value on the energy shell in order to invalidate the present conclusion.

We conclude that the cancellation between the $(\underline{\alpha})$ and $(\underline{\tilde{\beta}})$ terms which appeared when we kept only the lowest order in G may very likely be an accidental one, and a particular feature of the lowest order approximation; it corresponds to the fact that the cross sections for $P + \pi^+$ and for $P + \pi^-$ non-exchange scattering are equal in the lowest order approximation, whereas they are quite different in reality. We conclude moreover that, the hypothesis having been made that (α), $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$, $(\underline{\tilde{\beta}})$ are separately more important than all others, we are then justified in finally keeping only the $(\underline{\alpha})$ and $(\underline{\tilde{\alpha}})$ terms.

IV. The reaction $\pi^+ + P \to N + \pi^+ + \pi^+$ (2). Matrix elements and cross sections.

1. Calculation of the *K*-matrix element.

According to formula (10) of Chapter II the (exact) K-matrix element for the reaction is

$$\left.\begin{aligned} K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) \\ &+ \sum_{m_c} \frac{H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_1, m_c) K^{3/2}(\boldsymbol{k}_1, m_c; \boldsymbol{k}_i, m_i)}{E - E_1 - \varepsilon_1} \\ &+ \text{sym. terms } 1 \rightleftarrows 2 + \text{other terms.} \end{aligned}\right\} \tag{73}$$

Here, $E_1 = (M^2c^4 + k_1^2)^{1/2}$; $\varepsilon_1 = (\mu^2 c^4 + k_1^2)^{1/2}$; $E = E_{(i)} = E_i + \varepsilon_i$
$= (M^2c^4 + k_i^2)^{1/2} + (\mu^2 c^4 + k_i^2)^{1/2}$.

The first term in (73) is zero on the energy shell, the second term is represented by the set of graphs labelled ($\underline{\alpha}$) in the previous chapter, the third term by its symmetric ($\underline{\tilde{\alpha}}$) in $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. The "other terms" correspond to the graphs which we neglect according to Chapter III.

Taking as interaction Hamiltonian

$$H' = iG\,\bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha, \tag{74}$$

and writing $\varepsilon_2 = (\mu^2 c^4 + k_2^2)^{1/2}$; $E_{12} = [M^2c^4 + (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2]^{1/2}$, (75)

we have*

$$\left.\begin{aligned} &H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_1 m_c) = \\ &\frac{iG\sqrt{2}}{V^{1/2}}\,\frac{\hbar}{(2\varepsilon_2)^{1/2}}\left(\frac{M+E_{12}}{2E_{12}}\right)^{1/2}\left(\frac{M+E_1}{2E_1}\right)^{1/2}\left(u_f, \boldsymbol{\sigma}\left[\frac{\boldsymbol{k}_1+\boldsymbol{k}_2}{M+E_{12}} - \frac{\boldsymbol{k}_1}{M+E_1}\right]u_c\right), \end{aligned}\right\} \tag{76}$$

where u_f and u_c are the normalized spin wave functions of the nucleon in final and intermediate states, E_{12} is the final nucleon energy, V is as usual the volume of the normalizing box. For k_1 and $k_2 \ll M$, (76) reduces to

$$H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_1, m_c) = \frac{iG\hbar}{V^{1/2}}\,\frac{1}{2M}\,\frac{k_2}{\varepsilon_2^{1/2}}\,(u_f, \boldsymbol{\sigma}\boldsymbol{n}_2 u_c), \tag{77}$$

while, to the same approximation, the denominator in (73) reduces to

$$E - E_1 - \varepsilon_1 = E_i + \varepsilon_i - E_1 - \varepsilon_1 \approx \varepsilon_2. \tag{78}$$

For $K^{3/2}(\boldsymbol{k}_1, m_c;\ \boldsymbol{k}_i\, m_i)$ we have the expansion (66). Therefore

$$\left.\begin{aligned} &K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_i, m_i) = \frac{iG\hbar}{V^{1/2}}\,\frac{1}{2M} \\ &\times\left\{\frac{k_2}{\varepsilon_2^{3/2}}\left[\sum_{m_c}(u_f, \boldsymbol{\sigma}\boldsymbol{n}_2 u_c)\sum_{lJ}K_{lJ}^{3/2}(k_1, k_i)\sum_{M}\mathfrak{Y}_{Jl^{1/2}}^{M}(\boldsymbol{n}_1, m_c)\,\mathfrak{Y}_{Jl^{1/2}}^{M*}(\boldsymbol{n}_i, m_i)\right]\right. \\ &\left.\qquad + \text{sym. term } 1 \rightleftarrows 2\right\}. \end{aligned}\right\} \tag{79}$$

* We take units such that $c = 1$ in the following formulae.

2. From the *K*-matrix to the *T*-matrix.

The transition from the K-matrix to the T-matrix follows the pattern of Chapter II, § 4. With the same assumptions as in this paragraph ($|\boldsymbol{T}_{\beta\alpha}|$ small), (34) gives

$$\left.\begin{aligned} \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= \boldsymbol{K}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) \\ -i\pi \sum_{m_c} \int d\omega_c \boldsymbol{K}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_c, m_c) \boldsymbol{T}^{3/2}(\boldsymbol{k}_c, m_c; \boldsymbol{k}_i, m_i) \varrho, & \end{aligned}\right\} \quad (80)$$

where the density of the one-meson states in phase space is $\varrho\, d\omega$.

Expression (80), where the summation occurs only over the angles and the spin, is easily handled by using expansions (22) and (79), together with the orthonormality relation

$$\sum_{m_c} \int d\omega_c \mathcal{Y}^{M*}_{JI^{1/2}}(\boldsymbol{n}_c, m_c) \mathcal{Y}^{M'}_{J'I'^{1/2}}(\boldsymbol{n}_c, m_c) = \delta_{J,J'} \delta_{l,l'} \delta_{M,M'} \quad (81)$$

which follows directly from the definition (17) of $\mathcal{Y}$. The result is

$$\left.\begin{aligned} \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= \frac{iG\hbar}{V^{1/2}} \frac{1}{2M} \\ \times \left\{ \frac{k_2}{\varepsilon_2^{3/2}} \left[\sum_{m_c} (u_f, \sigma \boldsymbol{n}_2 u_c) \sum_{lJ} \mathfrak{X}^{3/2}_{lJ}(k_1, k_i) \sum_{M} \mathcal{Y}^{M}_{JI^{1/2}}(\boldsymbol{n}_1, m_c) \mathcal{Y}^{M*}_{JI^{1/2}}(\boldsymbol{n}_i, m_i) \right] \right. & \\ \left. + \text{Sym. term } 1 \rightleftarrows 2 \right\} & \end{aligned}\right\} \quad (82)$$

with

$$\mathfrak{X}^{(I)}_{lJ}(k_1, k_i) = [1 - i\pi\varrho\, T^I_{lJ}(k_i, k_i)]\, K^I_{lJ}(k_i, k_i) \quad (83\,a)$$

$$= \frac{K^I_{lJ}(k_1, k_i)}{K^I_{lJ}(k_i, k_i)} T^I_{lJ}(k_i, k_i) \quad (83\,b)$$

by virtue of (24). (82) and (83b) give the matrix element of the T-matrix (which is what we need) for the reaction in terms of the phase shifts for elastic scattering (via T^I_{lJ} and (25)) which are known from experiment, and also of the ratios $K^I_{lJ}(k_1, k_i)/K^I_{lJ}(k_i, k_i)$ which represent the change in the K-matrix element

for scattering when we proceed out of the energy shell*. These ratios are not experimentally measurable quantities. To get an estimate of their values one can approximate them by replacing the K's by their values in lowest order in G, i. e. by the B's (cf. Chapter III, eq. (70) and also Appendix I). Thus one gets

$$\frac{K^I_{0^1/_2}(k_1, k_i)}{K^I_{0^1/_2}(k_i, k_i)} \approx \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/_2}; \quad \frac{K^I_{1J}(k_1, k_i)}{K^I_{1J}(k_i, k_i)} \approx \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{3/_2} \frac{k_1}{k_i}. \tag{84}$$

3. Expression for the differential cross section in c. m. system.

The differential cross section for emission of one positive meson in the solid angle $d\omega_1$ with energy ε_1 and one positive meson in the solid angle $d\omega_2$ is, from (8),

$$d\sigma = \left(\frac{2\pi V}{\hbar v_i}\right) \cdot \left(\frac{1}{2} \sum_{m_i m_f} \left| \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) \right|^2\right) \cdot \varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1, \tag{85}$$

where v_i is the relative velocity and $\varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1$ is the density of the two-meson states in phase space. The calculation of the second factor in (85) is most easily done by first writing (82) in a somewhat different form which makes more apparent the invariance of (86) with respect to rotations of the coordinate system. In (82) we note the factors

$$X = \sum_{lJ} \mathfrak{X}^{(3/2)}_{lJ}(k_1, k_i) \sum_M \mathfrak{Y}^M_{Jl^1/_2}(\boldsymbol{n}_1, m_c)\, \mathfrak{Y}^{M*}_{Jl^1/_2} \boldsymbol{n}_i, m_i) \tag{86}$$

which are expansions of the same kind as (19), (22) or (66). It is shown in detail in Appendix II that such expressions can always be put in the form

$$\left.\begin{aligned} X = \sum_{lm} [U_l(k_1, k_i)\,(u_c, u_i) \\ + V_l(k_1, k_i)\,(u_c, (\sigma \cdot \boldsymbol{n}_1)\,(\sigma \cdot \boldsymbol{n}_i)\, u_i)]\; Y^m_l(\boldsymbol{n}_1)\; Y^{m*}_l(\boldsymbol{n}_i) \end{aligned}\right\} \tag{87}$$

with

$$U_l + V_{l+1} = \mathfrak{X}^{(3/2)}_{l,l+1/2}; \quad U_l + V_{l-1} = \mathfrak{X}^{(3/2)}_{l,l-1/2}. \tag{88}$$

* Cf. also B. d'Espagnat (1953).

It should be noted that (88) does not determine uniquely the U_l and V_l if the X's are known: we must add the requirement that U_l and V_l decrease with increasing l in the same way as do the X's themselves. One then gets the general formulae

$$\left.\begin{aligned} U_l &= \sum_{q=0}^{\infty} (\mathfrak{X}^{(3/2)}_{l+2q,\,l+2q+1/2} - \mathfrak{X}^{(3/2)}_{l+2q+2,\,l+2q+2-1/2}), \\ V_l &= \sum_{q=0}^{\infty} (\mathfrak{X}^{(3/2)}_{l+2q+1,\,l+2q+1-1/2} - \mathfrak{X}^{(3/2)}_{l+2q+1,\,l+2q+1+1/2}). \end{aligned}\right\} \tag{89}$$

Using (87) instead of (86) for X, the calculation of the second factor in (85) becomes quite trivial. Here, however, we shall give the explicit result only for the case where phase shifts of $l \geqslant 2$ are negligible compared to S and P phase shifts. This seems to be experimentally correct up to 135 Mev incident energy and we assume that, for the higher energies considered here, it is still approximately true. If this were contradicted by further experiments it would be easy to calculate (85) keeping $l = 2$.

The summation over m_c in (82) gives, with (87):

$$\left.\begin{aligned} \boldsymbol{T}^{++} = \frac{iG\hbar}{V^{1/2}} \frac{1}{2M} \Bigg\{ \frac{k_2}{\varepsilon_2^{3/2}} \sum_{lm} \Big[& U_l(k_1, k_i)\,(u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,u_i) \\ + V_l(k_1, k_i)\,(u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)\,u_i) \Big] & Y_l^m(\boldsymbol{n}_1)\,Y_l^{m*}(\boldsymbol{n}_i) \\ + \text{Sym. term } 1 \rightleftarrows 2 \Bigg\}, & \end{aligned}\right\} \tag{90}$$

while, on our assumption, (89) reduce to

$$U_0 = \mathfrak{X}^{(3/2)}_{0\,1/2}; \quad U_1 = \mathfrak{X}^{(3/2)}_{1\,3/2}; \quad V_0 = \mathfrak{X}^{(3/2)}_{1\,1/2} - \mathfrak{X}^{(3/2)}_{1\,3/2}, \tag{91}$$

other U_l and V_l being zero.

Now, by using (83b), (84) and (91), one sees that the coefficients of $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)$ and of $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)$, which appear in the symmetric term, are equal; therefore, using the identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1) + (\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2) = 2\,(\boldsymbol{n}_1\cdot\boldsymbol{n}_2), \tag{92}$$

(90) reduces to

$$\left.\begin{aligned} \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= [\lambda_2 + \mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)](\boldsymbol{\sigma} \cdot \boldsymbol{n}_2) \\ &+ [\lambda_1 + \mu(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)](\boldsymbol{\sigma} \cdot \boldsymbol{n}_1) + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{\sigma} \cdot \boldsymbol{n}_i) \end{aligned}\right\} \quad (93)$$

with

$$\lambda_2 = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{k_2}{\varepsilon_2^{3/2}} \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2} \frac{1}{2\pi\varrho} e_3, \quad (94\,\text{a})$$

$$\lambda_1 = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{k_1}{\varepsilon_1^{3/2}} \left(\frac{\varepsilon_i}{\varepsilon_2}\right)^{1/2} \frac{1}{2\pi\varrho} e_3, \quad (94\,\text{b})$$

$$\mu = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{3\,k_1 k_2}{(\varepsilon_1 \varepsilon_2)^{3/2}} \frac{\varepsilon_i^{3/2}}{k_i} \frac{1}{2\pi\varrho} e_{33}, \quad (94\,\text{c})$$

$$\nu = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{2\,k_1 k_2}{(\varepsilon_1 \varepsilon_2)^{3/2}} \frac{\varepsilon_i^{3/2}}{k_i} \frac{1}{2\pi\varrho} (e_{31} - e_{33}). \quad (94\,\text{d})$$

In (94 a, b, c, d) the notations (Anderson, Fermi, Martin, and Nagle, 1953):

$$\left.\begin{aligned} e_3 &= e^{2i\delta^{3/2}_{01/2}} - 1 \equiv -2\pi i\varrho\, T^{3/2}_{01/2}; \\ e_{3p} &= e^{2i\delta^{3/2}_{1p/2}} - 1 \equiv -2i\pi\varrho\, T^{3/2}_{1p/2};\ p = 1; 3 \end{aligned}\right\} \quad (95)$$

have been used.

Then

$$\left.\begin{aligned} &\frac{1}{2} \sum_{m_i m_f} \left| \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_i\, m_i) \right|^2 \\ &= \left| \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \mu\,[(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)\boldsymbol{n}_1 + (\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)\boldsymbol{n}_2] + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\boldsymbol{n}_i \right|^2. \end{aligned}\right\} \quad (96)$$

(96) and (85) give the required differential cross section for the process.

V. Results and discussion.

1. Total cross section.

We call, as before, $\varrho\, d\omega$ the density in phase space of the $1\,\pi$, $1\,\mathfrak{N}$ states (elastic scattering); $d\omega$ is the element of solid angle into which the meson is scattered. Similarly, let us call

$$\varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1$$

the density in phase space of the states in which two mesons are present (meson production). The energy of one of the mesons is between ε_1 and $\varepsilon_1 + d\varepsilon_1$ and its momentum is contained in the solid angle $d\omega_1$; the momentum of the second meson is contained in $d\omega_2$; the energy of the second meson is thereby fixed by the conservation laws.

The total cross section $\sigma^{+\,+}$ for two positive outgoing mesons is given by (85) integrated over $d\varepsilon_1$, $d\omega_1$, $d\omega_2$. Similarly, the total cross section $\sigma_{\text{el.}}$ for positive meson scattering is given by

$$\sigma_{\text{el.}} = \left(\frac{2\pi V}{\hbar v_i}\right) \cdot \int d\omega \frac{1}{2} \sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i, m_i) \right|^2 \varrho\,. \tag{97}$$

To perform the summation over spins one can, in analogy to the procedure of Chapter IV § 3, proceed as follows. First expand $\boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i, m_i)$ by formula (22) and use the above mentioned equivalence between expansions (86) and (87): this, using

$$\sum_m Y_l^m(\boldsymbol{n})\, Y_l^m(\boldsymbol{n}_i) = \frac{2l+1}{4\pi} P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i), \tag{98}$$

gives

$$\left.\begin{aligned} &\boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f;\ \boldsymbol{k}_i, m_i) \\ &= \left(u_f, \sum_l \left[U_l' + V_l'(\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)\right] \frac{2l+1}{4\pi} P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i)\, u_i\right), \end{aligned}\right\} \tag{99}$$

U_l' and V_l' being given in terms of $T_{lJ}^{3/2}$ by formula (89) (with $\mathfrak{T}$ replaced by T). In the case where only the S and P phase shifts are important, these formulae reduce to

$$\left.\begin{aligned} U_0' = T^{3/2}_{0^{1/2}} = (2\pi\varrho)^{-1} ie_3;\ \ U_1' = T^{3/2}_{1^{3/2}} = (2\pi\varrho)^{-1} ie_{33};\\ V_0' = T^{3/2}_{1^{1/2}} - T^{3/2}_{1^{3/2}} = (2\pi\varrho)^{-1} i(e_{31} - e_{33}), \end{aligned}\right\} \quad (100)$$

others being zero.
(99) now gives

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\cdots)\right|^2 \\ &\frac{1}{2}\,\mathrm{Trace}\left\{\left[\sum_l [\cdots]\frac{2l+1}{4\pi}P_l(\cdots)\right]\cdot\left[\sum_l [\cdots]^*\frac{2l+1}{4\pi}P_l(\cdots)\right]\right\}. \end{aligned}\right\} \quad (101)$$

Multiplying in the braces by $(\sigma\cdot\boldsymbol{n})^2 \equiv 1$, this simplifies to

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f;\ \boldsymbol{k}_i, m_i)\right|^2 \\ &= \left[\sum_l] U_l'\boldsymbol{n} + V_l'\boldsymbol{n}_i]\frac{2l+1}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i)\right] \\ &\times\left[\sum_l [U_l'^*\boldsymbol{n} + V_l'^*\boldsymbol{n}_i]\frac{2l+1}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i)\right]; \end{aligned}\right\} \quad (102)$$

in the case where only S and P phase shifts are important (100) and (102) give*

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i m_i)\right|^2 \\ &= (8\pi^2\varrho)^{-2}\left|[e_3 + 3(\boldsymbol{n}\cdot\boldsymbol{n}_i)e_{33}]\boldsymbol{n} + (e_{31} - e_{33})\boldsymbol{n}_i\right|^2. \end{aligned}\right\} \quad (103)$$

From now on we shall assume that, as seems likely in view of present experimental evidence, the S and $P_{1/2}$ phase shifts can also be neglected compared to the $P_{3/2}$ phase shift. (85) and (97) then give, with the help of (96), (94), (103) and (25),

* The reader will have no difficulty in verifying that (103) is a compact expression entirely equivalent to the expressions (12) of Anderson, Fermi, Martin, and Nagle (1953).

$$\left.\begin{aligned} \frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} &= \frac{G^2\hbar^2}{V\cdot 4M^2} \\ \frac{\int d\varepsilon_1 d\omega_1 d\omega_2 \frac{\varepsilon_i^3}{k_i^2}\frac{k_1^2 k_2^2}{\varepsilon_1^3\varepsilon_2^3}[3(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)\boldsymbol{n}_1+3(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)\boldsymbol{n}_2-2(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\boldsymbol{n}_i]^2\varrho_{12}}{\varrho\int d\omega\,[3(\boldsymbol{n}\cdot\boldsymbol{n}_i)\boldsymbol{n}-\boldsymbol{n}_i]^2}. \end{aligned}\right\} \quad (104)$$

In (104) the density of states ϱ_{12} is, because of the conservation of momentum, a complicated function of the directions of the two emitted mesons. If, however, following FERMI (1950), we use the approximation that the meson mass is much smaller than the nucleon mass, we can restrict the conservation of momentum to the nucleon only (cf. also, β-decay theory). Then,

$$\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2 = \frac{V^2}{(2\pi\hbar)^6 c^4} k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1\,d\omega_1\,d\omega_2. \quad (105)$$

Performing now the integrations over the angles in (104) and defining a "mean value" $\langle k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\rangle$ by

$$\langle k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\rangle = \int k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\cdot k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1 \Big/ \int k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1 \quad (106)$$

(both integrations performed on the energy shell: ε_2 is a function of ε_1) we have

$$\frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} = \frac{G^2\hbar^2}{V\cdot 4M^2}\,\frac{10}{3}\,\frac{\varepsilon_i^3}{k_i^2}\Big/\frac{k_1^2k_2^2}{\varepsilon_1^3\varepsilon_2^3}\Big\backslash \frac{\int\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2}{\varrho\int d\omega}. \quad (107)$$

(107) has been given a form which makes very easy a comparison with the Fermi approach (FERMI, 1950). In fact Fermi's $S(1, 2)$, $S(1, 1)$ (whose ratio should, in a Fermi approach, be equal to $\sigma_{++}/\sigma_{\mathrm{el.}}$) can be written*

$$S(1,1) = \Omega V^{-1}\int\varrho\,d\omega;\quad S(1,2) = \Omega^2 V^{-2}\int\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2. \quad (108)$$

It may be seen that, also in the present theory, one may write

$$\frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} = \frac{S(1,2)}{S(1,1)} \quad (109)$$

* $S(s, n)$ is the statistical weight for the emergence of s nucleons and n mesons, Ω being the interaction volume.

with the same definition for $S(1,2)$ and $S(1,1)$, provided Ω is defined by

$$\Omega = \frac{G^2 \hbar^2}{4 M^2} \frac{10}{3} \frac{\varepsilon_i^3}{k_i^2} \left\langle \frac{k_1^2 k_2^2}{\varepsilon_1^3 \varepsilon_2^3} \right\rangle . \tag{110}$$

It will be readily verified that this Ω is of roughly the same order of magnitude as Fermi's Ω; its dependence on the energy is not, however, exactly that which one would expect on the basis of the Fermi statistical approach: if, namely, we assumed that the statistical method proposed by FERMI for nucleon-nucleon collisions is also valid for meson-nucleon collisions, we should apparently be led to an Ω inversely proportional to the total energy (including rest mass) of the nucleon (Lorentz contraction of the nucleon). Instead, we have for Ω the expression (110) with (106). To evaluate (106) we make, according to Fermi, the extreme relativistic approximation for the mesons, ($\varepsilon_1 \approx c\, k_1$, $\varepsilon_2 \approx c\, k_2$). (106) then gives

$$\left\langle \frac{k_1^2 k_2^2}{\varepsilon_1^3 \varepsilon_2^3} \right\rangle \approx c^{-4} \int_0^T (T - \varepsilon_1)\, \varepsilon_1\, d\varepsilon_1 \Big/ \int_0^T (T - \varepsilon_1)^2\, \varepsilon_1^2\, d\varepsilon_1, \tag{111}$$

T being the total kinetic energy of the meson-nucleon system in the centre of mass system before collision:

$$T = \varepsilon_i - \mu c^2 + E_i - M c^2 \approx \varepsilon_i - \mu c^2 \quad \text{if} \quad \varepsilon_i \ll M c^2 \tag{112}$$

$$\left\langle \frac{k_1^2 k_2^2}{\varepsilon_1^3 \varepsilon_2^3} \right\rangle \approx 5\, c^{-4} T^{-2}. \tag{113}$$

Therefore (in a very crude approximation, however),

$$\Omega \sim \frac{G^2}{4 \pi \hbar c} \cdot \frac{50}{4} \left(\frac{\mu}{M} \right)^2 \cdot \frac{4 \pi}{3} \left(\frac{\hbar}{\mu c} \right)^3 \cdot \frac{\mu c^2}{T}. \tag{114}$$

(114) shows that the energy dependence of Ω is roughly T^{-1}, T being the energy of the incident meson. In conclusion, the ratio $\sigma_{++}/\sigma_{\text{el.}}$ predicted by our theory agrees roughly with

that which the Fermi statistical theory would give if it were applied to this problem, as far as orders of magnitude are concerned. However, our theory predicts an increase in the ratio $\sigma_{++}/\sigma_{\text{el.}}$ with increasing energy somewhat slower than that which the pure statistical approach would give, this being due to the more rapid decrease of Ω.

2. Angular distribution of emitted mesons.

The angular distribution is given by (96). Here also we first write down the simplified formulae obtained by neglecting all phase shifts except $P_{3/2}$: the relative probability that one π^+ should be emitted in the solid angle $d\omega_1$, and the other π^+ in the solid angle $d\omega_2$ is then, in the c.m. system,

$$\left[(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)\,\boldsymbol{n}_1 + (\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)\,\boldsymbol{n}_2 - \frac{2}{3}(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\,\boldsymbol{n}_i\right]^2 d\omega_1\, d\omega_2, \qquad (115)$$

$\boldsymbol{n}_i$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ being unit vectors in the directions of the incident and the two emergent π^+. This formula shows that

a) the distribution admits of a centre of symmetry, and therefore is also symmetrical with respect to a plane perpendicular to the direction of the incident meson (change $\boldsymbol{n}_1$ in $-\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ in $-\boldsymbol{n}_2$);

b) the distribution of one emergent meson, the direction of the other being given, also possesses property a) (change $\boldsymbol{n}_1$ in $-\boldsymbol{n}_1$ *or* $\boldsymbol{n}_2$ in $-\boldsymbol{n}_2$);

c) the mesons have a tendency to be emitted in a direction collinear with the direction $\boldsymbol{n}_i$ of the incident meson: $\pm\boldsymbol{n}_1 = \pm\boldsymbol{n}_2 = \boldsymbol{n}_i$.

Properties a) b) c) are expected to hold only approximately for they are valid only if the S and D ect. phase shifts are negligible.

Let us choose the z-axis in the direction of $\boldsymbol{n}_i$ and call θ_1 and θ_2 the angles between the direction $\boldsymbol{n}_i$ of the incident π^+ and the directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ of each of the emitted π^+. Then, the relative probability that θ_1 lies between θ_1 and $\theta_1 + d\theta_1$ and that simultaneously θ_2 lies between θ_2 and $\theta_2 + d\theta_2$, irrespective of the angle that both emitted mesons make with each other, is

$$\left.\begin{aligned} & P_r\{\theta_1, \theta_2\}\, d\theta_1\, d\theta_2 \\ &= \left[1 + \frac{7}{2}(\cos^2\theta_1 + \cos^2\theta_2)\right] \sin\theta_1\, d\theta_1 \cdot \sin\theta_2\, d\theta_2 \end{aligned}\right\} \quad (116)$$

and the angular distribution of one meson, irrespective of the direction of the other, is

$$P_r\{\theta\}\, d\theta = (13 + 21\cos^2\theta)\sin\theta\, d\theta. \quad (117)$$

It should be noted that these angular distributions are independent of the distribution of kinetic energy between the two emergent mesons.

These simple results, however, are only valid if all the other $T_{LJ}^{3/2}$ are negligibly small compared to $T_{1^3/_2}^{3/_2}$, i.e. if the sines of all the other phase-shifts are negligibly small compared to sin $\delta_{1^3/_2}^{3/_2}$. At the energies we are considering this may well not be the case: for instance, $\delta_{1^3/_2}^{3/_2}$ may go on increasing with increasing energy and cross the value $\pi/2$ (resonance); then sin $\delta_{1^3/_2}^{3/_2}$ starts decreasing while other sin $\delta_{LJ}^{3/_2}$ probably still increase — or, possibly, $\delta_{1^3/_2}^{3/_2}$ goes through a maximum and then decreases before having attained the value $\pi/2$. On account of these possibilities we quote here the general formula that one gets if one keeps the S, $P_{1/_2}$ and $P_{3/_2}$ phase shifts. We use the notation e_3, e_{31}, e_{33} of ANDERSON, FERMI, MARTIN and NAGLE (1953) (see chapter IV, eq. (95)).

Then, the relative probability that θ_1 lies between θ_1 and $\theta_1 + d\theta_1$ and that, simultaneously, θ_2 lies between θ_2 and $\theta_2 + d\theta_2$, irrespective of the angle that both emitted mesons make with each other, is

$$\left.\begin{aligned} & P_r\{\theta_1, \theta_2\}\, d\theta_1\, d\theta_2 \\ &= \{|a_2|^2 + |a_1|^2 + 2|e_{31} - e_{33}|^2 \\ &\quad + [a_2 e_{33}^* + a_2^* e_{33}]\, 3\cos\theta_1 \\ &\quad + [a_1 e_{33}^* + a_1^* e_{33}]\, 3\cos\theta_2 \\ &\quad + [9|e_{33}|^2 - 2|e_{31} - e_{33}|^2](\cos^2\theta_1 + \cos^2\theta_2) \\ &\quad + [a_1 (e_{33}^* + 2 e_{31}^*) + a_1^* (e_{33} + 2 e_{31})]\cos^2\theta_1 \cos\theta_2 \\ &\quad + [a_2 (e_{33}^* + 2 e_{31}^*) + a_2^* (e_{33} + 2 e_{31})]\cos\theta_1 \cos^2\theta_2 \\ &\quad + [|e_{31}|^2 + e_{31}^* e_{33} + e_{31} e_{33}^*]\, 6\cos^2\theta_1 \cos^2\theta_2\} \sin\theta_1\, d\theta_1 \cdot \sin\theta_2\, d\theta_2, \end{aligned}\right\} \quad (118)$$

where

$$a_1 = \frac{k_i}{\varepsilon_i}\frac{\varepsilon_2}{k_2}e_3; \quad a_2 = \frac{k_i}{\varepsilon_i}\frac{\varepsilon_1}{k_1}e_3, \tag{119}$$

ε_i, ε_1, ε_2 being as before the energies of the incident and emergent mesons in the c.m. system. The D phase shifts have not been included in formula (118), which should therefore be corrected if the D phase shifts happen not to be negligible at our energies.

3. Angular distribution when no field approximation is used.

We may first summarize the foregoing investigations as follows. An attempt has been made to study the inelastic meson-nucleon collisions that give rise to two emergent mesons, paying particular attention to the reaction in which a π^+ and a proton give rise to a neutron and two π^+: this process has, compared with the others, some simplifying features which are used in part of the present theory.

In Chapter II a rather general approach is developed that does not involve any field theoretical approximation, with the aim of throwing some light on analogies that may exist between such processes and ordinary nuclear reactions. Qualitatively the main conclusions of this chapter are:

a) The "damping" (i. e. the transition from the reaction matrix to the scattering matrix) does *not* appear to disfavour meson production as compared to elastic scattering; in fact, both processes seem to be damped in almost the same proportion, at least at energies where meson production is relatively small (cf. eq. (65)).

b) An analogy with nuclear reactions seems to require that the reaction matrix elements for meson production should be roughly proportional to the corresponding reaction matrix elements for elastic scattering, i. e. that the latter should become large at precisely the same energies for which the former happen to become large.

Since the approach of Chapter II is still somewhat formal we have supplemented it with a treatment involving some approximation of the field theory: this forms the subject of Chapters III and IV. In Chapter III the approximations are stated and discussed; although they appear rather natural, and less drastic

than the approximations of, e. g., the Tamm-Dancoff method*, they should nevertheless be considered a working hypothesis, as their reliability cannot, of course, be proved in any rigorous way. In particular, the emission process enters in the theory in a way that somewhat recalls weak-coupling approximations: this is certainly a not quite satisfactory feature of the theory and, consequently, the numerical values of the expressions containing explicitly the coupling constant G should not be taken too seriously. This applies especially to the total cross-section: formulae (108), (109), (110) . . . . . should be considered as giving nothing more than orders of magnitude.

Therefore, the question arises: "are there any results of the theory that can be considered to be fairly independent of the field theoretical approximation of Chapter III?" We show that, indeed, the results concerning the differential cross section (angular distributions) are likely to hold, even if the approximations of Chapter III appeared to be too crude.

To this purpose we want to restrict our investigations to the case in which both π^+ mesons are emitted with approximately equal energies in the c.m. system. Then, the most general pseudoscalar that can be formed with the three vectors $\boldsymbol{k}_i$, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and the pseudovector $\boldsymbol{\sigma}$, and that is symmetric with respect to interchange of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ (with $k_1 = k_2$), is

$$\left.\begin{aligned} &(\boldsymbol{\sigma}\cdot\boldsymbol{k}_2)\, f[(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)] \\ &+(\boldsymbol{\sigma}\cdot\boldsymbol{k}_1)\, f[(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)] \\ &+(\boldsymbol{\sigma}\cdot\boldsymbol{k}_i)\, g\,[(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)], \end{aligned}\right\} \qquad (120)$$

f and g being two functions of their three arguments. Now, suppose that f is expanded in a sum of products of three Legendre polynomials $P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)\cdot P_{l'}(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)\cdot P_{l''}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)$. We express $P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)$ as a sum of products of $Y_l^{m*}(\boldsymbol{n}_1)$ and $Y_l^m(\boldsymbol{n}_i)$, and similarly with $P_{l'}(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)$, and, finally, the products $Y_l^m(\boldsymbol{n}_i)\, Y_{l'}^{m'}(\boldsymbol{n}_i)$ we express as a sum of terms involving only one $Y_L^M(\boldsymbol{n}_i)$ with $|l-l'| \leqslant L \leqslant l+l'$. Now we make the rather natural assumption that, at not too high energies, only incident

* Some a posteriori justification for these approximations may be found in the fact that they yield the same result (b) as the theory of Chapter II.

waves with small angular momenta, say $L \leqslant 1$, take a significant part in the process: this gives $l + l' \leqslant 1$.

The same assumption, made on the emergent meson waves, gives $l' + l'' \leqslant 1$, $l'' + l \leqslant 1$; therefore, either $l = l' = l'' = 0$ or one only of l, l' and l'' is 1, both others being zero, which gives

$$f = a + b\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i) + c\,(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i) + d\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2), \tag{121}$$

a, b, c, d being some functions of $k_1 = k_2$ and k_i. Similarly,

$$g = \alpha + \beta\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i) + \gamma\,(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i) + \delta\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2). \tag{122}$$

Now it is easy to see by a similar argumentation that $(\boldsymbol{\sigma} \cdot \boldsymbol{n}_2)$ $(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)$ involves in fact spherical harmonics of $\boldsymbol{n}_2$ of order two, $Y_2^M(\boldsymbol{n}_2)$, which we must also reject: therefore $c = 0$ and, similarly, $d = 0$, $\beta = 0$, $\gamma = 0$. Finally, we are left with

$$\left.\begin{aligned} &(\boldsymbol{\sigma} \cdot \boldsymbol{n}_2)\,[a + b\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)] \\ &+ (\boldsymbol{\sigma} \cdot \boldsymbol{n}_1)\,[a + b\,(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)] + (\boldsymbol{\sigma} \cdot \boldsymbol{n}_i)\,[\alpha + \delta\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)]. \end{aligned}\right\} \tag{123}$$

Except for the term $\alpha\,(\boldsymbol{\sigma} \cdot \boldsymbol{n}_i)$, this is just the expression (93) with

$$\lambda_1 = \lambda_2 = a; \quad \mu = b; \quad \nu = \delta. \tag{124}$$

Working backward from expression (93) one then sees that the matrix element of the T-matrix can quite generally be put into the form exhibited by the square bracket in (82), plus a term proportional to $(\boldsymbol{\sigma} \cdot \boldsymbol{k}_i)$.

This result can also be obtained by starting from expansion (62) and limiting the possible values of l, l_1, l_2 to small numbers, say 0 and 1. The possible combinations of l, J, L, l_1, l_2 are then, on account of (61) and the symmetry between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$,

$$\begin{aligned} &l = 0 \quad J = 1/2 \quad L = 1 \qquad l_1 = 0, l_2 = 1 \text{ or } l_1 = 1, l_2 = 0 \\ &l = 1 \quad J = 1/2 \quad L = 0 \qquad \begin{cases} l_1 = l_2 = 0 \\ l_1 = l_2 = 1 \end{cases} \\ &l = 1 \quad J = 3/2 \quad L = 2 \qquad l_1 = l_2 = 1. \end{aligned}$$

We now express the function $\Phi^M_{l_1 l_2(L)J}$ appearing in (62) in terms of the functions $\Psi^M_{l_1, l_2 s(j), J}$ when l_2 and the spin s are coupled together to give j, and then l_1 and j coupled together to give J (RACAH, 1943). This gives

$$\left.\begin{aligned} -\left(\frac{4}{3}\right)^{1/2} \Phi^M_{11(0)^1/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) \\ = \Psi^M_{1,1^1/_2(^1/_2),^1/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) + \Psi^M_{1,1^1/_2(^1/_2),^1/_2}(\boldsymbol{n}_2, \boldsymbol{n}_1, m_f); \\ \left(\frac{10}{3}\right)^{1/2} \Phi^M_{11(2)^3/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) \\ = \Psi_{1,1^1/_2(^1/_2),^3/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) + \Psi^M_{1,1^1/_2(^1/_2),^3/_2}(\boldsymbol{n}_2, \boldsymbol{n}_1, m_f). \end{aligned}\right\} \quad (124)$$

Moreover, it is easily shown that

$$\Psi^M_{1,1^1/_2(^1/_2),J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) = -(4\pi)^{-1/2} \sum_{m'} (u_f, (\boldsymbol{\sigma} \cdot \boldsymbol{n}_2)\, u')\, \mathcal{Y}^M_{J1^1/_2}(\boldsymbol{n}_1, m'). \quad (125)$$

Use of the general formula of Appendix II then finally gives a general expression for the cross section which includes no approximation, except the neglect of D etc. waves*, and which is

$$\left.\begin{aligned} d\sigma = \frac{2\pi V_i}{\hbar v} \frac{1}{(4\pi)^3} \big| a_2 \boldsymbol{n}_2 + a_1 \boldsymbol{n}_1 + \alpha \cdot \boldsymbol{n}_i + b\,[(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)\boldsymbol{n}_2 + (\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)\boldsymbol{n}_1] \\ + \delta\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\,\boldsymbol{n}_i \big|^2 \varrho_{12}\, d\varepsilon_1\, d\omega_1\, d\omega_2, \end{aligned}\right\} \quad (126)$$

where a_1, a_2, α, b, δ are functions of the $T_{lJ\, l_1 l_2}$.

If one writes

$$\frac{T_{lJl_1l_2}(k_1, k_2; k_i) + T_{lJl_2l_1}(k_2, k_1; k_i)}{T_{lJ}(k_i; k_i)} = c_{lJl_1l_2}(k_1, k_2, k_i) \quad (127)$$

and

$$\frac{\sigma^{++}_{lJ}}{\sigma_{lJ\,\mathrm{el.}}} = \frac{1}{\varrho} \sum_{l_1 l_2} \int (c_{lJl_1l_2})^2 \varrho_{12}\, d\varepsilon_1, \quad (128)$$

the integration being performed on the energy shell. In (128), σ^{++}_{lJ} is the contribution to σ^{++} arising from the incident l, J wave. If the approach of Chapter II is valid, then the functions $c_{lJl_1l_2}(k_1, k_2, k_i)$ are real and smoothly varying.

* See, however, text following eq. (61), p. 18.

The calculations leading to (126) yield

$$\left.\begin{aligned}
2\pi i\varrho\cdot a_1 &= e_3\, c_{0\,1/2\,1\,0}(k_1,k_2,k_i)\\
2\pi i\varrho\cdot a_1 &= e_3\, c_{0\,1/2\,0\,1}(k_1,k_2,k_i)\\
2\pi i\varrho\cdot \alpha &= e_3\, c_{0\,1/2\,0\,0}(k_1,k_2,k_i)\\
2\pi i\varrho\cdot b &= 3\left(\frac{3}{10}\right)^{1/2} e_{33}\, c_{1\,3/2\,1\,1}(k_1,k_2,k_i)\\
2\pi i\varrho\cdot \delta &= -\frac{2}{3}(2\pi i\varrho\cdot b)-3^{1/2} e_{31} c_{1\,1/2\,1\,1}(k_1,k_2,k_i).
\end{aligned}\right\} \quad (129)$$

It is easily seen that formula (96), giving the angular distribution derived from the field theoretical treatment, is a special case of (126). Whether it is a good approximation or not depends largely on whether the $\sigma_{IJ}^{++}/\sigma_{IJ\,\mathrm{el.}}$ are all of the same order of magnitude or not. If only the $P_{3/2}$ wave is significant, then again (116) and (117) follow; these equations therefore do not depend on the approximation of Chapter III.

4. Concluding remarks.

(126), (128), (129) seem to yield a convenient and mathematically consistent formalism for the analysis of the phenomena where two positive mesons emerge at not too high energies and a possibility of connecting them with elastic scattering data; moreover, it could be extended to all meson-nucleon collisions in which two mesons emerge with only minor modifications.

As regards the total cross section, the situation is less satisfactory, though it seems that a statistical approach of the Fermi type might yield correct orders of magnitude. It is, however, quite possible that the value of Ω should be changed when processes involving more than two emergent mesons are taken into consideration.

The present investigation has been carried out during my stay with the Theoretical Study Group of CERN, and I have greatly benefited from frequent discussions with the other members of the Group as well as with members and guests of the

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Appendix I.

Formulation of the Tamm-Dancoff method using the reaction matrix.

Since the basic assumption of the Tamm-Dancoff method is not very reliable, even for elastic scattering problems, it was felt necessary to avoid the use of this method in the main body of this paper. However, for reference, a short summary of the Tamm-Dancoff method applied to the K-matrix is given in the following.

Let us call $(p\,\pi, q\,\mathfrak{N})$ a state containing p mesons and q nucleons. The basic assumption of the (lowest order) Tamm-Dancoff method applied to meson scattering is that only states $(1\,\pi, 1\,\mathfrak{N})$, $(0\,\pi, 1\,\mathfrak{N})$, $(2\,\pi, 1\,\mathfrak{N})$, $(0\,\pi, 3\,\mathfrak{N})$ $(2\,\pi, 3\,\mathfrak{N})$ interact directly with each other (cf. Fig. (I. 1)).

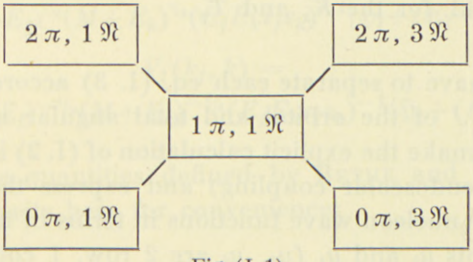

Fig. (I. 1).

Write eq. (10) of the text choosing for (a) a $(1\,\pi, 1\,\mathfrak{N})$ state and for (b), successively, each of the five kinds of states just mentioned; then, between these five equations eliminate all K_{ba} except those where (b) is also a $(1\,\pi, 1\,\mathfrak{N})$ state. The result is, (a), (b), (c) being $(1\,\pi, 1\,\mathfrak{N})$ states,

$$K_{ba} = B_{ba} + P \sum_{(a)} \frac{B_{bc}\, K_{ca}}{E_{(c)} - E_{(a)}} + \lambda K_{ba}, \qquad \text{(I. 1)}$$

where λ is an infinite quantity of the nature of a renormalization which (without much justification) we discard. B_{ba} is given by

$$B_{ba} = \sum_{(d)} \frac{H'_{bd} H'_{da}}{E_{(a)} - E_{(d)}}. \tag{I. 2}$$

Denote by $\boldsymbol{B}_{ba}$ the quantity B_{ba} for $E_{(a)} = E_{(b)}$; then $\boldsymbol{B}_{ba}$ is just the ordinary lowest order matrix element for scattering.

The next step is the separation of (I. 1) according to total isotopic spin I = 1/2 or 3/2. This is easily done with the help of expansions such as (20), with the result that

$$K^I_{ba} = B^I_{ba} + P \sum_{(c)} \frac{B^I_{bc} K^I_{ca}}{E_{(a)} - E_{(c)}} \tag{I. 3}$$

$$\left.\begin{aligned} &B(\pi^+ + P \to \pi^+ + P) = B^{3/2} \\ &B(\pi^- + P \to \pi^- + P) = \frac{1}{3} B^{3/2} + \frac{2}{3} B^{1/2} \\ &B(\pi^- + P \to \pi^0 + N) = \frac{\sqrt{2}}{3} (B^{3/2} - B^{1/2}) \\ &\text{id. for the } K'_s \text{ and } T'_s. \end{aligned}\right\} \tag{I. 4}$$

We now have to separate each eq. (I. 3) according to eigenvalues l and J of the orbital and total angular momenta. For this purpose, make the explicit calculation of (I. 2) in c.m. system (assuming pseudoscalar coupling) and express the small components of the nucleon wave functions in terms of the normalized big components u_i and u_f (u_i, u_f are 2 row, 1 column-matrices $|u_i|^2 = |u_f|^2 = 1$). The result is

$$B^I_{fi} = f[k_f, k_i, (\boldsymbol{n}_f \cdot \boldsymbol{n}_i)]\,(u_f, u_i) + g[k_f, k_i, (\boldsymbol{n}_f \cdot \boldsymbol{n}_i)]\,(u_f, (\sigma \cdot \boldsymbol{n}_f)(\sigma \cdot \boldsymbol{n}_i)\, u_i),$$

k_i, k_f being the magnitude of the relative momenta in initial and final states (not necessarily of the same total energy), $\boldsymbol{n}_i$, $\boldsymbol{n}_f$ their directions (unit vectors), f and g two known expressions. f and g can both be expanded in Legendre polynomials $P_l(\boldsymbol{n}_f \cdot \boldsymbol{n}_i) = 4\pi(2l+1)^{-1} \sum_m Y^m_l(\boldsymbol{n}_f)\, Y^{m*}_l(\boldsymbol{n}_i)$. Finally, one has

$$B^I_{fi} = \sum_{lm} [U_l \cdot (u_f, u_i) + V_l \cdot (u_f, (\sigma \cdot \boldsymbol{n}_f)(\sigma \cdot \boldsymbol{n}_i) u_i)] \, Y^m_l(\boldsymbol{n}_f) \, Y^{m*}_l(\boldsymbol{n}_i) \qquad \text{(I 5)}$$

which, using the result of Appendix II, can be written

$$B^I_{fi} = \sum_{lJ} B^I_{lJ}(k_f, k_i) \sum_M \mathcal{Y}^M_{Jl_{1/2}}(\boldsymbol{n}_f, m_f) \, \mathcal{Y}^{M*}_{Jl_{1/2}}(\boldsymbol{n}_i, m_i) \qquad \text{(I. 6)}$$

with

$$B^I_{l,l\pm 1/2}(k_f, k_i) = U_l(k_f, k_i) + V_{l\pm 1}(k_f, k_i). \qquad \text{(I. 7)}$$

Introducing (I. 6) and (21) in (I. 3) gives a one variable integral equation for each K^I_{lJ} (k_i is a parameter):

$$K^I_{lJ}(k_f, k_i) = B^I_{lJ}(k_f, k_i) + P \int \frac{V k^2 \, dk}{(2\pi\hbar)^3} \frac{B^I_{lJ}(k_f, k) \, K^I_{lJ}(k, k_i)}{E_k + \varepsilon_k - E_i - \varepsilon_i} \qquad \text{(I. 8)}$$

with $\varepsilon_k = (\mu^2 c^4 + k^2)^{1/2}$; $E_k = (M^2 c^4 + k^2)^{1/2}$; $\varepsilon_i \equiv \varepsilon_{k_i}$; $E_i \equiv E_{k_i}$; V being the volume of the normalizing box. Only the equations with $I = {}^3/_2$ can have a meaning (with $I = {}^1/_2$ they have no finite solution). For $I = {}^3/_2$ the actual expressions for U_l and V_l are (we take $c = 1$):

$$\left.\begin{aligned} U_l(k_f, k) = \\ G^2 \hbar^2 \pi V^{-1} (M + E_f)^{1/2} (M + E_k)^{1/2} (E_f E_k \varepsilon_f \varepsilon_k)^{-1} [\mathfrak{H}_l + (E_f + E_k - M) \mathfrak{k}_l] \\ V_l(k_f, k) = \\ G^2 \hbar^2 \pi V^{-1} k_f k (M + E_f)^{-1/2} (M + E_k)^{-1/2} (E_f E_k \varepsilon_f \varepsilon_k)^{-1} [\mathfrak{H}_l + (E_f + E_k + M) \mathfrak{k}_l], \end{aligned}\right\} \text{(I 9)}$$

$\mathfrak{H}$ and $\mathfrak{k}$ being quantities defined by Bethe and Dyson (1953) which we rewrite here for convenience:

$$\left.\begin{aligned} \mathfrak{H}_l &= \frac{1}{2\bar{E} r} \int_{1-r}^{1+r} \left(\frac{c}{c+z} - \frac{b}{b+z} \right) P_l\left(\frac{z^2 - 1 - r^2}{2r} \right) dz \\ \mathfrak{k}_l &= \frac{1}{2\bar{E}^2 r} \int_{1-r}^{1+r} \left(\frac{1}{c+z} + \frac{1}{b+z} \right) P_l\left(\frac{z^2 - 1 - r^2}{2r} \right) dz \end{aligned}\right\} \text{(I 10)}$$

$$\left.\begin{aligned} E_s = [M^2 + (k_f + k)^2]^{1/2}; \; E_d = [M^2 + (k_f - k)^2]^{1/2}; \; r = \frac{E_s - E_d}{E_s + E_d}; \; \bar{E} = \frac{E_s + E_d}{2} \\ b = (\varepsilon_f + \varepsilon_k - E_i - \varepsilon_i) \bar{E}^{-1}; \quad c = (E_f + E_k - E_i - \varepsilon_i) \bar{E}^{-1}. \end{aligned}\right\} \text{(I 11)}$$

To leading order in the small quantity r,

$$\left.\begin{aligned} \mathfrak{H}_0 &= \bar{E}^{-1}[(1+b)^{-1}-(1+c)^{-1}]; \\ \mathfrak{k}_0 &= \bar{E}^{-2}[(1+b)^{-1}+(1+c)^{-1}]; \\ \mathfrak{H}_1 &= \frac{1}{3}k_f k \bar{E}^{-3}[(1+c)^{-2}-(1+b)^{-2}]; \\ \mathfrak{k}_1 &= -\frac{1}{3}\bar{E}^{-4}[(2+c)(1+c)^{-2}+(2+b)(1+b)^{-2}]. \end{aligned}\right\} \text{(I. 12)}$$

Equations (I. 8) can be very much simplified if one introduces a cut-off $\ll M$ on the integration over k; $v(k)$ being a cut-off function with $v(0) = 1$ and y^I_{IJ} being defined by (69) the integral equations for the y^I_{IJ} are ($I = {}^3/_2$)

$$\left.\begin{aligned} & y_{0^1/_2}(k_f, k_i) = \\ & 1-(4\pi^2\hbar)^{-1}G^2\left(M-\frac{\varepsilon_i}{2}\right)^{-1}P\int_\mu^\infty d\varepsilon_k \frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i} v^2(k)\, y_{0^1/_2}(k, k_i), \\ & y_{1^3/_2}(k_f, k_i) = \\ & 1+(12\pi^2\hbar)^{-1}G^2M^{-2}P\int_\mu^\infty d\varepsilon_k \frac{k^3}{\varepsilon_k(\varepsilon_k+\varepsilon_f-\varepsilon_i)}\frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i} v^2(k)\, y_{1^3/_2}(k, k_i), \\ & y_{1^1/_2}(k_f, k_i) = \\ & 1-(24\pi^2\hbar)^{-1}G^2M^2P\int_\mu^\infty d\varepsilon_k \frac{k^3}{\varepsilon_k(\varepsilon_k+\varepsilon_f-\varepsilon_i)}\frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i} v^2(k)\, y_{1^1/_2}(k, k_i). \end{aligned}\right\} \text{(I. 13)}$$

The first one shows that $y_{0^1/_2}(k_f, k_i)$ is in fact independent of k_f and thus equal to $y_{0^1/_2}(k_i, k_i)$. The two others are "true" integral equations. Chew (1953) has solved them approximately by a variation method (cf. also Fubini, 1953).

Appendix II.

Let (m = component of nucleon spin on z-axis):

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}; \quad u_+ = \delta_{m,1/2}; \quad u_- = \delta_{m,-1/2}$$

and

$$\left.\begin{aligned} \mathcal{Y}^M_{Jl^{1/2}}(\boldsymbol{n}, m) &= \sum_{\nu\nu'} C_{l_1}(J, M, \nu, \nu')\, Y^\nu_l(\boldsymbol{n})\, \delta_{\nu, m} \\ &\equiv C_{l_1}(J, M, M-m, m)\, Y^{M-m}_l(\boldsymbol{n}). \end{aligned}\right\} \quad \text{(II. 1)}$$

Theoretical calculations often give the scattering matrix elements as functions of the angles and the spin operator $\boldsymbol{\sigma}$. The following formula (II. 2) is a convenient and general tool for expanding such results according to eigenvalues of l and J (phase shift analysis).

$$\left.\begin{aligned} &(u, \sum_l [U_l + V_l(\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n}')] \sum_\nu Y^\nu_l(\boldsymbol{n})\, Y^{\nu *}_l(\boldsymbol{n}')\, u') \\ &\quad = \sum_{l,J,M} \mathfrak{X}_{l,J}\, \mathcal{Y}^M_{Jl^{1/2}}(\boldsymbol{n}, m)\, \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}', m') \end{aligned}\right\} \quad \text{(II. 2)}$$

with

$$J = l \pm \frac{1}{2}; \quad \mathfrak{X}_{l, l\pm 1/2} = U_l + V_{l\pm 1}.$$

Conversely, (II. 2) also provides a convenient means for the evaluation of, e.g., spin averages over quantities given as expansions in l and J (see the compact expression (102) or (103) for the angular distribution in $\pi^+ + P$ scattering).

(II. 2) can be proved by elementary calculation. The first step is to write $(u, (\boldsymbol{\sigma}\cdot\boldsymbol{n})\, u'')$ as a linear combination of $Y^\mu_1(\boldsymbol{n})$. $(u', (\boldsymbol{\sigma}\cdot\boldsymbol{n})\, u'')\, Y^\nu_l(\boldsymbol{n})$ can then be expressed as a linear combination of $Y^M_L(\boldsymbol{n})$ with $L = l \pm 1$. The resulting expression for the product $(u, (\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n}')\, u') \sum_\nu Y^\nu_l(\boldsymbol{n})\, Y^{\nu *}_l(\boldsymbol{n}')$ is such that by using simple numerical equalities between products of Clebsch-

Gordan coefficients it can be brought into the form

$$\sum_M [\mathcal{Y}^M_{l+1/2,l+1,1/2}(\boldsymbol{n}, m)\, \mathcal{Y}^{M*}_{l+1/2,l+1,1/2}(\boldsymbol{n}', m) + \mathcal{Y}^M_{l-1/2,l-1,1/2}(\boldsymbol{n}, m)\, \mathcal{Y}^{M*}_{l-1/2,l-1,1/2}(\boldsymbol{n}', m')]$$

from which (II. 2) is easily derived.

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CONFIGURATION SPACE REPRESENTATION FOR NON-LINEAR FIELDS

BY

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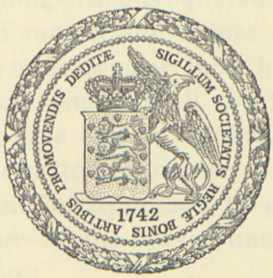

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Introduction.

In the theory of nucleons and mesons we deal with a situation in which the coupling between the two fields is not small. It is, therefore, of importance for the treatment of such problems to develop methods more powerful than perturbation theory. The divergence difficulties inherent in current field theory necessitate a formulation of the non-perturbation approaches which allow for an incorporation of the idea of renormalization of mass and charge. In practice, this implies as a necessary condition that the formalism must be covariant.

The method proposed by SALPETER and BETHE [1], [10] for the treatment of the two-body problem is an example of such an approach. A general theory of a similar kind has been initiated by SCHWINGER [2]. In this theory, one starts from the consideration of certain combinations of vacuum expectation values of time ordered products of field operators, the so-called Green's functions. In general such quantities obey inhomogeneous equations of motion. It can be seen that the study of the oscillating solutions of the corresponding homogeneous equations provides information about the energy and momentum values of stationary states of the system. According to Schwinger, these homogeneous equations apply to scattering problems as well. By his method, equations of the Bethe-Salpeter type can be established without reference to the limit of no interaction. However, it seems rather difficult by means of this kind of approach to obtain a clear understanding of the nature of the wave functions which obey the homogeneous equation.

Partly to overcome this problem, HEISENBERG [5] and FREESE [4] have proposed to start directly from a definition of the wave function for the problem. In the general formalism

developed by Freese it is shown how, for each state of the system, one can construct an infinite set of wave functions from free field Green's functions and matrix elements of time ordered products of field operators. The construction is such that the discontinuities in the matrix elements are compensated by corresponding discontinuities in the free field Green's functions. Consequently, Freese's wave functions obey homogeneous equations of motion. The infinite set of wave functions constitutes a generalization of the Fock representation in the configuration space for free fields to the case of interacting fields. For some problems one can substitute the infinite set of wave functions by essentially one function, only. The equation obtained for this function is of a similar structure as the equation of the Bethe-Salpeter type following from Schwinger's theory, but is in general not identical with Schwinger's equation. One reason for this may be found in the fact that free field concepts enter in Freese's representation.

In the present paper, an attempt is made at modifying the ideas of Heisenberg and Freese so as to unify their theory with that of Schwinger, and thus to combine the advantages of both formalisms. To this purpose, we employ the technique of variation of external sources developed by PEIERLS and SCHWINGER [6]. In Section 1, a survey of this method is given in a form which is convenient for our purpose. In Section 2, we relate to any state of the system a functional of the sources. The variational derivatives of this functional with respect to the sources define an infinite set of amplitudes. These are shown in Section 3 to generalize the Fock representation to non-linear fields. No reference to free field concepts is made in the definition of the state vector amplitudes. Several simple properties of the Fock representation are maintained in the non-linear case.

The problem of the construction of the scalar product of two states given in this configuration space representation has not been solved. Until further progress is made one must, therefore, use the term representation with some reservation. The equations of motion, in the configuration space representation, are derived in Section 4. Finally, in Section 5, a preliminary discussion is given of the one-nucleon problem and of the two-nucleon problem.

The corresponding equations of motion become identical with those following from Schwinger's theory.

Much of the discussion given by Freese can directly be taken over to the present formalism and is not repeated here. In particular for the discussion of scattering problems, the reader may be referred to Freese's paper.

All considerations below are of a highly formal character in so far as we have completely neglected the divergence difficulties. However, the renormalization theory, for instance in the form given by KÄLLÉN [8], can easily be incorporated in the present formalism.

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1. The field equations including coupling to external sources.

With the aim to illustrate the general method we consider the example of a spin one-half field (nucleons) coupled to a scalar neutral meson field. With a suitable symmetrization of the interaction terms, the equations of motion are

$$\left.\begin{aligned} (\partial\!\!\!/ + M)\,\psi_0(x) + (\lambda/2)\left\{ u_0(x), \psi_0(x) \right\} &= 0, \\ (\bar{\partial}\!\!\!/ + M)\,\bar{\psi}_0(x) + (\lambda/2)\left\{ u_0(x), \bar{\psi}_0(x) \right\} &= 0, \\ (-\Box + m^2)\,u_0(x) + (\lambda/2)\left[\bar{\psi}_0(x), \psi_0(x)\right] &= 0. \end{aligned}\right\} \tag{1.1}$$

Here, λ is the coupling parameter, and $\partial\!\!\!/$ and $\bar{\partial}\!\!\!/$ denote

$$\partial\!\!\!/ = \gamma_\mu\,\partial/\partial x_\mu, \quad \bar{\partial}\!\!\!/ = -\gamma_\mu^T\,\partial/\partial x_\mu,$$

where γ_μ^T is the transposed of the matrix γ_μ. The index μ runs from one to four and $x_\mu = (x_1, x_2, x_3, x_4)$, $x_4 = it$. As usual, $\overline{\psi}_0$ is defined in terms of ψ_0^*, the Hermitian conjugate of ψ_0, as $\overline{\psi}_0 = \psi_0^* \gamma_4$. The units chosen are such that $\hbar = c = 1$.

As mentioned in the introduction, we employ the method of variation of external sources developed by PEIERLS and SCHWINGER [6]. Therefore, we introduce external sources for all three kinds of fields and thus modify the equations (1.1) to

$$\left.\begin{aligned} (\partial + M)\,\psi(x) + (\lambda/2)\,\{u(x), \psi(x)\} + \varphi(x) &= 0, \\ (\partial + M)\,\overline{\psi}(x) + (\lambda/2)\,\{u(x), \overline{\psi}(x)\} + \overline{\varphi}(x) &= 0, \\ (-\Box + m^2)\,u(x) + (\lambda/2)\,[\overline{\psi}(x), \psi(x)] + I(x) &= 0. \end{aligned}\right\} \quad (1.2)$$

By omitting in these equations the subscript attached to the field operators in (1.1) we distinguish the source-dependent field variables from the usual ones describing the closed system. In the following, we assume that the sources vanish for both $|\vec{x}|$ and $|t|$ tending to infinity. With this restriction the equations (1.2) can be supplemented by a boundary condition which requires that the source-dependent field operators become identical with the usual source-free fields in the infinite past. Considering such solutions only we can regard the field variables as functionals of the sources. As no other solutions of the equations (1.2) will be considered in the following, it is superfluous to discriminate by any label this retarded solution from other possible ones.

We take $I(x)$, the external source of the meson field, as a c-number. Of course, one could also treat the external spinor sources as c-numbers. However, this is not what we shall do. In order that the external sources be useful, one should take the spinor sources as the analogue of c-numbers for the fermion case, i. e. as quantities such that

$$\{\varphi(x), \varphi(x')\} = \{\varphi(x), \overline{\varphi}(x')\} = \{\overline{\varphi}(x), \overline{\varphi}(x')\} = 0, \quad (1.3)$$

and

$$\left.\begin{aligned} \{\varphi(x), \psi_0(x')\} = \{\overline{\varphi}(x), \psi_0(x')\} &= 0 \\ \{\varphi(x), \overline{\psi}_0(x')\} = \{\overline{\varphi}(x), \overline{\psi}_0(x')\} &= 0, \end{aligned}\right\} \quad (1.4)$$

while φ and $\bar{\varphi}$ commute with I (and of course with any other c-number). For the further specification of the manifold of pairs of spinor sources it is advantageous to write φ and $\bar{\varphi}$ in the form

$$\left.\begin{aligned} \varphi(x) &= \Theta_0 f(x), \\ \bar{\varphi}(x) &= \Theta_0 g(x), \end{aligned}\right\} \tag{1.5}$$

where Θ_0 is a constant operator which commutes with f and g and anticommutes with ψ_0 and $\bar{\psi}_0$. Hence, due to (1.4) and (1.3), f and g commute with ψ_0 and $\bar{\psi}_0$ and satisfy

$$\{f(x), f(x')\} = \{f(x), g(x')\} = \{g(x), g(x')\} = 0. \tag{1.6}$$

As is well known, essentially only one such quantity Θ_0 exists, viz. the parity of the difference ΔN between the number of nucleons and the number of anti-nucleons. In terms of the field operators, ΔN is

$$\Delta N = \frac{1}{2}\int [\psi_0^*(\vec{x}, t), \psi_0(\vec{x}, t)]\, d^3\vec{x}. \tag{1.7}$$

We choose Θ_0 as

$$\Theta_0 = (-1)^{\Delta N} = \text{parity of } \Delta N, \tag{1.8}$$

thereby normalizing Θ_0 so that $\Theta_0^2 = 1$ and $\Theta_0 | 0 \rangle = | 0 \rangle$, where $| 0 \rangle$ is the vacuum state of the source-free system.

Corresponding to any pair f, g, we define the domain of pairs of sources obtained by allowed variations as the totality of pairs of the form $f + \delta f, g + \delta g$, where δf and δg are infinitesimal and anti-commute with f and g, i. e.

$$\left.\begin{aligned} \{\delta f(x), f(x')\} &= \{\delta g(x), f(x')\} = 0, \\ \{\delta f(x), g(x')\} &= \{\delta g(x), g(x')\} = 0. \end{aligned}\right\} \tag{1.9}$$

It should be noted that (1.9) is not a consequence of (1.6). For any pair f, g we now require the manifold of allowed variations to be so large that we, from a relation of the type

$$\int [\delta f(x) K(x) + \delta g(x) L(x)]\, d^4x = 0, \tag{1.10}$$

holding for all pairs of allowed variations, can conclude that $K(x)$ and $L(x)$ vanish identically. In (1.10), K and L are considered as quantities of the same nature as f and g.

This last mentioned property of the spinor sources, together with (1.6), is all we need for the formal calculations below. The consistency of all requirements is demonstrated in Appendix I by the construction of an example of a possible domain of pairs f, g. As shown there, one can imagine the quantities f, g, or as we shall say, the f-number pairs, to be infinite matrices. It should, however, be emphasized that the f-number pairs will be treated as a kind of numbers and not as operators. In other words, all matrix elements are matrix elements in the space of the source-free operators only, and are, for the rest, quantities of the same nature as the f-number pairs. Thus, corresponding to (1.5) and the fact that the parity of the vacuum state of the source-free system is unity, we write

$$\langle 0 \mid \varphi(x) \mid \Psi \rangle = f(x) \langle 0 \mid \Theta_0 \mid \Psi \rangle = f(x) \langle 0 \mid \Psi \rangle. \quad (1.11)$$

In this relation $\mid \Psi \rangle$ can be any state.

By means of the field equations one can easily see that the spinor sources anticommute with the source-dependent spinor fields and commute with u. This statement is based on the essential property of the source-dependent fields that ψ and $\bar{\psi}$ are odd functionals of quantities which anticommute with the spinor sources, while u is an even functional of such quantities. Thus

$$\left.\begin{aligned} \{\varphi(x), \psi(x')\} = \{\bar{\varphi}(x), \psi(x')\} = 0, \\ \{\varphi(x), \bar{\psi}(x')\} = \{\bar{\varphi}(x), \bar{\psi}(x')\} = 0. \end{aligned}\right\} \quad (1.12)$$

Similarly, it can be verified that allowed variations $\delta\varphi$ and $\delta\bar{\varphi}$ anticommute with ψ and $\bar{\psi}$ and commute with u. By allowed variations we here understand variations of the form $\delta\varphi = \Theta_0 \delta f$ and $\delta\bar{\varphi} = \Theta_0 \delta g$, where δf and δg satisfy (1.9).

Conversely, we could also have started from (1.12) instead of (1.4), as (1.4) follows from (1.12), the field equations (1.2) and the retarded boundary condition.

One further remark may be useful here. It can easily be verified from (1.2) and the boundary condition that the canonical commutation relations

$$\left.\begin{aligned} \{\psi_\alpha(\vec{x}, t), \bar{\psi}_\beta(\vec{x}', t)\} &= (\gamma_4)_{\alpha\beta}\,\delta(\vec{x} - \vec{x}') \\ [u(\vec{x}, t), \dot{u}(\vec{x}', t)] &= i\,\delta(\vec{x} - \vec{x}') \end{aligned}\right\} \tag{1.13}$$

hold in the source-dependent case also.

We can now formulate the following main theorem as regards the dependence of the fields on the sources. For any infinitesimal variation δI of the meson field source, and for any pair of allowed variations $\delta\varphi$ and $\delta\bar{\varphi}$ of the spinor sources, the corresponding variations of the fields are given by

$$\left.\begin{aligned} \delta\psi(x) &= i\,[\int_{-\infty}^{t} \delta W(x')\, d^4x',\ \psi(x)], \\ \delta\bar{\psi}(x) &= i\,[\int_{-\infty}^{t} \delta W(x')\, d^4x',\ \bar{\psi}(x)], \\ \delta u(x) &= i\,[\int_{-\infty}^{t} \delta W(x')\, d^4x',\ u(x)], \end{aligned}\right\} \tag{1.14}$$

where the infinitesimal operator δW is

$$\delta W(x) = \delta\bar{\psi}(x)\,\psi(x) + \bar{\psi}(x)\,\delta\varphi(x) + u(x)\,\delta I(x). \tag{1.15}$$

The statement (1.14) is included in the general variation principle for quantized systems formulated by Schwinger [6]. It is, however, quite easy to prove (1.14) directly from the field equations. Evidently, (1.14) is in accordance with the boundary condition. Therefore, we only need to show that the variations (1.14) satisfy the varied field equations. For instance, from the first equation (1.14), we get

$$(\partial + M)\,\delta\psi(x) = i\,[\int_{-\infty}^{t} \delta W(x')\, d^4x',\ (\partial + M)\,\psi(x)]$$

$$+\,[\int d^3\vec{x}'\,\delta W(\vec{x}', t),\ \gamma_4\,\psi(x)].$$

which by the field equations, the properties of the sources, and the canonical commutators becomes

$$(\partial + M)\,\delta\psi(x) + (\lambda/2)\{u(x), \delta\psi(x)\} + (\lambda/2)\{\delta u(x), \psi(x)\} + \delta\varphi(x) = 0.$$

This is precisely the equation one would have obtained by varying the first equation (1.2). In a similar manner one obtains the other varied field equations, and this verifies (1.14).

In concluding this section we shall reexpress the contents of the variational equations (1.14), using the notion of variational derivatives. Consider a functional, $\Phi[\varphi, \bar{\varphi}, I]$ say, of the sources. Assume, that one can write the variation of this functional in the form

$$\delta\Phi[\varphi, \bar{\varphi}, I] = \int \left(\delta f(x) A(x) + \delta g(x) B(x) + \delta I(x) C(x)\right) d^4x = 0,$$

holding for any infinitesimal allowed variations of the sources. Then, the quantities A, B, and C are uniquely determined. This follows from the conclusion drawn from (1.10). We can thus define A, B, and C as the variational derivatives of the functional Φ corresponding to variations of f, g, and I, respectively. It is convenient to introduce the notation

$$A(x) = \delta\Phi[\varphi, \bar{\varphi}, I]/\delta f(x),$$
$$B(x) = \delta\Phi[\varphi, \bar{\varphi}, I]/\delta g(x),$$
$$C(x) = \delta\Phi[\varphi, \bar{\varphi}, I]/\delta I(x).$$

It should be emphasized that, for instance, $\delta f(x)$ and $A(x)$ do not commute in general. The variational derivatives introduced here are thus left-hand derivatives. In a similar way, one could introduce right-hand variational derivatives.

As above, let $|\,0\,\rangle$ be the vacuum state of the source-free system and let $|\,\Psi\,\rangle$ be any other source-independent state. From (1.14) we get

$$\delta\langle\, 0 \,|\, \psi(x) \,|\, \Psi\,\rangle = i \int_{-\infty}^{\infty} \frac{1 - \varepsilon(x' - x)}{2} \times \langle\, 0 \,|\, [\bar{\psi}(x')\,\delta\varphi(x') + \delta\bar{\psi}(x')\cdot\psi(x') + \delta I(x')\cdot u(x'), \psi(x)] \,|\, \Psi\,\rangle\, d^4x'.$$

As usual, $\varepsilon(x'-x)$ is the step function $(t'-t)/|t'-t|$. From this equation we infer, using the properties of allowed variations and relations like

$$\langle 0 \mid \delta\varphi(x) = \delta f(x) \langle 0 \mid \Theta_0 = \delta f(x) \langle 0 \mid, \tag{1.16}$$

that

$$\left.\begin{aligned} i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta g(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|\{\psi(x'),\psi(x)\}|\Psi\rangle, \\ -i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta f(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|\{\bar{\psi}(x'),\psi(x)\}|\Psi\rangle, \\ i\frac{\delta\langle 0|\psi(x)|\Psi\rangle}{\delta I(x')} &= -\frac{1-\varepsilon(x'-x)}{2}\langle 0|[u(x'),\psi(x)]|\Psi\rangle. \end{aligned}\right\} \tag{1.17}$$

In a similar manner, one may obtain expressions for the variational derivatives of matrix elements of the other field variables. The minus sign on the left-hand side of the second equation originates from the reordering $\bar{\psi}\delta\varphi = -\delta\varphi\bar{\psi}$ necessary to obtain the left-hand variational derivative with respect to f.

2. Generating functionals for ordered products of field operators.

The ordered products considered in this section can all be constructed from one operator $\boldsymbol{T}$ which, as we shall see, is the generator of the time ordered product as defined by WICK [9].

i) *The time ordered product.*

We introduce an operator $\boldsymbol{T}$ by the variational equation

$$\delta\,\boldsymbol{T} = -i\,\boldsymbol{T}\int_{-\infty}^{\infty}\delta W(x)\,dx \tag{2.1}$$

and the boundary condition $\boldsymbol{T} = 1$ in the limit of vanishing

sources*. The infinitesimal operator δW, defined by (1.15), is closely connected to the total variation of the operator

$$W(x) = \overline{\psi}(x)\,\varphi(x) + \overline{\varphi}(x)\,\psi(x) + I(x)\,u(x). \tag{2.2}$$

It follows from the properties of the sources that

$$[\delta W(x'), \varphi(x)] = [\delta W(x'), \overline{\varphi}(x)] = [\delta W(x'), I(x)] = 0, \tag{2.3}$$

whence, by (1.14),

$$\delta_{\text{total}} W(x) = \delta W(x) + i\int_{-\infty}^{x} [\delta W(x'),\, W(x)]\, dx'. \tag{2.4}$$

We shall verify in detail that the solution of the variational equation (2.1) is given by

$$\left.\begin{aligned} \boldsymbol{T} &= \overline{P} \exp\left\{-i\int_{-\infty}^{\infty} W(x)\, dx\right\} \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx' \cdots \int_{-\infty}^{\infty} dx^{(n)}\, \overline{P}\{W(x') \cdots W(x^{(n)})\}, \end{aligned}\right\} \tag{2.5}$$

where $\overline{P}$ orders the W-factors in the reverse sense of DYSON's [3] chronologically ordering operator. Thus, if $x^{(\mu)}$ antedates $x^{(\nu)}$, then $W(x^{(\nu)})$ appears to the right of $W(x^{(\mu)})$ in the $\overline{P}$-ordered product. To prove (2.5) we first consider the variation of an ordinary product of W-factors. By (2.4) we get

$$\left.\begin{aligned} &\delta\{W(x')\,W(x'')\ldots W(x^{(n)})\} \\ &\quad = i\left(\int_{-\infty}^{x'} \delta W(x)\, dx\right) W(x')\,W(x'')\ldots W(x^{(n)}) \\ &\quad + i\,W(x')\left(\int_{x'}^{x''} \delta W(x)\, dx\right) W(x'')\ldots W(x^{(n)}) \\ &\quad + \ldots \\ &\quad + i\,W(x')\,W(x'')\ldots W(x^{(n)})\left(\int_{x^{(n)}}^{\infty} \delta W(x)\, dx\right) + \end{aligned}\right\} \tag{2.6}$$

* For a more general discussion it might be of advantage to consider another solution of the variational equation (2.1) corresponding to the boundary condition $\boldsymbol{T} = S$ for $\varphi = \overline{\varphi} = I = 0$, where S is the S-matrix for the closed system. All considerations in the following remain valid for this choice of solution.

$$
\left.
\begin{aligned}
&+ \delta W(x') \,.\, W(x'') \ldots W(x^{(n)}) \\
&+ \ldots \\
&+ W(x') \, W(x'') \ldots \delta W(x^{(n)}) \\
&- i \, W(x') \, W(x'') \ldots W(x^{(n)}) \cdot \int_{-\infty}^{\infty} \delta W(x) \, dx ,
\end{aligned}
\right\} \tag{2.6}
$$

where we have collected the contributions from the commutators between δW and W in an obvious manner. The complete symmetry of the $\bar{P}$-ordered product allows us to write the variation of the general term in series (2.5) in the somewhat simpler form

$$
\left.
\begin{aligned}
&\delta \int_{-\infty}^{\infty} dx' \ldots \int_{-\infty}^{\infty} dx^{(n)} \, \bar{P} \{ W(x') \, W(x'') \ldots W(x^{(n)}) \} \\
&\quad = i \int_{-\infty}^{\infty} dx' \ldots \int_{-\infty}^{\infty} dx^{(n)} \int_{-\infty}^{\infty} dx \, \bar{P} \{ \delta W(x) \, W(x') \ldots W(x^{(n)}) \} \\
&\quad - i \int_{-\infty}^{\infty} dx' \ldots \int_{-\infty}^{\infty} dx^{(n)} \, \bar{P} \{ W(x') \ldots W(x^{(n)}) \} \cdot \int_{-\infty}^{\infty} \delta W(x) \, dx \\
&\quad + n \int_{-\infty}^{\infty} dx' \ldots \int_{-\infty}^{\infty} dx^{(n)} \, \bar{P} \{ \delta W(x') \cdot W(x'') \ldots W(x^{(n)}) \} .
\end{aligned}
\right\} \tag{2.7}
$$

If we introduce this expression into the variation of $\boldsymbol{T}$ obtained from (2.5), we see that the contributions from the first and the third term on the right-hand side of (2.7) cancel, and that the sum of the remaining terms equals the right-hand side of (2.1). This verifies (2.5) as this expression obviously is in accordance with the boundary condition.

All allowed variations commute with W. Thus, by (2.5), also $\boldsymbol{T}$ commutes with these variations and we can write (2.1) in the form

$$
\delta \boldsymbol{T} = - i \int_{-\infty}^{\infty} dx \{ \delta \bar{\varphi} \boldsymbol{T} \psi - \delta \varphi \boldsymbol{T} \bar{\psi} + \delta I \boldsymbol{T} u \} . \tag{2.8}
$$

Consequently, for any source-independent state $| \Psi \rangle$, we have by (1.16)

$$\left.\begin{aligned} i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta I(x)} &= \langle 0\,|\,\boldsymbol{T}u(x)\,|\,\Psi\rangle \\ i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta g(x)} &= \langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle \\ -i\frac{\delta\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle}{\delta f(x)} &= \langle 0\,|\,\boldsymbol{T}\bar{\psi}(x)\,|\,\Psi\rangle. \end{aligned}\right\} \tag{2.9}$$

From (2.8) and the variational equations (1.14) we get for the variation of, for instance, the right-hand side of the second equation (2.9)

$$\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle =$$

$$-i\left[\int_x^\infty \delta I(x')\langle 0\,|\,\boldsymbol{T}u(x')\psi(x)\,|\,\Psi\rangle + \int_{-\infty}^x \delta I(x')\langle 0\,|\,\boldsymbol{T}\psi(x)u(x')\,|\,\Psi\rangle\right]$$

$$-i\left[\int_x^\infty \delta g(x')\langle 0\,|\,\boldsymbol{T}\psi(x')\psi(x)\,|\,\Psi\rangle - \int_{-\infty}^x \delta g(x')\langle 0\,|\,\boldsymbol{T}\psi(x)\psi(x')\,|\,\Psi\rangle\right]$$

$$+i\left[\int_x^\infty \delta f(x')\langle 0\,|\,\boldsymbol{T}\bar{\psi}(x')\psi(x)\,|\,\Psi\rangle - \int_{-\infty}^x \delta f(x')\langle 0\,|\,\boldsymbol{T}\psi(x)\bar{\psi}(x')\,|\,\Psi\rangle\right],$$

whence

$$\left.\begin{aligned} i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta I(x')} &= \langle 0\,|\,\boldsymbol{T}\,T\big(u(x')\,\psi(x)\big)\,|\,\Psi\rangle \\ i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta g(x')} &= \langle 0\,|\,\boldsymbol{T}\,T\big(\psi(x')\,\psi(x)\big)\,|\,\Psi\rangle \\ -i\frac{\delta\langle 0\,|\,\boldsymbol{T}\psi(x)\,|\,\Psi\rangle}{\delta f(x')} &= \langle 0\,|\,\boldsymbol{T}\,T\big(\bar{\psi}(x')\,\psi(x)\big)\,|\,\Psi\rangle, \end{aligned}\right\} \tag{2.10}$$

where $T\,(\cdots)$ designates Wick's time-ordered product. The expressions (2.9) and (2.10) are special cases of the general formula

$$\left.\begin{aligned} &i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})}\,i\frac{\delta}{\delta g(y')}\cdots i\frac{\delta}{\delta g(y^{(l)})} \\ &\times\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-i\frac{\delta}{\delta f(z^{(m)})}\right)\langle 0\,|\,\boldsymbol{T}\,|\,\Psi\rangle = \\ &\langle 0|\boldsymbol{T}\,T\big(u(x')\cdots u(x^{(k)})\psi(y')\cdots\psi(y^{(l)})\bar{\psi}(z')\cdots\bar{\psi}(z^{(m)})\big)|\Psi\rangle, \end{aligned}\right\} \tag{2.11}$$

which reveals $\boldsymbol{T}$ as the generator of the T-product. To prove (2.11) denote, for fixed values of the space time points, $x' \cdots x^{(k)}$ $y' \cdots y^{(l)}\, z' \cdots z^{(m)}$, the chronologically ordered sequence of the same points by $x_1, x_2, \cdots x_n$, $n = k + l + m$. Further, let χ denote any of the field variables ψ, $\overline{\psi}$, and u. With this notation, we have

$$\begin{aligned}
&\delta \langle 0 \mid \boldsymbol{T}\, \chi(x_1) \cdots \chi(x_n) \mid \Psi \rangle = \\
&- i \langle 0 \mid \boldsymbol{T} \int_{x_1}^{\infty} \delta W(x)\, dx\, \chi(x_1) \cdots \chi(x_n) \mid \Psi \rangle, \\
&- i \langle 0 \mid \boldsymbol{T}\, \chi(x_1) \int_{x_2}^{x_1} \delta W(x)\, dx \cdots \chi(x_n) \mid \Psi \rangle, \\
&- i \cdots \\
&- i \langle 0 \mid \boldsymbol{T}\, \chi(x_1) \cdots \chi(x_n) \int_{-\infty}^{x_n} \delta W(x)\, dx \mid \Psi \rangle,
\end{aligned}$$

in virtue of (2.1) and the variational equations (1.14). If we displace all source variations to the extreme left we get

$$\left.\begin{aligned}
&\delta \langle 0 \mid \boldsymbol{T}\, \chi(x_1) \cdots \chi(x_n) \mid \Psi \rangle = \\
&- i \int_{-\infty}^{\infty} \delta g(x)\, (\pm) \langle 0 \mid \boldsymbol{T}\, P \{ \psi(x)\, \chi(x_1) \cdots \chi(x_n) \} \mid \Psi \rangle\, dx, \\
&+ i \int_{-\infty}^{\infty} \delta f(x)\, (\pm) \langle 0 \mid \boldsymbol{T}\, P \{ \overline{\psi}(x)\, \chi(x_1) \cdots \chi(x_n) \} \mid \Psi \rangle\, dx, \\
&- i \int_{-\infty}^{\infty} \delta I(x) \qquad \langle 0 \mid \boldsymbol{T}\, P \{ u(x)\, \chi(x_1) \cdots \chi(x_n) \} \mid \Psi \rangle\, dx,
\end{aligned}\right\} \quad (2.12)$$

where P is Dyson's chronologically ordering operator. The $(\pm)$ factor in the two first terms on the right-hand side of (2.12) is the parity of the number of permutations between the nucleon operators and the variations of the spinor field sources. Evidently, the number of these permutations equals the number of permutations of spinor fields required to bring the field variables ψ and $\overline{\psi}$, respectively, from the place indicated in (2.12) to the position required by the P-operator. Thus, $(\pm)$ is the change of sign characterizing Wick's T-product as compared with Dyson's P-product and, hence,

$$\left.\begin{aligned} & i\,\delta\langle 0\,|\,\boldsymbol{T}\,T\,(u(x')\cdots\overline{\psi}(z^{(m)}))\,|\,\Psi\rangle = \\ & \int_{-\infty}^{\infty}\delta g(x)\langle 0\,|\,\boldsymbol{T}\,T\,(\psi(x)\,u(x')\cdots\overline{\psi}(z^{(m)}))\,|\,\Psi\rangle\,dx, \\ & -\int_{-\infty}^{\infty}\delta f(x)\langle 0\,|\,\boldsymbol{T}\,T\,(\overline{\psi}(x)\,u(x')\cdots\overline{\psi}(z^{(m)}))\,|\,\Psi\rangle\,dx, \\ & +\int_{-\infty}^{\infty}\delta I(x)\langle 0\,|\,\boldsymbol{T}\,T\,(u(x)\,u(x')\cdots\overline{\psi}(z^{(m)}))\,|\,\Psi\rangle\,dx. \end{aligned}\right\} \tag{2.13}$$

The minus sign is due to the occurrence of $\delta\varphi$ to the right of $\overline{\psi}$ in the expression for δW. The proof of (2.11) is now easily completed by an induction argument.

ii) *Matrix elements of normal products.*

The formula (2.11) demonstrates the convenience of Schwinger's formalism for the introduction of ordered products of field operators, but adds nothing new. The normal product*, however, is not defined for non-linear fields and it is, therefore, more interesting that we by this formalism can give a general definition of the normal product. The detailed discussion of the normal product as introduced here, and in particular the proof that this product is a generalization of that introduced by Wick for free fields, will be given in the following section.

The generator for the N-product is the operator $\boldsymbol{N}$ which is connected with the $\boldsymbol{T}$-operator by

$$\boldsymbol{N} = \langle 0\,|\,\boldsymbol{T}\,|\,0\rangle^{-1}\,\boldsymbol{T}. \tag{2.14}$$

We shall regard

$$\left.\begin{aligned} & \Psi(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}) = i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})} \\ & i\frac{\delta}{\delta g(y')}\cdots i\frac{\delta}{\delta g(y^{(l)})}\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-i\frac{\delta}{\delta f(z^{(m)})}\right)\langle 0\,|\,\boldsymbol{N}\,|\,\Psi\rangle \end{aligned}\right\} \tag{2.15}$$

as the matrix element between the states $\langle 0|$ and $|\Psi\rangle$ of the N-ordered product of the field variables corresponding to the

* By Wick [9] denoted as the S-product. To avoid the use of the letter S for too many purposes we shall, henceforward, use the term N-product.

space-time points indicated[1]. The relation (2.14) implies

$$\langle 0 \mid \boldsymbol{N} \mid \Psi \rangle = \langle 0 \mid \boldsymbol{T} \mid 0 \rangle^{-1} \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle \tag{2.16}$$

and

$$\langle 0 \mid \boldsymbol{T} \mid 0 \rangle \langle 0 \mid \boldsymbol{N} \mid \Psi \rangle = \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle. \tag{2.17}$$

From these two expressions two relations originate between the matrix elements of N-products and the matrix elements of T-products. To express these relations in a compact form we introduce some conventions about notation.

Let $\xi', \xi'', \cdots \xi^{(\varkappa)}$, $\varkappa \leqq k$ denote some of the space-time points $x', x'', \cdots x^{(k)}$. By

$$x', x'', \cdots x^{(k)}; \xi', \xi'', \cdots \xi^{(\varkappa)} \tag{2.18}$$

we denote the sequence of space-time points obtained by omitting the space-time points $\xi', \xi'', \cdots \xi^{(\varkappa)}$ from the sequence $x', x'', \cdots x^{(k)}$. Thus, for example, $x', x'', x''', x''''; x'', x''' = x', x''''$. In the same way, we introduce symbols such as $y', y'', \cdots y^{(l)}; \eta', \eta'', \cdots \eta^{(\lambda)}$ and $z', z'', \cdots z^{(m)}; \zeta', \zeta'', \cdots \zeta^{(\mu)}$.

We also introduce a notation for matrix elements of T-products similar to that we use for matrix elements of N-products. For instance, we write the right-hand side of (2.11) as

$$T_{\Psi}(x' \cdots x^{(k)} \mid y' \cdots y^{(l)} \mid z' \cdots z^{(m)}). \tag{2.19}$$

If $\mid \Psi \rangle$ is the vacuum state, we denote the vacuum expectation value of the T-product by

$$T_0 (x' \cdots x^{(k)} \mid y' \cdots y^{(l)} \mid z' \cdots z^{(m)}). \tag{2.20}$$

For completeness, we note that, in the special case $k = l = m = 0$, we write

$$\left.\begin{aligned} \langle 0 \mid \boldsymbol{T} \mid \Psi \rangle &= T_{\Psi}(\mid), \\ \langle 0 \mid \boldsymbol{N} \mid \Psi \rangle &= \Psi(\mid). \end{aligned}\right\} \tag{2.21}$$

[1] It is evident how to generalize this definition and the formula (2.11) to matrix elements of the N- and T-products, respectively, between any two (source-independent) states of the system.

Also, with the notation (2.20), we have

$$\langle 0 \,|\, \boldsymbol{T} \,|\, 0 \rangle = T_0 (|\,|). \tag{2.22}$$

As mentioned in Section 1, matrix elements of field operators are in general not c-numbers. This introduces some minor complications in the following considerations, but is the price we have to pay in order that all three kinds of sources appear in a symmetric manner in the variational equations (1.14).

Still, any T_0-function with an even total number of spinor space-time points is effectively a c-number in the theory. Any f-number commutes with such "even" T_0-functions. The general relation for, for instance, δf is

$$\left.\begin{aligned} &\delta f \, T_0 (x' \cdots | \, y' \cdots y^{(l)} \, | z' \cdots z^{(m)}) \\ &= (-1)^{l+m} \, T_0 (x' \cdots | \, y' \cdots y^{(l)} \, | z' \cdots z^{(m)}) \, \delta f, \end{aligned}\right\} \tag{2.23}$$

and is easily proved by the use of (1.16) and the anti-commutativity of $\delta\varphi$ with all spinor fields. A similar relation holds for δg, f, and g. Hence, even T_0-functions commute with any functional of f and g and, in particular, with any other T_0-function. Thus,

$$[T_0 (x' \cdots | \, y' \cdots y^{(l)} \, | z' \cdots z^{(m)}), \; T_0 (x_1 \cdots | \, y_1 \cdots y_\lambda \, | z_1 \cdots z_\mu)] = 0$$

if $\lambda + \mu$ is even. The variational derivative of this equation with respect to $g\,(y)$ gives, for the case of $l + m$ being odd,

$$\begin{aligned} &[T_0 (x' \cdots | \, y \, y' \cdots y^{(l)} \, | z' \cdots z^{(m)}), \; T_0 (x_1 \cdots | \, y_1 \cdots y_\lambda \, | z_1 \cdots z_\mu)] \\ &- \{T_0 (x' \cdots | \, y' \cdots y^{(l)} | z' \cdots z^{(m)}), T_0 (x_1 \cdots | \, y \, y_1 \cdots y_\lambda | z_1 \cdots z_\mu)\} = 0. \end{aligned}$$

The appearance of an anti-commutator is a consequence of the anti-commutativity of δg with odd T_0-functions. The first term vanishes and we infer that two T_0-functions, both having an odd total number of spinor space-time points, anti-commute. In particular, $T_0 (|\,|)$ commutes with all matrix elements and

$$\left.\begin{aligned} \{T_0(|y|),\ T_0(|y'|)\} &= 0, \\ \{T_0(|y|),\ T_0(||z)\} &= 0, \\ \{T_0(||z),\ T_0(||z')\} &= 0. \end{aligned}\right\} \tag{2.24}$$

We are now prepared to prove the first of the relations mentioned above. From (2.17) follows

$$\left.\begin{aligned} & T_\Psi(x'\cdots x^{(k)}|y'\cdots y^{(l)}|z'\cdots z^{(m)}) \\ &= \sum_{\varkappa\lambda\mu} \frac{1}{\varkappa!} \sum_{\xi'\cdots\xi^{(\varkappa)}} \frac{1}{\lambda!} \sum_{\eta'\cdots\eta^{(\lambda)}} \frac{1}{\mu!} \sum_{\zeta'\cdots\zeta^{(\mu)}} \\ & \times(\pm)\ T_0(\xi'\cdots\xi^{(\varkappa)}|\eta'\cdots\eta^{(\lambda)}|\zeta'\cdots\zeta^{(\mu)}) \\ & \Psi(x'\cdots x^{(k)};\ \xi'\cdots\xi^{(\varkappa)}|y'\cdots y^{(l)};\ \eta'\cdots\eta^{(\lambda)}|z'\cdots z^{(m)};\ \xi'\cdots\xi^{(\mu)}), \end{aligned}\right\} \tag{2.25}$$

where the summation is taken over $\varkappa = 0, 1, \cdots k$, $\lambda = 0, 1, \cdots l$, and $\mu = 0, 1, \cdots m$ while the ξ's run independently over all the space-time points $x'\cdots x^{(k)}$, etc. The factorials take into account that we sum over all permutations of the sets $\xi'\cdots$, $\eta'\cdots$ and $\zeta'\cdots$. Apart from the factor $(\pm)$ in front of the general term, (2.25) is easily recognized as the usual formula for the iterated derivative of a product, viz. the product on the left-hand side of (2.17). Thus, (2.25) is correct if we interpret the sign factor $(\pm)$ correctly. From (2.23) it follows, however, that the factor $(\pm)$ is the parity of the permutation of spinor space-time points involved in the substitution

$$\left.\begin{aligned} & (x'x''\cdots x^{(k)}|y'y''\cdots y^{(l)}|z'z''\cdots z^{(m)}) \to \\ & (\xi'\cdots\xi^{(\varkappa)}|\eta'\cdots\eta^{(\lambda)}|\zeta'\cdots\zeta^{(\mu)}) \\ & (x'\cdots x^{(k)};\ \xi'\cdots\xi^{(\varkappa)}|y'\cdots y^{(l)};\ \eta'\cdots\eta^{(\lambda)}|z'\cdots z^{(m)};\ \zeta'\cdots\zeta^{(\mu)}). \end{aligned}\right\} \tag{2.26}$$

To illustrate (2.25) we note a few examples which also later will serve for reference:

$$\left.\begin{aligned} T_\Psi(x'x''|\,|) = {} & T_0(|\,|)\,\Psi(x'x''|\,|) + T_0(x'|\,|)\,\Psi(x''|\,|) \\ & + T_0(x''|\,|)\,\Psi(x'|\,|) + T_0(x'x''|\,|)\,\Psi(|\,|) \end{aligned}\right\} \tag{2.27}$$

$$\left.\begin{aligned} T_\Psi(x\,|y|) = {} & T_0(|\,|)\,\Psi(x\,|y|) + T_0(x|\,|)\,\Psi(|y|) \\ & + T_0(|y|)\,\Psi(x|\,|) + T_0(x\,|y|)\,\Psi(|\,|) \end{aligned}\right\} \tag{2.28}$$

$$\left.\begin{aligned} T_\Psi(x|\,|z) = {} & T_0(|\,|)\,\Psi(x|\,|z) + T_0(x|\,|)\,\Psi(|\,|z) \\ & + T_0(|\,|z)\,\Psi(x|\,|) + T_0(x|\,|z)\,\Psi(|\,|), \end{aligned}\right\} \tag{2.29}$$

and, finally, to illustrate the $(\pm)$ factor,

$$\left.\begin{aligned} T_\Psi(|y'y''|) = {} & T_0(|\,|)\,\Psi(|y'y''|) + T_0(|y'|)\,\Psi(|y''|) \\ & - T_0(|y''|)\,\Psi(|y'|) + T_0(|y'y''|)\,\Psi(|\,|). \end{aligned}\right\} \tag{2.30}$$

The formula (2.25) may be looked upon as a recursion formula which implicitly expresses the Ψ-functions in terms of matrix elements of T-products. The resulting formula may, however, be obtained directly from (2.16) if we introduce the functions

$$\left.\begin{aligned} & C(x'\cdots x^{(k)}|y'\cdots y^{(l)}|z'\cdots z^{(m)}) = i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta I(x^{(k)})} \\ & i\frac{\delta}{\delta g(y')}\cdots i\frac{\delta}{\delta g(y^{(l)})}\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\left(-\frac{\delta}{\delta f(z^{(m)})}\right)C(|\,|), \end{aligned}\right\} \tag{2.31}$$

where

$$C(|\,|) = \langle 0\,|\,\boldsymbol{T}\,|0\rangle^{-1}. \tag{2.32}$$

By an argument similar to that by which (2.25) was obtained, we get from (2.16)

$$\left.\begin{aligned} & \Psi(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}) \\ & = \sum_{\varkappa\lambda\mu}{}^{\prime}\frac{1}{\varkappa!}\sum_{\xi'\cdots\xi^{(\varkappa)}}{}^{\prime}\frac{1}{\lambda!}\sum_{\eta'\cdots\eta^{(\lambda)}}{}^{\prime}\frac{1}{\mu!}\sum_{\zeta'\cdots\zeta^{(\mu)}}{}^{\prime} \\ & (\pm)\,C(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}) \\ & T_\Psi(x'\cdots x^{(k)};\,\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\;\zeta'\cdots\zeta^{(\mu)}). \end{aligned}\right\} \tag{2.33}$$

An important property of the Ψ-functions follows from (2.14). In the special case where $|\Psi\rangle$ is the vacuum state of the source-free system, we have $\langle 0|\boldsymbol{N}|0\rangle = 1$, independent of the sources. Hence, all Ψ-functions vanish, except the one corresponding to $k = l = m = 0$. Thus, in this case, (2.33) reduces to

$$\left.\begin{aligned} &\sum_{\varkappa\lambda\mu}{}' \frac{1}{\varkappa!\lambda!\mu!} \sum_{\xi'\cdots\xi^{(\varkappa)}}{}' \sum_{\eta'\cdots\eta^{(\lambda)}}{}' \sum_{\zeta'\cdots\zeta^{(\mu)}}{}' \\ &(\pm)\, C(\xi'\cdots\xi^{(\varkappa)}|\eta'\cdots\eta^{(\lambda)}|\zeta'\cdots\zeta^{(\mu)}) \\ T_0(x'\cdots x^{(k)};\ &\xi'\cdots\xi^{(\varkappa)}|y'\cdots y^{(l)};\ \eta'\cdots\eta^{(\lambda)}|z'\cdots z^{(m)};\ \zeta'\cdots\zeta^{(\mu)}) \\ &= \delta_{0k}\delta_{0l}\delta_{0m}, \end{aligned}\right\} \tag{2.34}$$

and this is a recursion formula expressing the C-functions in terms of vacuum expectation values of T-products[1].

In the following, the formulas (2.25), (2.33), and (2.34) will serve as a basis for the discussion of the properties of the matrix elements of N-products. It will be shown that these expressions generalize the algebraic relations between T- and N-products for free fields to the case of non-linear fields.

3. Properties of matrix elements of N-products.

In the Fock representation [7] in configuration space for free fields, one characterizes a state of the system by an infinite set of many-particle wave functions. As long as one considers free fields, this representation may in a trivial way be extended to a multiple time representation. If we use the notion of a normal product, we can write the many-time wave functions, or as we prefer to say here, the state vector amplitudes, in the form

$$\left.\begin{aligned} &\Psi(x'\cdots x^{(k)}|y'\cdots y^{(l)}|z'\cdots z^{(m)}) \\ &= \langle 0|N(u(x')\cdots u(x^{(k)})\psi(y')\cdots\psi(y^{(l)})\bar{\psi}(z')\cdots\bar{\psi}(z^{(m)}))|\Psi\rangle. \end{aligned}\right\} \tag{3.1}$$

The results of Wick's discussion of the properties of T- and N-products of free field operators are expressed in the Appendix

[1] The formula (2.34) could also have been obtained directly from the identity $C(|\,|)\,T_0(|\,|) = 1$.

II with the aid of a notation which is convenient for our purpose. In Section 2, we derive relations connecting matrix elements of the normal product of field operators for non-linear fields with matrix elements of T-products. If we compare the formula (2.25) with Wick's formula (Ap. II. 11), we see that the N-product for non-linear fields, as defined by (2.15), is a generalization of the N-product for free fields, as the formula (2.25) in the limit $\varphi = \bar{\varphi} = I = \lambda = 0$ reduces to the corresponding formula for free fields given in the Appendix II.

The equation (2.15) may, therefore, be taken as the general definition, valid also for non-linear fields, of the state vector amplitudes which represent any given state $|\Psi\rangle$. After a discussion, in this section, of some of the simplest properties of the state vector amplitudes, we shall in the following section derive the equations of motion in this new representation. It will then be seen that the state vector amplitudes are closely connected to the "wave functions" which enter in the homogeneous equations of motion following from Schwinger's theory.

The following simple properties of the state vector amplitudes are independent of the magnitude of the coupling constant.

i) The ground state of the source-free system has the representation $\Psi(||) = 1$, while all other amplitudes vanish. As already remarked at the end of the last section, this follows in a trivial way from the definitions (2.15) and (2.14). The fact that the simplest state of the system has the simplest possible representation is in accordance with the expectation that the present formalism provides us with a convenient description of the lowest lying levels of the system.

ii) It is easily verified from (2.25), by means of well-known properties of T-products, that the state vector amplitudes are symmetric functions in all meson coordinates and anti-symmetric in as well all nucleon coordinates as all anti-nucleon coordinates. So far we have not introduced Ψ-functions such that we can speak about symmetry properties when interchanges of, for instance, nucleons and anti-nucleons are involved. It is, however, evident how one could generalize (2.15) to cover such cases also. One would then obtain state vector amplitudes which,

in the general case, possess all the well-known symmetry properties of the free field wave function (3.1). The most direct way to see this is to observe that we formally can use the relations

$$\left.\begin{aligned} \left\{ i\frac{\delta}{\delta g(y')}, \; i\frac{\delta}{\delta g(y'')} \right\} &= 0, \\ \left\{ i\frac{\delta}{\delta g(y')}, -i\frac{\delta}{\delta f(z')} \right\} &= 0, \\ \left\{ -i\frac{\delta}{\delta f(z')}, -i\frac{\delta}{\delta f(z'')} \right\} &= 0, \end{aligned}\right\} \tag{3.2}$$

and commutativity of $i\delta/\delta I$ with all variational derivative operators when the objects of operation are matrix elements of T- and N-products. To this remark we shall come back in the next section. To illustrate (3.2) we evaluate

$$\left.\begin{aligned} &\left(-i\frac{\delta}{\delta f(z)}\right)\langle 0 \mid \boldsymbol{T}\, T(\psi(y')\cdots\psi(y^{(l)})) \mid \Psi\rangle \\ &= \langle 0 \mid \boldsymbol{T}\, T(\bar{\psi}(z)\,\psi(y')\cdots\psi(y^{(l)})) \mid \Psi\rangle \\ &= (-1)^l\, T_\Psi(\mid y'\cdots y^{(l)} \mid z). \end{aligned}\right\} \tag{3.3}$$

Here we have used (2.13) and the symmetry properties of T-products.

iii) The state vector amplitudes are continuous functions of the coordinates. This is not quite trivial, because matrix elements of T-products are, in general, discontinuous functions. The discontinuous character of the T_Ψ-functions is made apparent by the δ-terms in the equations of motion for these functions (Ap. III. 3, 4 and 5). It can, however, be seen that the application of the differential operators occurring in the field equations to Ψ-functions does not give rise to such δ-functions. This can, for instance, be proved by induction using (2.25). In the following section, we find that the Ψ-functions satisfy homogeneous equations of motion, and this constitutes another verification of the continuity of these functions[1].

[1] The first derivative of the Ψ-functions with respect to a meson coordinate is also continuous. This difference between spinor field variables and meson field variables reflects the difference in the equations of motion for the two kinds of fields, the nucleon equations being of the first order, while the meson equation is of the second order.

iv) If the state $|\Psi\rangle$ represented by the infinite set of state vector amplitudes is a stationary state, corresponding to the eigenvalues P_μ for the total energy momentum vector of the closed system, then, in the source-free limit, the Ψ-functions oscillate according to

$$\Psi(x' \cdots | y' \cdots | z' \cdots) \sim \exp i P_\mu X_\mu. \tag{3.4}$$

Here, the X_μ's are any "center of gravity" coordinate. For instance, one can take X_μ as the average value of the coordinates $x'_\mu, \cdots y'_\mu, \cdots z'_\mu, \cdots$.[1] This follows immediately from (2.33) and the fact that T_Ψ-functions possess this property. The property (3.4) is of course the basis for the application of the present formalism to bound state problems.

v) The configuration space representation. The state vector amplitudes corresponding to a state $|\Psi\rangle$ provide us with a generalization of the Fock representation for free fields. As we have seen above, several of the simple properties of the Fock representation are maintained in the general case. One might, therefore, consider the set of state vector amplitudes as a representation of the state $|\Psi\rangle$. We shall take such a point of view in the following, and speak of this representation as the configuration space representation. Alternatively, we can also consider the functional $\Psi(||)$ which generates the state vector amplitudes as representing the state in question. In this way we speak of the functional representation. For the sake of convenience, we denote these two representations by the CSR and the FR, respectively.

To make full use of the CSR one should know, at least in principle, how to construct the scalar product of two states represented by their state vector amplitudes. This problem could not be solved and we have not even been able to prove that the CSR is a complete representation. Until further progress is

[1] Cf. FREESE [4]. As mentioned by FREESE, the most general definition of X_μ is

$$X_\mu = \alpha' x'_\mu + \cdots + \beta' y'_\mu + \cdots + \gamma' z'_\mu + \cdots,$$

where the α, β, and γ's are subject to the condition

$$\alpha' + \cdots + \beta' + \cdots + \gamma' + \cdots = 1.$$

made, application of the present formalism must therefore be based on an assumption of the completeness of the representation.

A comparison of the CSR with other better known representations might offer a possibility of discussing the completeness problem. The fact that $\langle 0|$ is the vacuum state of the source-free system has been used in the discussion of the oscillating behaviour of the amplitudes representing stationary states. It is easily seen that all other considerations remain valid for any choice of $\langle 0|$ if only this state coincides with the free-field vacuum in the limit of no coupling. An example of another possible choice of this state is provided by the vacuum state $\langle 0, \sigma|$ for the free fields $u(x, \sigma)$, $\psi(x, \sigma)$ which coincide with the source independent fields on a space-like surface σ. Moreover, it can be seen that one can choose different states in the definition of the functional $\Psi(||)$. Thus, instead of (2.15), we could have defined

$$\Psi^{\sigma''}_{\sigma'}(||) = \frac{\langle 0, \sigma''| \boldsymbol{T} |\Psi\rangle}{\langle 0, \sigma'| \boldsymbol{T} |0, \sigma'\rangle},$$

where $|0, \sigma''\rangle$ and $|0, \sigma'\rangle$ may be different.

The choice

$$\Psi^{\sigma}_{\sigma}(||) = \frac{\langle 0, \sigma| \boldsymbol{T} |\Psi\rangle}{\langle 0, \sigma| \boldsymbol{T} |0, \sigma\rangle} \tag{3.5}$$

leads to a representation in which the state vector amplitudes for all space-time points on σ coincide with the Tamm-Dancoff representation.

As is well known, one can consider the state $|0\rangle$ as the limit of $|0, \sigma\rangle$ in the sense of a certain limiting process, usually referred to as the adiabatic switching-on of the coupling at $t = -\infty$. In the sense of the same limiting process, one can regard the CSR representation as the limit of the representation based on (3.5) for $\sigma \to -\infty$. The coincidence of the representation (3.5) with the Tamm-Dancoff representation on σ tells us that (3.5) is a complete representation. There might, therefore, be a possibility of discussing the completeness of the CSR by a comparison with the Tamm-Dancoff representation. The complexity of the limiting process involved, however, does not make this a very promising prospect.

Another representation could be based on the functional

$$\Psi_\sigma(||) = \frac{\langle 0 | \boldsymbol{T} | \Psi \rangle}{\langle 0, \sigma | \boldsymbol{T} | 0, \sigma \rangle}. \tag{3.6}$$

The corresponding state vectors can be seen to coincide on σ with the representation given by DYSON [12].

4. The equations of motion.

In the preceding section, we have introduced two new representations, the functional representation and the configuration space representation. The simplest way to obtain the equations of motion in these two representations is first to derive the equations of motion in the FR. As we shall see, the equations of motion in the CSR can be obtained from those in the FR by a simple procedure.

i) The equations of motion in the FR. To determine the dependence of the functional Ψ on the sources we must try to set up variational equations making use of the field equations. The Ψ-functions depending on one space-time point only are given by the expressions

$$\left.\begin{aligned}
\Psi(x||) &= T_0(||)^{-1} T_\Psi(x||) - T_0(||)^{-1} T_0(x||)\, T_0(||)^{-1} T_\Psi(||), \\
\Psi(|y|) &= T_0(||)^{-1} T_\Psi(|y|) - T_0(||)^{-1} T_0(|y|)\, T_0(||)^{-1} T_\Psi(||), \\
\Psi(||z) &= T_0(||)^{-1} T_\Psi(||z) - T_0(||)^{-1} T_0(||z)\, T_0(||)^{-1} T_\Psi(||).
\end{aligned}\right\} \tag{4.1}$$

These equations are special cases of the formula (2.33), but can also easily be verified directly from the definition (2.15). As shown in the Appendix II, the T_Ψ-functions depending on one space-time point satisfy

$$\left.\begin{aligned}
(-\Box_x + m^2)\, T_\Psi(x||) - \lambda\, T_\Psi(|x|x) + I(x)\, T_\Psi(||) &= 0, \\
(\partial_y + M)\, T_\Psi(|y|) + \lambda\, T_\Psi(y|y|) + f(y)\, T_\Psi(||) &= 0, \\
(\bar{\partial}_z + M)\, T_\Psi(||z) + \lambda\, T_\Psi(z||z) + g(z)\, T_\Psi(||) &= 0.
\end{aligned}\right\} \tag{4.2}$$

These equations are of course also satisfied in the special case of $|\Psi\rangle$ being the vacuum state, i. e. for T_0-functions. Combining (4.1) and (4.2) we get

$$\left.\begin{aligned}
&(-\Box_x + m^2)\, \Psi(x\,|\,|) - \lambda\, T_0(|\,|)^{-1}\, T_\Psi(|x|x) \\
&\quad + \lambda\, T_0(|\,|)^{-1}\, T_0(|x|x)\, \Psi(|\,|) = 0, \\
&(\partial_y + M)\, \Psi(|y|) + \lambda\, T_0(|\,|)^{-1}\, T_\Psi(y|y|) \\
&\quad - \lambda\, T_0(|\,|)^{-1}\, T_0(y|y|)\, \Psi(|\,|) = 0, \\
&(\bar{\partial}_z + M)\, \Psi(|\,|z) + \lambda T_0(|\,|)^{-1}\, T_\Psi(z|\,|z) \\
&\quad - \lambda\, T_0(|\,|)^{-1}\, T_0(z|\,|z)\, \Psi(|\,|) = 0,
\end{aligned}\right\} \quad (4.3)$$

where the sources do no longer explicitly appear. To express (4.3) as linear equations in Ψ and the variational derivatives of Ψ, we eliminate the T_Ψ-functions by use of expressions of the type (2.28), (2.29). The resulting equations contain as factors certain combinations of T-functions for which we introduce the notation

$$\left.\begin{aligned}
\eta(x\,|\,|) &= T_0(|\,|)^{-1}\, T_0(x\,|\,|), \\
\eta(|y|) &= T_0(|\,|)^{-1}\, T_0(|y|), \\
\eta(|\,|z) &= T_0(|\,|)^{-1}\, T_0(|\,|z).
\end{aligned}\right\} \quad (4.4)$$

By the aid of these η-functions we can write the resulting linear differential variational equations for Ψ in the form

$$\left.\begin{aligned}
&(-\Box_x + m^2)\, \Psi(x\,|\,|) - \lambda\eta(|x|)\, \Psi(|\,|x) \\
&\quad + \lambda\eta(|\,|x)\, \Psi(|x|) - \lambda\Psi(|x|x) = 0, \\
&(\partial_y + M)\, \Psi(|y|) + \lambda\eta(y|\,|)\, \Psi(|y|) \\
&\quad + \lambda\eta(|y|)\, \Psi(y|\,|) + \lambda\Psi(y|y|) = 0, \\
&(\bar{\partial}_z + M)\, \Psi(|\,|z) + \lambda\eta(z|\,|)\, \Psi(|\,|z) \\
&\quad + \lambda\eta(|\,|z)\, \Psi(z|\,|) + \lambda\Psi(z|\,|z) = 0.
\end{aligned}\right\} \quad (4.5)$$

Thus, for any state $|\Psi\rangle$, the corresponding functional Ψ satisfies (4.5). The problem which restrictions (if any) must be imposed on the solutions of (4.5) to guarantee that the functional obtained represents a state of the system has not been solved in general. Hereto we shall return later.

ii) The *η-functions.* By a similar method we obtain equations of motion for the η-functions. Combining (4.2) and (4.4) we get

$$\left.\begin{aligned}(-\Box_x + m^2)\,\eta(x\,|\,|) - \lambda\, T_0(|\,|)^{-1}\, T_0(|x|x) + I(x) &= 0,\\ (\partial_y + M)\,\eta(|y|) + \lambda\, T_0(|\,|)^{-1}\, T_0(y|y|) + f(y) &= 0,\\ (\bar{\partial}_z + M)\,\eta(|\,|z) + \lambda\, T_0(|\,|)^{-1}\, T_0(z|\,|z) + g(z) &= 0.\end{aligned}\right\} \tag{4.6}$$

It is convenient to introduce a functional η by

$$\eta(|\,|) = \log \langle 0 |\, \boldsymbol{T}\, | 0 \rangle. \tag{4.7}$$

The η-functions (4.4) are contained as special cases in the following general definition of η-functions:

$$\left.\begin{aligned}&\eta(x'\cdots|y'\cdots|z'\cdots)\\ &= i\frac{\delta}{\delta I(x')}\cdots i\frac{\delta}{\delta g(y')}\cdots\left(-i\frac{\delta}{\delta f(z')}\right)\cdots\eta(|\,|).\end{aligned}\right\} \tag{4.8}$$

Thus, for instance, η-functions depending on two space-time points are given by

$$\left.\begin{aligned}\eta(|y|z) &= T_0(|\,|)^{-1}\, T_0(|y|z) - \eta(|y|)\,\eta(|\,|z),\\ \eta(x|y|) &= T_0(|\,|)^{-1}\, T_0(x|y|) - \eta(x|\,|)\,\eta(|y|),\\ \eta(x|\,|z) &= T_0(|\,|)^{-1}\, T_0(x|\,|z) - \eta(x|\,|)\,\eta(|\,|z).\end{aligned}\right\} \tag{4.9}$$

With the aid of these formulas we can eliminate the T_0-functions in (4.6) and obtain

$$\left.\begin{aligned}(-\Box_x + m^2)\,\eta(x\,|\,|) - \lambda\eta(|x|)\,\eta(|\,|x) - \lambda\eta(|x|x) + I(x) &= 0,\\ (\partial_y + M)\,\eta(|y|) + \lambda\eta(y|\,|)\,\eta(|y|) + \lambda\eta(y|y|) + f(y) &= 0,\\ (\bar{\partial}_z + M)\,\eta(|\,|z) + \lambda\eta(z|\,|)\,\eta(|\,|z) + \lambda\eta(z|\,|z) + g(z) &= 0.\end{aligned}\right\} \tag{4.10}$$

These equations are variational differential equations satisfied by the η-functional in the FR.

Contrary to the Ψ-functional which depends on the particular state considered, the η-functional is uniquely determined in the theory. We must, therefore, supplement the η-equations by

boundary conditions characterizing the particular solution of (4.10) which enters in the equations of motion for the Ψ-functional.

For the discussion of this problem we need an interpretation of the operator $\boldsymbol{T}$. Let $\boldsymbol{T}(t)$ be the transformation which connects the source-free fields and the source-dependent fields according to

$$\left.\begin{aligned} u(x) &= \boldsymbol{T}(t)^{-1}\, u_0(x)\, \boldsymbol{T}(t), \\ \psi(x) &= \boldsymbol{T}(t)^{-1}\, \psi_0(x)\, \boldsymbol{T}(t). \end{aligned}\right\} \tag{4.11}$$

As may be seen from (1.14), $\boldsymbol{T}(t)$ satisfies the variational equation

$$\delta\, \boldsymbol{T}(t) = -\, i\, \boldsymbol{T}(t) \int_{-\infty}^{t} \delta W(x')\, dx', \tag{4.12}$$

and the boundary condition $\boldsymbol{T}(t) = 1$ in the limit of vanishing sources. Hence, we see that the operator $\boldsymbol{T}$, as defined by (2.1), can be interpreted as the transformation which connects the source-independent fields with the complete source-dependent fields in the infinite future, i. e.

$$\lim_{t \to \infty} (\boldsymbol{T}^{-1}\, u_0(x)\, \boldsymbol{T} - u(x)) = 0,$$

and similar relations for the two other fields. We shall use these relations in the form

$$\left.\begin{aligned} \lim_{t \to \infty} (u_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, u(x)) &= 0, \\ \lim_{t \to \infty} (\psi_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, \psi(x)) &= 0, \\ \lim_{t \to \infty} (\bar{\psi}_0(x)\, \boldsymbol{T} - \boldsymbol{T}\, \bar{\psi}(x)) &= 0. \end{aligned}\right\} \tag{4.13}$$

Assume now, as we already tacitly have done in the previous considerations, that the source-independent system by a suitable renormalization has been cast into a form such that a state of lowest energy, the vacuum state, exists and that the energy and momentum of this state is zero. It follows that any stationary state of the system corresponds to an energy momentum vector lying inside the half cone in momentum space characterized by $p_\mu p_\mu < 0$ and $p_0 > 0$. Evidently, time-like momenta corresponding to negative energy are excluded by the assumption made. How-

ever, also space-like momenta are excluded since, by a suitable Lorentz transformation, any space-like momentum vector might be brought into a form with $p_0 < 0$, i. e. with negative energy. Corresponding to the invariant decomposition of momentum space into the three subspaces: the positive frequency part (or the (+)-part) characterized by $p_\mu\, p_\mu < 0, p_0 > 0$, the negative frequency part (or the (—)-part) characterized by $p_\mu\, p_\mu < 0, p_0 < 0$, and the (0)-part with $p_\mu\, p_\mu > 0$, we can split any field variable into three parts, the (+)- the (—)- and the (0)-part. For instance, if we define $u(p)$ by

$$u(x) = (2\pi)^{-2} \int u(p)\, e^{ipx}\, dp, \tag{4.14}$$

we have

$$\left.\begin{aligned} u^{(+)}(x) &= (2\pi)^{-2} \int\limits_{p^2 < 0,\ p_0 > 0} u(p)\, e^{ipx}\, dp, \\ u^{(-)}(x) &= (2\pi)^{-2} \int\limits_{p^2 < 0,\ p_0 < 0} u(p)\, e^{ipx}\, dp, \\ u^{(0)}(x) &= (2\pi)^{-2} \int\limits_{p^2 > 0} u(p)\, e^{ipx}\, dp. \end{aligned}\right\} \tag{4.15}$$

From our assumption it follows that

$$u_0(x)\,|0\rangle = u_0^{(-)}(x)\,|0\rangle. \tag{4.16}$$

Hence also, as $\lim\limits_{t \to -\infty} (u(x) - u_0(x)) = 0$,

$$\lim_{t \to -\infty} (u(x) - u_0^{(-)}(x))\,|0\rangle = 0, \tag{4.17}$$

i. e. $u(x)\,|0\rangle$ contains only negative frequencies in the infinite past. The corresponding statement about the asymptotic behaviour of the field variables when multiplied by $\langle 0|\, \boldsymbol{T}$ from the left follows from (4.13). By the same kinds of arguments as those leading to (4.17) we get

$$\lim_{t \to +\infty} \langle 0|\, \boldsymbol{T}\, (u(x) - u_0^{(+)}(x)) = 0. \tag{4.18}$$

Thus, $\langle 0|\, \boldsymbol{T}\, u(x)$ contains only positive frequencies in the in-

finite future. The same result applies to the two other kinds of field variables.

Consider any T_0-function

$$T_0(x' \cdots x^{(\nu)} \cdots x^{(k)} \,|\, y' \cdots |\, z' \cdots),$$

say. In the limit where one of the space-time points, for instance $x^{(\nu)}$, tends to $-\infty$, we have, considering all other space-time points as fixed,

$$\left.\begin{aligned} &\lim_{x_0^{(\nu)} \to -\infty} [T_0(x' \cdots x^{(\nu)} \cdots x^{(k)} | y' \cdots | z' \cdots) \\ &- \langle 0 | \boldsymbol{T}\, T(u(x') \cdots u(x^{(\nu-1)})\, u(x^{(\nu+1)}) \cdots u(x^{(k)}) \cdots) \\ &\times u(x^{(\nu)}) \,|\, 0\rangle] = 0. \end{aligned}\right\} \quad (4.19)$$

Hence, we infer from (4.17) that in the limit $x_0^{(\nu)} \to -\infty$ the T_0-function contains only negative frequencies in a Fourier decomposition with respect to $x^{(\nu)}$. The same property holds for any other space-time point occurring in a T_0-function. In the opposite limit, we get by a similar argument that T_0 contains only positive frequencies corresponding to any space-time point approaching the infinite future. Using a terminology which is suggestive in connection with the discussion, given by STUECKELBERG, FEYNMAN and FIERZ [11], of the properties of the causal Green's functions, we say that T_0-functions obey causal boundary conditions. The possibility of expressing the η-functions in terms of T_0-functions (as, for instance, expressed by (4.8) and (4.7)) implies that also η-functions satisfy causal boundary conditions.

The equations for the η-functions (4.10) are of the second order in the variational derivatives. We must therefore supplement the boundary conditions with the value of the functional η and its first variational derivative in the limit of vanishing sources. In this limit, however, $\boldsymbol{T} = 1$. Hence, $\eta(x\,|\,|) = \langle 0 \,|\, u_0(x) \,|\, 0\rangle = 0$, in virtue of (4.16). Similarly, in the same limit, $\eta(|\,y\,|) = \eta(|\,|\,z) = 0$. Finally, by the definition (4.7) we have chosen $\eta(|\,|)\,|_{I=\varphi=\bar{\varphi}=0} = 0$.

Similar considerations apply to the state vector amplitudes in the infinite future. This is obvious from (2.33) or alternatively from the definition (2.15). Hence, Ψ-functions obey causal

boundary conditions in the infinite future. In the infinite past, however, the behaviour of the Ψ-functions depend on the particular state considered. One more information about the Ψ-functions follows from the considerations in Section 3, iv). If we consider the Ψ-functions for all time variables equal, then, in the source-free limit, only positive frequencies are allowed with respect to this common time.

It is not known whether more conditions must be imposed on the state vector amplitudes to guarantee that a solution of the equations of motion (4.5) actually represents a state of the system. The solution of this problem is of course connected with the likewise unsolved problem of the completeness of the CSR.

iii) The equations of motion in the CSR. Having thus obtained the equations of motion in the FR it becomes a simple matter to derive the equations of motion in the CSR. As mentioned in the Appendix III, in connection with the derivation of the equations of motion of the time ordered products, the differential operators occurring in the field equations commute with all variational operators. We can, therefore, obtain an infinite set of linear differential equations for the Ψ-functions by variational derivation of the equations (4.5). For instance, by applying the variational operator $i\,\delta/\delta I\,(x')$ to the first equation (4.5), we get

$$\left.\begin{aligned}&(-\Box_x + m^2)\,\Psi(xx'\,|\,|) - \lambda\eta\,(|x|)\,\Psi(x'\,|\,|x)\\&+\lambda\eta\,(|\,|x)\,\Psi(x'\,|x|) - \lambda\eta\,(x'\,|x|)\,\Psi(|\,|x)\\&+\lambda\eta\,(x'\,|\,|x)\,\Psi(|x|) - \lambda\,\Psi(x'\,|x|x) = 0.\end{aligned}\right\} \quad (4.20)$$

Similarly, from the second equation (4.5), we infer

$$\left.\begin{aligned}&(\partial_y + M)\,\Psi(x'\,|y|) + \lambda\eta\,(y\,|\,|)\,\Psi(x'\,|y|)\\&+\lambda\eta\,(|y|)\,\Psi(x'y\,|\,|) + \lambda\eta\,(yx'\,|\,|)\,\Psi(|y|)\\&+\lambda\eta\,(x'\,|y|)\,\Psi(y\,|\,|) + \lambda\,\Psi(x'y\,|y|) = 0.\end{aligned}\right\} \quad (4.21)$$

Proceeding, and taking variational derivatives, one can construct equations involving Ψ-functions with an arbitrary number of meson space-time coordinates. Equations involving one more

nucleon space time point and one more anti-nucleon space time point are obtained by applying the operators $i\delta/\delta g(y')$ and $-i\delta/\delta f(z')$, respectively. Only, when varying the spinor sources, one should remember the anti-commutativity (3.2) of the nucleon variational operators. Thus,

$$i\frac{\delta}{\delta g(y')}(\eta(y||)\,\Psi(|y|)) = \eta(y|y'|)\,\Psi(|y|) - \eta(y||)\,\Psi(|yy'|),$$

$$i\frac{\delta}{\delta g(y')}(\eta(|y|)\,\Psi(y||)) = -\eta(|yy'|)\,\Psi(y||) - \eta(|y|)\,\Psi(y|y'|),$$

$$i\frac{\delta}{\delta g(y')}\,\Psi(y|y|) = -\Psi(y|yy'|).$$

Observing this, we get by applying $i\delta/\delta g(y')$ to the second equation (4.5)

$$\left.\begin{aligned}(\partial_y + M)\,\Psi(|yy'|) + \lambda\eta(y||)\,\Psi(|yy'|) + \lambda\,\Psi(y|yy'|)\\ -\lambda\eta(y|y'|)\,\Psi(|y|) + \lambda\eta(|yy'|)\,\Psi(y||)\\ +\lambda\eta(|y|)\,\Psi(y|y'|) = 0.\end{aligned}\right\} \quad (4.22)$$

By a similar procedure one obtains equations connecting the various η-functions. For later reference we note a few examples:

$$\left.\begin{aligned}(-\Box_x + m^2)\,\eta(xx'||) - \lambda\eta(x'|x|)\,\eta(||x) - \lambda\eta(|x|)\,\eta(x'||x)\\ -\lambda\eta(x'|x|x) + i\,\delta(x-x') = 0,\end{aligned}\right\} \quad (4.23)$$

$$\left.\begin{aligned}(\partial_y + M)\,\eta(|y|z) + \lambda\eta(y||)\,\eta(|y|z) - \lambda\eta(y||z)\,\eta(|y|)\\ +\lambda\eta(y|y|z) + i\,\delta(y-z) = 0,\end{aligned}\right\} \quad (4.24)$$

$$\left.\begin{aligned}(\bar\partial_z + M)\,\eta(|y|z) + \lambda\eta(z||)\,\eta(|y|z) + \lambda\eta(z|y|)\,\eta(||z)\\ +\lambda\eta(z|y|z) + i\,\delta(y-z) = 0,\end{aligned}\right\} \quad (4.25)$$

and, finally, an equation involving three space time points

$$\left.\begin{aligned}(\partial_y + M)\,\eta(x|y|z) + \lambda\eta(y||)\,\eta(x|y|z) + \lambda\eta(xy|y|z)\\ + \lambda\eta(xy||)\,\eta(|y|z) - \lambda\eta(xy||z)\,\eta(|y|)\\ - \lambda\eta(y||z)\,\eta(x|y|) = 0\,.\end{aligned}\right\} \quad (4.26)$$

The last equation can, for instance, be obtained by operating with $i\delta/\delta I(x)$ on the equation (4.24).

In the CSR it would seem most natural to represent the state under consideration by the state vector amplitudes taken in the limit of vanishing sources. There is, however, as emphasized by SCHWINGER [2], some advantage of postponing the limiting process $I(x) \to 0$ to a later stage in the considerations. If we, instead of considering meson theory, had taken electrodynamics as an example of illustrating the general scheme developed here, we would have had an obvious reason for doing this, as in electrodynamics the external source of the electromagnetic field has a direct interpretation in terms of a classical distribution of current and charge interacting with the system. Such a justification can hardly be found in our case. Still, we shall find it mathematically convenient in the following considerations to keep the meson field source in the theory.

We, thus, consider the limit of vanishing spinor sources. In this case, simplifications arise due to the fact that the difference ΔN between the total number of nucleons and the total number of anti-nucleons is then a constant of the motion. This implies a selection rule for T_0-functions. Only those T_0-functions are different from zero which contain the same number of nucleon and anti-nucleon space time points. If no $\boldsymbol{T}$ operator appeared in the definition

$$T_0(x'\cdots|y'\cdots|x'\cdots) = \langle 0|\,\boldsymbol{T}\cdot T(u(x')\cdots\psi(y')\cdots\bar{\psi}(z')\cdots)\,|0\rangle,$$

this selection rule would follow in the usual way from $\Delta N|0\rangle = 0$. However, it is easily seen from (2.5), remembering that in the limit considered we have $W = I\,u$, that ΔN commutes with $\boldsymbol{T}$ and, thus, the selection rule is not influenced by the presence of the $\boldsymbol{T}$-operator.

With this result, we can write (4.5) in the simpler form

$$\left.\begin{aligned}(-\Box_x + m^2)\,\Psi(x\,|\,|) - \lambda\,\Psi(|\,x\,|\,x) &= 0,\\ (\partial_y + M)\,\Psi(|\,y\,|) + \lambda\eta(y\,|\,|)\,\Psi(|\,y\,|) + \lambda\,\Psi(y\,|\,y\,|) &= 0,\\ (\bar{\partial}_z + M)\,\Psi(|\,|\,z) + \lambda\eta(z\,|\,|)\,\Psi(|\,|\,z) + \lambda\,\Psi(z\,|\,|\,z) &= 0,\end{aligned}\right\} \quad (4.27)$$

the limit $\varphi = \bar{\varphi} = 0$ being understood in these equations. It may be of some interest in the following to compare these equations with the equations obtained from (4.23), (4.24), and (4.25), taking the same limit, viz.

$$\left.\begin{aligned}(-\Box_x + m^2)\,\eta(x x'\,|\,|) - \lambda\eta(x'\,|\,x\,|\,x) + i\delta(x - x') &= 0,\\ (\partial_y + M)\,\eta(|\,y\,|\,z) + \lambda\eta(y\,|\,|)\,\eta(|\,y\,|\,z) + \lambda\eta(y\,|\,y\,|\,z) + i\delta(y - z) &= 0,\\ (\bar{\partial}_z + M)\,\eta(|\,y\,|\,z) + \lambda\eta(z\,|\,|)\,\eta(|\,y\,|\,z) + \lambda\eta(z\,|\,y\,|\,z) + i\delta(y - z) &= 0.\end{aligned}\right\} \quad (4.28)$$

These two sets of equations are of very much the same structure. The main difference is that the equations for the state vector amplitudes are homogeneous equations, while those for the η-functions are inhomogeneous ones. We shall discuss the relations between these two sets of equations more closely in the next section. Here, we only mention that the second equation (4.27) and the second equation (4.28) may be written as

$$\left.\begin{aligned}\left(\partial_y + M + \lambda\eta(y\,|\,|) + i\lambda\frac{\delta}{\delta I(y)}\right)\Psi(|\,y\,|) &= 0,\\ \left(\partial_y + M + \lambda\eta(y\,|\,|) + i\lambda\frac{\delta}{\delta I(y)}\right)\eta(|\,y\,|\,z) &= -i\delta(y - z),\end{aligned}\right\} \quad (4.29)$$

respectively. Thus, we see that, in a certain sense, $\Psi(|\,y\,|)$ obey the homogeneous equation of motion corresponding to the equation for $\eta(|\,y\,|\,z)$.

5. The equations for the one-and two-nucleon problems.

As mentioned in the Introduction, the present formalism combines the theory of Schwinger with that of Heisenberg and Freese. To illustrate this we shall briefly discuss the formal properties of the one- and two-nucleon equations from the point of view of the CSR. For the sake of completeness, and in order

to introduce some convenient notations, we first summarize the derivation of the one-nucleon equation given by SCHWINGER [2].

i) *The one-nucleon equation.*

In the limit of vanishing spinor sources the simplest η-functions satisfy, according to (4.10) and (4.28), the equations of motion

$$\left.\begin{aligned} (-\Box + m^2)\,\eta(x\,|\,|) - \lambda\eta(|x|x) + I(x) &= 0, \\ (-\Box + m^2)\,\eta(xx'\,|\,|) - \lambda\eta(x'|x|x) + i\delta(x-x') &= 0, \\ \left(\partial_y + m + \lambda\eta(y\,|\,|)\right)\eta(|y|z) + \lambda\eta(y|y|z) + i\delta(y-z) &= 0. \end{aligned}\right\} \quad (5.1)$$

To simplify the notation, and also to distinguish the η-functions in this limit from the general ones, we introduce

$$\left.\begin{aligned} U(x) &= \eta(x\,|\,|), \\ \Delta_c'(x,x') &= i\eta(xx'\,|\,|), \\ S_c'(x,x') &= -i\eta(|x|x'), \end{aligned}\right\} \quad (5.2)$$

and, consequently, write the equations (5.1) in the form

$$\left.\begin{aligned} (-\Box + m^2)\,U(x) - i\lambda S_c'(x,x) + I(x) &= 0, \\ (-\Box + m^2)\,\Delta_c'(x,x') + i\lambda\frac{\delta}{\delta I(x')}S_c'(x,x) &= \delta(x-x'), \\ \left(\partial + M + \lambda U(x)\right)S_c'(x,x') + i\lambda\frac{\delta}{\delta I(x)}S_c'(x,x') &= -\delta(x-x'). \end{aligned}\right\} \quad (5.3)$$

As discussed in Section 4.ii, the η-functions satisfy causal boundary conditions. Hence, in the limit $I = \lambda = 0$, we have

$$\left.\begin{aligned} \Delta_c'(x,x') &= \Delta_c(x-x'), \\ S_c'(x,x') &= S_c(x-x'), \end{aligned}\right\} \quad (5.4)$$

where Δ_c and S_c are the well-known causal solutions of

$$\left.\begin{aligned} (-\Box + m^2)\,\Delta_c(x-x') &= \delta(x-x'), \\ (\partial + M)\,S_c(x-x') &= -\delta(x-x'). \end{aligned}\right\} \quad (5.5)$$

The equations (5.3) are, for our case of nucleons interacting with scalar neutral mesons, the analogue of the equations for the Green's functions in electrodynamics studied in Schwinger's paper. Following his method we substitute the variational derivative operators in (5.3) by polarization operators Π_c^* and Σ_c^* defined by

$$\left.\begin{aligned} i\lambda \frac{\delta}{\delta I(x')} S_c'(x, x) &= \Pi_c^*(x, 1)\, \Delta_c'(1, x') \\ i\lambda \frac{\delta}{\delta I(x)} S_c'(x, x') &= \Sigma_c^*(x, 1)\, S_c'(1, x'). \end{aligned}\right\} \tag{5.6}$$

Here, and in the following, numbers occurring twice denote variables of integration. Thus, for instance,

$$\Sigma_c^*(x, 1)\, S_c'(1, x') = \int \Sigma_c^*(x, \xi')\, S_c'(\xi', x')\, d\xi'. \tag{5.7}$$

By (5.6) the equations (5.3) take the form

$$\left.\begin{aligned} (-\Box + m^2)\, U(x) - i\lambda\, S_c'(x, x) &= -I(x), \\ (-\Box + m^2)\, \Delta_c'(x, x') + \Pi_c^*(x, 1)\, \Delta_c'(1, x') &= \delta(x - x'), \\ (\partial + M + \lambda U(x))\, S_c'(x, x') + \Sigma_c^*(x, 1)\, S_c'(1, x') &= -\delta(x - x'). \end{aligned}\right\} \tag{5.8}$$

Operating on the last of these equations with $i\lambda\delta/\delta I(x'')$ we get, after integration and taking into account the causal boundary conditions,

$$i\lambda \frac{\delta}{\delta I(x'')} S_c'(x', x''') = S_c'(x', 1)\, \Phi(1, 2, 3)\, \Delta_c'(2, x'')\, S_c'(3, x'''). \tag{5.9}$$

The kernel Φ depending on three space time points is given by

$$\left.\begin{aligned} &\Phi(x', x'', x''') \\ &= -i\lambda^2 \delta(x' - x'')\, \delta(x'' - x''') - i\lambda \frac{\delta \Sigma_c^*(x', x''')}{\delta U(x'')}. \end{aligned}\right\} \tag{5.10}$$

In the derivation of (5.9) use has been made of the fact that I does not appear explicitly in the last equation (5.8), whence

$$\frac{\delta}{\delta I(x)} \Sigma_c^* (x', x''') = -\frac{\delta \Sigma_c^* (x', x''')}{\delta U(2)} \Delta_c' (2, x''). \quad (5.11)$$

On comparison of (5.6) and (5.9) we infer integro variational equations characterizing the polarization operators, viz.

$$\left.\begin{aligned} \Pi_c^* (x, x') &= S_c' (x, 1)\, \Phi (1, x', 3)\, S_c' (3, x), \\ \Sigma_c^* (x, x') &= S_c' (x, 1)\, \Phi (1, 2, x')\, \Delta_c' (2, x). \end{aligned}\right\} \quad (5.12)$$

For later reference we mention that, from the equation conjugate to the last equation (5.1), viz.

$$\left(\bar{\partial}_z + M + \lambda \eta (z|\,|)\right) \eta (|\,y\,|\,z) + \lambda \eta (z|\,y\,|\,z) + i\, \delta (y - z) = 0, \quad (5.13)$$

we get by arguments similar to those leading to (5.8) an equation of the form

$$\left(\bar{\partial}' + M + \lambda\, U (x')\right) S_c' (x, x') + S_c' (x, 1)\, \bar{\Sigma}_c^* (1, x') = -\delta (x - x'). \quad (5.14)$$

The polarization operator in this equation is given by

$$\bar{\Sigma}_c^* (x, x') = \Phi (x, 2, 3)\, \Delta_c' (2, x')\, S_c' (3, x'). \quad (5.15)$$

According to Schwinger, the one-nucleon equation is obtained as the homogeneous equation of motion corresponding to the inhomogeneous equation (5.8) for the Green's function S_c'. Thus, denoting the one-nucleon "wave function" by χ, the equation reads

$$\left(\partial + M + \lambda\, U (x)\right) \chi (x) + \Sigma_c^* (x, 1)\, \chi (1) = 0. \quad (5.16)$$

As shown in the previous section, the equation of motion for the state vector amplitude depending on one nucleon coordinate is

$$\left(\partial_y + M + \lambda\, U (y)\right) \Psi (|\,y\,|) + i\, \lambda \frac{\delta}{\delta I (y)} \Psi (|\,y\,|) = 0. \quad (5.17)$$

The similarity between this equation and the inhomogeneous equation for the Green's function S_c' makes it natural to investigate under which conditions solutions of (5.16) also satisfy (5.17). For this to be true we must have

$$i\lambda \frac{\delta}{\delta I(x)} \chi(x) = \Sigma_c^*(x, 1)\, \chi(1). \tag{5.18}$$

By derivation of (5.16) with respect to $I(x')$ we get, after integration, an expression for the variational derivative of χ, viz.

$$i\lambda \frac{\delta \chi(x)}{\delta I(x')} = \chi^{(0)}(x'; x) + S_c'(x, 1)\, \Phi(1, 2, 3)\, \Delta_c'(2, x')\, \chi(3). \tag{5.19}$$

The function $\chi^{(0)}$ is a so far undetermined solution of

$$\left(\partial + M + \lambda U(x)\right) \chi^{(0)}(x'; x) + \Sigma_c^*(x, 1)\, \chi^{(0)}(x'; 1) = 0. \tag{5.20}$$

Comparing (5.18) with (5.19) we see that χ is a solution of (5.17), provided that $\chi^{(0)}$ vanishes.

We thus have the result that any solution of the coupled equations

$$\left.\begin{aligned} &\left(\partial_y + M + \lambda U(y)\right) \Psi_{(1)}(|y|) + \Sigma_c^*(y, 1)\, \Psi_{(1)}(|1|) = 0, \\ &\Psi_{(1)}(x|y|) = \lambda^{-1} S_c'(y, 1)\, \Phi(1, 2, 3)\, \Delta_c'(2, x)\, \Psi_{(1)}(|3|) \end{aligned}\right\} \tag{5.21}$$

also satisfies (5.17). The reverse statement is of course not true. We have, therefore, attached a subscript to the state vector amplitudes in these equations to indicate that a solution in the form (5.21) is possible for a restricted class of states only, the one-nucleon states.

From (5.17) we get by variational derivation an infinite system of coupled equations for the state vector amplitudes. The first of the equations derived from (5.17) reads

$$\left.\begin{aligned} &\left(\partial_y + M + \lambda U(y)\right) \Psi(x|y|) - i\lambda \Delta_c'(x, y)\, \Psi(|y|) \\ &\qquad + \lambda\, \Psi(xy|y|) = 0. \end{aligned}\right\} \tag{5.22}$$

Let us now follow, in the present version of the CSR, the suggestion by Freese and try to eliminate all amplitudes depending on one or more meson coordinates from the infinite set of equations. The states for which this elimination process is possible might, alternatively, be called the one-nucleon states. To get an idea how the resulting equation will look we convert the infinite system of equations into a finite one by the approximation assumption that $\Psi(xy|y|)$ can be neglected in (5.22). We can then solve (5.22) by the aid of the Green's function satisfying

$$\left(\partial_y + M + \lambda\, U(y)\right) S_c^{(U)}(y, y') = -\,\delta(y - y'), \quad (5.23)$$

and causal boundary conditions. The approximate solution of (5.22) is then

$$\Psi(x\,|\,y\,|) = \varphi^{(0)}(x; y) - i\,\lambda\, S_c^{(U)}(y, 1)\, \Delta_c'(1, x)\, \Psi(|\,1\,|), \quad (5.24)$$

where $\varphi^{(0)}$ is a solution of

$$\left(\partial_y + M + \lambda\, U(y)\right) \varphi^{(0)}(x; y) = 0. \quad (5.25)$$

To obtain an equation of the form (5.16) we choose $\varphi^{(0)} = 0$. With this choice we get, instead of (5.17),

$$\left(\partial_y + M + \lambda\, U(y)\right) \Psi(|\,y\,|) - i\,\lambda^2\, S_c^{(U)}(y, 1)\, \Delta_c'(1, y)\, \Psi(|\,1\,|) = 0. \quad (5.26)$$

In the next approximation one would keep all amplitudes with less than two meson space time points. Proceeding in this way one can, in principle, construct an exact equation of the form (5.16), provided that the procedure converges. The polarization operator ${}'\Sigma_c^*$, say, obtained in this way is characterized by the requirement that the resulting equation

$$\left(\partial_y + M + \lambda\, U(y)\right) \Psi(|\,y\,|) + {}'\Sigma_c^*(y, 1)\, \Psi(|\,1\,|) = 0$$

is consistent with (5.17), i. e. that

$${}'\Sigma_c^*(y, 1)\, \Psi(|\,1\,|) = i\,\lambda\, \frac{\delta}{\delta I(y)}\, \Psi(|\,y\,|).$$

By arguments similar to those above it can easily by verified that ${}'\Sigma_c^*$ is, in fact, identical with Σ_c^*. Thus the resulting one-nucleon equation is identical with Schwinger's equation.

The advantage of the equations (5.21) as compared with the infinite system of equations obtained from (5.17) becomes obvious when we pass to the physically interesting limit of vanishing external sources. For $I = 0$, the second equation (5.21) and the equations obtained therefrom by variational derivation become explicit expressions for the state vector amplitudes with one and more meson space time coordinates. Therefore, for $I = 0$, any solution of the one-nucleon equation provides us

with a corresponding solution of the equations of motion in the configuration space representation.

ii) *The two-nucleon equation.*

According to Schwinger, the two-nucleon Green's function is defined by

$$G(y, y'; z, z') = \frac{\delta}{\delta f(z')} \frac{\delta}{\delta f(z)} \frac{T_0(|yy'|)}{T_0(|\,|)}\Bigg|_{\varphi=\bar{\varphi}=0.} \tag{5.27}$$

Using the formulas in Section 4, it is easily verified that

$$\left.\begin{aligned} &G(y, y'; z, z') \\ &= \eta(|yy'|zz') - \eta(|y|z)\,\eta(|y'|z') + \eta(|y|z')\,\eta(|y'|z), \end{aligned}\right\} \tag{5.28}$$

the limit $\varphi = \bar{\varphi} = 0$ being understood in this formula. An equation of motion for $\eta(|yy'|zz')$ can be obtained from (4.10) by taking appropriate variational derivatives. From the equation obtained in this way, and by (5.3), we get

$$\left.\begin{aligned} &\left(\partial_y + M + \lambda U(y)\right) G(y, y'; z, z') + i\lambda \frac{\delta}{\delta I(y)} G(y, y'; z, z') \\ &\quad = -\delta(y - z)\, S_c'(y', z') + \delta(y - z')\, S_c'(y', z). \end{aligned}\right\} \tag{5.29}$$

Using (5.3) we see that

$$\left.\begin{aligned} &\left(\partial_{y'} + M + \lambda U(y) + \lambda \frac{\delta}{\delta I(y)}\right)\left(\partial_y + M + \lambda U(y') + i\lambda \frac{\delta}{\delta I(y')}\right) G(y, y'; z, z') \\ &\quad = \delta(y - z)\,\delta(y' - z') - \delta(y - z')\,\delta(y' - z). \end{aligned}\right\} \tag{5.30}$$

Following Schwinger, we introduce an interaction operator W by

$$\left.\begin{aligned} &F(y)\,F(y')\,G(y, y'; z, z') - W(y, y'; 1, 2)\,G(1, 2; z, z') \\ &\quad = \delta(y - z)\,\delta(y' - z') - \delta(y - z')\,\delta(y' - z). \end{aligned}\right\} \tag{5.31}$$

The symbol F is an abbreviation of the integral differential operator entering in the equation for the one-nucleon Green's function, i. e.

$$F(y)\,\xi(y) = \left(\partial_y + M + \lambda U(y)\right)\xi(y) + \Sigma_c^*(y, 1)\,\xi(1). \tag{5.32}$$

As G satisfies causal boundary conditions we can integrate (5.31) by means of S'_c. Combining the resulting equation, viz.

$$\left.\begin{aligned} F(y)\,G(y,y';z,z') + S'_c(y',2)\,W(y,2;3,4)\,G(3,4;z,z') \\ = -\,\delta(y-z)\,S'_c(y',z') + \delta(y-z')\,S'_c(y',z), \end{aligned}\right\} \quad (5.33)$$

with (5.29) we infer a condition on the interaction operator:

$$\left.\begin{aligned} i\lambda\frac{\delta}{\delta I(y)}\,G(y,y';z,z') = \Sigma^*_c(y,1)\,G(1,y';z,z') \\ + S'_c(y',2)\,W(y,2;3,4)\,G(3,4;z,z'). \end{aligned}\right\} \quad (5.34)$$

Integrating (5.33) once more, we find that

$$\left.\begin{aligned} G(y,y';z,z') - S'_c(y,1)\,S'_c(y',2)\,W(1,2;3,4)\,G(3,4;z,z') \\ = S'_c(y,z)\,S'_c(y',z') - S'_c(y,z')\,S'_c(y',z). \end{aligned}\right\} \quad (5.35)$$

From this equation one gets an expression for the variational derivative of G with respect to $I(y)$ which, together with (5.34), gives Schwinger's characterization of the interaction operator, viz.

$$\left.\begin{aligned} W(y,y';1,2)\,G(1,2;z,z') \\ = \Phi(y',1,2)\,\Delta'_c(1,y)\,G(y,2;z,z') \\ + S'_c(y,1)\,i\lambda\frac{\delta}{\delta I(y)}\,[W(1,y';3,4)\,G(3,4;z,z')]. \end{aligned}\right\} \quad (5.36)$$

For W we shall use another equation which does not depend explicitly on the variational derivative of the two-nucleon Green's function. From (5.35) one gets, using (5.9),

$$\left.\begin{aligned} i\lambda\frac{\delta}{\delta I(x)}\,G(y,y';z,z') \\ = \frac{1}{2}\lambda\,G(y,y';1,2)\left[i\frac{\delta}{\delta I(x)}\,W(1,2;3,4)\right]G(3,4;z,z') \\ + G(y,y';1,2)\,\Phi(1,3,4)\,\Delta'_c(3,x)\,S'_c(4,5)\,F(5)\,F(2)\,G(5,2;z,z'). \end{aligned}\right\} \quad (5.37)$$

The combination of this expression with (5.34) gives the alternative characterization of W, viz.

$$\left.\begin{aligned}\frac{1}{2}\left(\Sigma_c^*(y,z)\,\delta(y'-z') - \Sigma_c^*(y,z')\,\delta(y'-z)\right) + S_c'(y',2)\,W(y,2;z,z') \\ = \frac{1}{2}\lambda\,G(y,y';1,2)\,i\frac{\delta}{\delta I(y)}\,W(1,2;z,z') \\ -\frac{1}{2}\overline{F}(z')\,G(y,y';1,z')\,\Phi(1,2,z)\,\Delta_c'(2,y) \\ -\frac{1}{2}\overline{F}(z)\,G(y,y';z,1)\,\Phi(1,2,z')\,\Delta_c'(2,y).\end{aligned}\right\} \quad (5.38)$$

Here, $\overline{F}$ denotes the operator entering in the equation of motion for the one-nucleon Green's function in the form given by (5.14), i. e.

$$\overline{F}(z)\,\zeta(z) = \left(\bar{\partial}_z + M + \lambda\,U(z)\right)\zeta(z) + \zeta(1)\,\overline{\Sigma}_c^*(1,z). \quad (5.39)$$

The equations of motion for the state vector amplitude depending on two nucleon space time points obtained from the equation (4.22) by passing to the limit of vanishing spinor sources read

$$\left.\begin{aligned}\left(\partial_y + M + \lambda\,U(y)\right)\Psi(|yy'|) + \lambda\,\Psi(y|yy'|) = 0, \\ \left(\partial_{y'} + M + \lambda\,U(y')\right)\Psi(|yy'|) + \lambda\,\Psi(y'|yy'|) = 0,\end{aligned}\right\} \quad (5.40)$$

whence also

$$\left(\partial_y + M + \lambda\,U(y) + i\lambda\frac{\delta}{\delta I(y)}\right)\left(\partial_{y'} + M + \lambda\,U(y') + i\lambda\frac{\delta}{\delta I(y')}\right)\Psi(|yy'|) = 0. \quad (5.41)$$

This equation is a homogeneous equation of motion corresponding to (5.30) in the same sense as the equation (5.17) for the one-nucleon amplitude is the homogeneous equation corresponding to the equation for the one-nucleon Green's function (5.3).

According to Schwinger, the two-nucleon equation is the homogeneous equation corresponding to the equation (5.31), i. e.

$$F(y)\,F(y')\,\chi(y,y') - W(y,y';1,2)\,\chi(1,2) = 0. \quad (5.42)$$

It seems to be difficult to establish any general connection between the solutions of this equation and the solutions of (5.41). If, however, we take instead of (5.42) the two integrated equations

$$\left.\begin{aligned} &F(y)\,\chi(y,y') + S_c'(y',2)\,W(y,2;3,4)\,\chi(3,4) = 0, \\ &F(y')\,\chi(y,y') + S_c'(y,1)\,W(1,y';3,4)\,\chi(3,4) = 0, \end{aligned}\right\} \quad (5.43)$$

where the inhomogeneous terms have been dropped, then one can rather easily find the connection between the solutions of these equations and the solutions of (5.40). Evidently, the condition for compatibility of (5.43) and (5.40) is that

$$\left.\begin{aligned} &\Sigma_c^*(y,1)\,\chi(1,y') + S_c'(y',2)\,W(y,2;3,4)\,\chi(3,4) \\ &\qquad = i\lambda\frac{\delta}{\delta I(y)}\,\chi(y,y'), \\ &\Sigma_c^*(y',2)\,\chi(y,2) + S_c'(y,1)\,W(1,y';3,4)\,\chi(3,4) \\ &\qquad = i\lambda\frac{\delta}{\delta I(y')}\,\chi(y,y'). \end{aligned}\right\} \quad (5.44)$$

By integration of (5.43) we get

$$\chi(y,y') - S_c'(y,1)\,S_c'(y',2)\,W(1,2;3,4)\,\chi(3,4) = \varphi(y,y'), \quad (5.45)$$

where φ is any solution of

$$F(y)\,\varphi(y,y') = F(y')\,\varphi(y,y') = 0, \quad (5.46)$$

i. e. φ has one-particle properties with respect to both coordinates. From this equation we infer by arguments similar to those used in the derivation of the one-particle equation that

$$\left.\begin{aligned} &i\lambda\frac{\delta}{\delta I(x)}\,\varphi(y,y') = \varphi^{(0)}(x;y,y') \\ &\quad + S_c'(y',1)\,\Phi(1,2,3)\,\Delta_c'(2,x)\,\varphi(y,3) \\ &\quad + S_c'(y,1)\,\Phi(1,2,3)\,\Delta_c'(2,x)\,\varphi(3,y'), \end{aligned}\right\} \quad (5.47)$$

where

$$F(y)\,\varphi^{(0)}(x;y,y') = F(y')\,\varphi^{(0)}(x;y,y') = 0. \quad (5.48)$$

Using this we find from (5.45) an expression for the variational derivative of χ with respect to $I(x)$, viz.

$$\left.\begin{aligned} i\lambda\frac{\delta}{\delta I(x)}\,\chi(y,y') = \frac{1}{2}\lambda\,G(y,y';1,2)\left[i\frac{\delta}{\delta I(x)}\,W(1,2;3,4)\right]\chi(3,4) \\ -\left(\bar{F}(4)\,G(y,y';1,4)\right)\Phi(1,2,3)\,\Delta_c'(2,x)\,\chi(3,4) + R^{(0)}, \end{aligned}\right\} \quad (5.49)$$

where $R^{(0)}$ is a contribution which vanishes for $\varphi^{(0)}$ equal to zero. On comparison with (5.44) and using (5.38) we find that χ satisfies (5.40), provided that $\varphi^{(0)}$ vanishes.

Hence, corresponding to (5.21), we have the result for the two-nucleon system: Any solution of the coupled equations

$$\left.\begin{aligned} &\Psi_{(2)}(|yy'|) - S'_c(y,1)\,S'_c(y',2)\,W(1,2;3,4)\,\Psi_{(2)}(|34|) = \varphi(y,y') \\ &i\lambda\frac{\delta}{\delta I(x)}\Psi_{(2)}(|yy'|) = \frac{1}{2}\lambda G(y,y';1,2)\left[i\frac{\delta}{\delta I(x)}W(1,2;3,4)\right]\Psi_{(2)}(|34|) \\ &\quad - (\overline{F}(4)\,G(y,y';1,4))\,\Phi(1,2,3)\,\Delta'_c(2,x)\,\Psi_{(2)}(|34|), \end{aligned}\right\} \quad (5.50)$$

where $\varphi(y,y')$ satisfies (5.46), is a solution of (5.40). In particular, passing to the limit $I = 0$, the second equation (5.50) and its variational derivatives become explicit expressions for the state vector amplitudes depending on one and more meson coordinates besides the two nucleon space time coordinates. It is thus possible in a unique way to relate to any solution of the Bethe-Salpeter equation a solution of the equations of motion in the configuration space representation.

Summary.

A reformulation of quantum field theory is given, in which any state of the system considered is represented by a functional depending on external sources. The variational derivatives of this functional provide us with a generalization of the Fock representation in configuration space to the case of non-linear fields. The representing amplitudes can be expressed entirely in terms of matrix elements of time ordered products of field operators and possess several simple properties which are independent of the magnitude of the coupling constant. It is shown that these amplitudes satisfy homogeneous equations of motion which can be derived in a simple manner. The equations of the Bethe-Salpeter type following herefrom become identical with those following from Schwinger's theory of Green's functions. Our representation has many properties in common with that given by Freese.

Appendix I.

The sources of the spinor fields.

In Section 1 we have assumed that the domain of the external sources can be chosen so large that variational derivatives with respect to allowed variations of f and g can be defined in a unique way. This property together with the anti-commutativity (1.6) is all we need for the development of the configuration space representation. It is, maybe, not quite trivial that the requirements to the sources are consistent. We shall, therefore, construct an example of a possible domain of allowed f-number pairs.

Let a_n and b_n, $n = 1, 2, \cdots$ be two sets of infinite matrices which satisfy the commutation rules

$$\left.\begin{aligned} \{a_n, a_m^\dagger\} &= \delta_{nm} \\ \{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} &= 0, \\ \{b_n, b_m^\dagger\} &= \delta_{nm} \\ \{b_n, b_m\} = \{b_n^\dagger, b_m^\dagger\} &= 0, \end{aligned}\right\} \quad \text{(Ap. I. 1)}$$

while all the a's anti-commute with all the b's. As is well known, there exists a matrix which anti-commutes with all the a's and with all the b's and with their adjoints. This matrix Ω, say, is the parity of the matrix $\Sigma\,(a_n^\dagger\, a_n + b_n^\dagger\, b_n)$. We choose Ω hermitian and unitary, i. e.

$$\Omega^\dagger = \Omega, \quad \Omega^2 = 1. \qquad \text{(Ap. I. 2)}$$

For the construction of the f-number pairs we further need two complete orthonormal sets of functions in four-dimensional space, $f_n(x)$ and $g_n(x)$, such that any function, $\xi(x)$ say, can be expanded in either of the forms

$$\xi(x) = \Sigma\, \xi_n^{(f)}\, f_n(x)$$

or

$$\xi(x) = \Sigma\, \xi_n^{(g)}\, g_n(x).$$

Let c_1 and c_2 be complex numbers. Then,

$$\left.\begin{aligned} f(x) &= c_1 \Sigma\, a_n f_n(x) \\ g(x) &= c_2 \Sigma\, b_n g_n(x), \end{aligned}\right\} \quad \text{(Ap. I. 3)}$$

is a possible allowed f-number pair. In fact, due to (Ap. I. 1)

$$\{f(x), f(x')\} = \{f(x), g(x')\} = \{g(x), g(x')\} = 0. \quad \text{(Ap. I. 4)}$$

A domain of f-number pairs can be obtained from the particular pair (Ap. I. 3) by unitary transformations in the a, b-space. In particular we are interested in infinitesimal unitary transformations such that the corresponding variations of the f-number pair form a pair of allowed variations in the sense of Section 1, i. e. such that

$$\left.\begin{aligned} \{\delta f(x), f(x')\} = \{\delta f(x), g(x')\} &= 0, \\ \{\delta g(x), f(x')\} = \{\delta g(x), g(x')\} &= 0. \end{aligned}\right\} \quad \text{(Ap. I. 5)}$$

Such variations can be obtained by means of the matrix

$$A = \Sigma\,(a_n^\dagger\, \Omega\, \alpha_n - \alpha_n^*\, \Omega\, a_n + b_n^\dagger\, \Omega\, \beta_n - \beta_n^*\, \Omega\, b_n), \quad \text{(Ap. I. 6)}$$

where the α_n's and the β_n's are infinitesimal complex numbers. By the properties of Ω, A is anti-hermitian, whence $1 + A$ is unitary. By this transformation the a_n's and the b_n's vary according to

$$\left.\begin{aligned} \delta a_n &= -[A, a_n] = \Omega \alpha_n, \\ \delta b_n &= -[A, b_n] = \Omega \beta_n. \end{aligned}\right\} \quad \text{(Ap. I. 7)}$$

The corresponding variation of the f-number pairs is

$$\left.\begin{aligned} \delta f(x) &= c_1\, \Omega\, \Sigma\, a_n f_n(x), \\ \delta g(x) &= c_2\, \Omega\, \Sigma\, \beta_n g_n(x). \end{aligned}\right\} \quad \text{(Ap. I. 8)}$$

Obviously we have here an example of a pair of allowed variations for any set of infinitesimal α_n's and β_n's. Thus, all variations of the form

$$\left.\begin{aligned} \delta f(x) &= \Omega\, \delta\xi(x), \\ \delta g(x) &= \Omega\, \delta\eta(x), \end{aligned}\right\} \quad \text{(Ap. I. 9)}$$

where $\delta\xi$ and $\delta\eta$ are infinitesimal functions are included among the allowed variations. Therefore, if an expression of the form (1.10) holds for an arbitrary pair of allowed variations, we have in particular

$$\Omega \int \left(\delta\xi\,(x)\,K\,(x) + \delta\eta\,(x)\,L\,(x)\right)\delta x = 0\,, \qquad \text{(Ap. I. 10)}$$

with arbitrary $\delta\,\xi$ and $\delta\,\eta$. As Ω is non-singular we conclude that $K\,(x)$ as well as $L\,(x)$ vanish identically.

Appendix II.

Reformulation of a theorem due to Wick.

Let $u\,(x)$ be the field operator of a free scalar neutral meson field. We shall use Dyson's notation

$$N\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)})\right) \qquad \text{(Ap. II. 1)}$$

to designate the product of the u's ordered such that all absorption operators stand to the right of all emission operators. This product we call the normal product of the u's indicated. As shown by Wick [9], any time ordered product can be decomposed into a sum of normal constituents according to

$$T\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)})\right) = \sum_{\nu=0}^{n} N^{(\nu)}. \qquad \text{(Ap. II. 2)}$$

For ν odd $N^{(\nu)}$ vanishes. For ν even, $N^{(\nu)}$ is a sum of terms, one term for each possible pairing of ν factors u. Let for ν even, $\xi', \xi'', \cdots, \xi^{(\nu)}$ be some of the space time points $x', x'', \cdots, x^{(n)}$. For a definite pairing (ξ', ξ''), (ξ''', ξ''''), $\cdots$ $(\xi^{(\nu-1)}, \xi^{(\nu)})$ the contribution to $N^{(\nu)}$ is

$$\left.\begin{aligned} &\langle 0\,|\,T\left(u\,(\xi')\,u\,(\xi'')\right)|\,0\rangle\langle 0\,|\,T\left(u\,(\xi''')\,u\,(\xi'''')\right)|\,0\rangle\cdots \\ &\qquad\times\langle 0\,|\,T\left(u\,(\xi^{(\nu-1)})\,u\,(\xi^{(\nu)})\right)|\,0\rangle \\ &\times N\left(u\,(x')\,u\,(x'')\cdots u\,(x^{(n)});\,\xi', \xi'', \cdots \xi^{(\nu)}\right). \end{aligned}\right\} \qquad \text{(Ap. II. 3)}$$

Here, $N\,(\cdots)$ denotes the normal product of the unpaired u's. For instance,

$$N\left(u\,(x')\,u\,(x'')\,u\,(x''')\,u\,(x'''');\,x''\,x''''\right) \equiv N\left(u\,(x')\,u\,(x''')\right).$$

To obtain $N^{(\nu)}$ from terms of the form (Ap. II. 3) one must add all contributions from possible pairings of the space time points $\xi', \xi'', \cdots \xi^{(\nu)}$ and, further, sum over all subsets of ν field operators u. Hence, we can write $N^{(\nu)}$ in the form

$$\left.\begin{aligned} N^{(\nu)} &= \sum_{\xi', \xi'', \cdots \xi^{(\nu)}} C(\xi', \xi'', \cdots \xi^{(\nu)}) \\ & N(u(x')\, u(x'') \cdots u(x^{(n)}); \xi', \xi'', \cdots \xi^{(\nu)}), \end{aligned}\right\} \quad \text{(Ap. II. 4)}$$

where the C's are certain c-number functions not depending on n. The summation runs over all subsets $\xi', \xi'', \cdots \xi^{(\nu)}$. In particular, we note that, for n even,

$$N^{(n)} = C(x', x'', \cdots x^{(n)}). \qquad \text{(Ap. II. 5)}$$

Combining (Ap. II. 4) and (Ap. II, 2) we have

$$\left.\begin{aligned} & T\left(u(x') \cdots u(x^{(n)})\right) = \\ & \sum_{\nu=0}^{n} \sum_{\xi' \cdots \xi^{(\nu)}} C(\xi', \cdots \xi^{(\nu)})\, N\left(u(x') \cdots u(x^{(n)}); \xi', \cdots \xi^{(\nu)}\right). \end{aligned}\right\} \text{(Ap. II. 6)}$$

We include formally odd ν's in the summation and choose vanishing corresponding C's.

The vacuum expectation value of any N-product is zero. Thus, from (Ap. II. 6) for n even, we get explicit expressions for the C's, viz.

$$C(x', x'', \cdots x^{(n)}) = T_0(x', x'', \cdots x^{(n)}). \qquad \text{(Ap. II. 7)}$$

As in (2.20), $T_0(x', \cdots)$ stands for the vacuum expectation value of the T-product. Wick's theorem now takes the form

$$\left.\begin{aligned} & T\left(u(x') \cdots u(x^{(n)})\right) = \\ & \sum_{\nu=0}^{n} \sum_{\xi' \cdots \xi^{(\nu)}} T_0(\xi', \cdots \xi^{(\nu)})\, N\left(u(x') \cdots u(x^{(n)}); \xi', \cdots \xi^{(\nu)}\right). \end{aligned}\right\} \text{(Ap. II. 8)}$$

It should be noted that (Ap. II. 7) also holds for ν odd as the vacuum expectation value of the product of an odd number of free field operators vanishes.

In case also other types of fields are considered, the definition of the N-product is slightly modified. Each term in the N-product

should now be multiplied by a factor $(\pm)$ which has the value $+1$ if the permutation of spinor operators involved in the ordering process is even, and -1 if this permutation is odd.

Consider the case of a free meson field u and a free spinor field described by the field operators $\bar{\psi}$ and ψ. Similar to (Ap. II. 8) one can write Wick's theorem for this case in the form

$$\left.\begin{aligned} &T\left(u\left(x'\right)\cdots u\left(x^{(k)}\right)\psi\left(y'\right)\cdots\psi\left(y^{(l)}\right)\bar{\psi}\left(z'\right)\cdots\bar{\psi}\left(z^{(m)}\right)\right)= \\ &\sum_{\varkappa\lambda\mu}\sum_{\xi'\cdots\xi^{(\varkappa)}}\sum_{\eta'\cdots\eta^{(\lambda)}}\sum_{\zeta'\cdots\zeta^{(\mu)}}(\pm)\,T_0\left(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}\right) \\ &\times N\left(u\left(x'\right)\cdots u\left(x^{(k)}\right);\,\xi'\cdots\xi^{(\varkappa)}\,|\,\psi\left(y'\right)\cdots\psi\left(y^{(l)}\right);\,\eta'\cdots\eta^{(\lambda)}\,|\right. \\ &\left.\bar{\psi}\left(z'\right)\cdots\bar{\psi}\left(z^{(m)}\right);\,\zeta'\cdots\zeta^{(\mu)}\right), \end{aligned}\right\}\quad\text{(Ap. II. 9)}$$

where $(\pm)$ is the parity of the permutation

$$\left\{\begin{array}{c}\left(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}\right)\\ \left(x'\cdots x^{(k)};\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\,\zeta'\cdots\zeta^{(\mu)}\right)\\ \rightarrow\left(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}\right).\end{array}\right\}$$

We introduce the notation

$$\Psi\left(x'\cdots\,|\,y'\cdots\,|\,z'\cdots\right)=\langle 0\,|\,N\left(u\left(x'\right)\cdots\psi\left(y'\right)\cdots\bar{\psi}\left(z'\right)\cdots\right)|\,\Psi\rangle.\quad\text{(Ap. II. 10)}$$

If, further, we use the notation of Section 2 (p. 17), we get from (Ap. II. 9)

$$\left.\begin{aligned} &T_{\Psi}\left(x'\cdots x^{(k)}\,|\,y'\cdots y^{(l)}\,|\,z'\cdots z^{(m)}\right)= \\ &\sum_{\varkappa\lambda\mu}\frac{1}{\varkappa!}\sum_{\xi'\cdots\xi^{(\varkappa)}}\frac{1}{\lambda!}\sum_{\eta'\cdots\eta^{(\lambda)}}\frac{1}{\mu!}\sum_{\zeta'\cdots\zeta^{(\mu)}}(\pm)\,T_0\left(\xi'\cdots\xi^{(\varkappa)}\,|\,\eta'\cdots\eta^{(\lambda)}\,|\,\zeta'\cdots\zeta^{(\mu)}\right) \\ &\times\Psi\left(x'\cdots x^{(k)};\,\xi'\cdots\xi^{(\varkappa)}\,|\,y'\cdots y^{(l)};\,\eta'\cdots\eta^{(\lambda)}\,|\,z'\cdots z^{(m)};\,\zeta'\cdots\zeta^{(\mu)}\right). \end{aligned}\right\}\quad\text{(Ap. II. 11)}$$

The factorials take into account that we now perform the summation such that the ξ's, η's, and ζ's run independently over the x's, y's, and z's, respectively.

As is well known, the functions Ψ are the representing amplitudes for the state $|\,\Psi\rangle$ in the Fock representation in the configuration space. We can thus regard (Ap. II. 11) as a re-

cursion formula expressing the Fock amplitudes in terms of matrix elements of time ordered products.

Appendix III.

The equations of motion for the T_Ψ-functions.

The equations of motion for the T_Ψ-functions depending on one space time point only are easily obtained from the field equations (1.2). One finds[1]

$$\left.\begin{aligned}
&(-\Box_x + m^2)\, T_\Psi(x|\,|) - \lambda\, T_\Psi(|\,x\,|\,x) + I(x)\, T_\Psi(|\,|) = 0\,,\\
&(\partial_y + M)\, T_\Psi(|\,y\,|) + \lambda\, T_\Psi(y\,|\,y\,|) + f(y)\, T_\Psi(|\,|) = 0\,,\\
&(\bar{\partial}_z + M)\, T_\Psi(|\,|\,z) + \lambda\, T_\Psi(z\,|\,|\,z) + g(z)\, T_\Psi(|\,|) = 0\,.
\end{aligned}\right\} \text{(Ap. III. 1)}$$

From the variational equations (1.14) and the canonical commutators it follows that

$$(\partial_y + M)\, \delta\, \psi(y) = \delta\, [(\partial_y + M)\, \psi(y)]\,,$$

and similar relations for the other field variables. Hence, the differential operators appearing in the field equations commute with all variational derivative operators. We can thus obtain equations of motion for T_Ψ-functions depending on more than one space time point simply by taking variational derivatives of the equations (Ap. III. 1). For instance, applying $-i\,\delta/\delta f(z)$ to the second of these equations, we get

$$\left.\begin{aligned}
&(\partial_y + M)\, T_\Psi(|\,y\,|\,z) + \lambda\, T_\Psi(y\,|\,y\,|\,z)\\
&+ f(y)\, T_\Psi(|\,|\,z) + i\,\delta(y - z)\, T_\Psi(|\,|) = 0\,.
\end{aligned}\right\} \text{(Ap. III. 2)}$$

One should note that, for instance,

$$-i\,\frac{\delta}{\delta f(z)}\, T_\Psi(|\,y\,|) = -\,T_\Psi(|\,y\,|\,z)\,.$$

[1] The T-product of $\psi(x)$ and $\bar{\psi}(x')$ for $x = x'$ is chosen as

$$T(\psi(x)\,\bar{\psi}(x)) = \frac{1}{2}\,[\psi(x), \bar{\psi}(x)]\,.$$

Hence the minus sign in the first equation (Ap. III. 1).

The following equations hold

$$\left.\begin{aligned}
&(-\Box_{x'}+m^2)\,T_\Psi(x'\cdots|y'\cdots y^{(l)}|z'\cdots)\\
&-(-1)^l\,\lambda\,T_\Psi(x''\cdots|x'y'\cdots y^{(l)}|x'z'\cdots)\\
&+I\,x')\,T_\Psi(x''\cdots|y'\cdots y^{(l)}|z'\cdots)\\
&+i\sum_\xi\delta(x'-\xi)\,T_\Psi(x''\cdots;\xi|y'\cdots y^{(l)}|z'\cdots)=0.
\end{aligned}\right\}\quad\text{(Ap. III. 3)}$$

$$\left.\begin{aligned}
&(\partial_{y'}+M)\,T_\Psi(x'\cdots|y'\cdots|z'\cdots)\\
&+\lambda\,T_\Psi(x'\cdots y'|y'\cdots|z'\cdots)\\
&+f(y')\,T_\Psi(x'\cdots|y''\cdots|z'\cdots)\\
&+i\sum_\zeta(\pm)\,\delta(y'-\zeta)\,T_\Psi(x'\cdots|y''\cdots|z'\cdots;\zeta)=0.
\end{aligned}\right\}\quad\text{(Ap. III. 4)}$$

$$\left.\begin{aligned}
&(\bar\partial_{z'}+M)\,T_\Psi(x'\cdots|y'\cdots y^{(l)}|z'\cdots)\\
&+\lambda\,T_\Psi(x'\cdots z'|y'\cdots y^{(l)}|z'\cdots)\\
&+g(z')\,(-1)^l\,T_\Psi(x'\cdots|y'\cdots y^{(l)}|z''\cdots)\\
&+i\sum_\eta(\pm)\,\delta(\eta-z')\,T_\Psi(x'\cdots|y'\cdots y^{(l)};\eta|z''\cdots)=0.
\end{aligned}\right\}\quad\text{(Ap. III. 5)}$$

The $(\pm)$ factors have the same meaning as, f. inst., in (2.25).

One can verify these formulas by induction on the number of space time points. To illustrate: if we apply $i\,\delta/\delta\,g\,(y)$ on (Ap. III.5), we get

$$\left.\begin{aligned}
&(\bar\partial_{z'}+M)\,T_\Psi(x'\cdots|yy'\cdots y^{(l)}|z'\cdots)\\
&+\lambda\,T_\Psi(x'\cdots z'|yy'\cdots y^{(l)}|z'\cdots)\\
&-g(z')\,(-1)^l\,T\Psi(x'\cdots|yy'\cdots y^{(l)}|z'\cdots)\\
&+i\,\delta(y-z')\,(-1)^l\,T_\Psi(x'\cdots|y'\cdots y^{(l)}|z''\cdots)\\
&+i\sum_{\eta\neq y}(\pm)\,\delta(\eta-z')\,T_\Psi(x'\cdots|yy'\cdots y^{(l)};\eta|z''\cdots)=0.
\end{aligned}\right\}\quad\text{(Ap. III. 6)}$$

As the number of nucleon space time points has now increased by one, the third term has the required sign factor. The factor $(-1)^l$ appearing in the fourth term is in accordance with the

convention as regards the value of the parity factor ($\pm$). Hence, the two last terms in (Ap. III. 6) combine to give a term of the form of the last term in (Ap. III. 5) and we see that (Ap. III. 6) is again of the form (Ap. III. 5).

The above equations have been derived by FREESE [4] for the source-free case by means of other methods.

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ON ALMOST PERIODIC MOVEMENTS

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Introduction.

Let $\mathfrak{M}$ denote a metric space. The points of $\mathfrak{M}$ are denoted by $x, y, \cdots$ and the distance by $[x, y]$. A continuous function $x = f(t)$, $-\infty < t < \infty$, $x \in \mathfrak{M}$, will be called a movement in $\mathfrak{M}$. An almost periodic movement in $\mathfrak{M}$ is defined in close analogy to the classical complex-valued almost periodic functions introduced by H. Bohr [3], [5]. If $x = g(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$; $x \in \mathfrak{M}$, is a continuous function with the period 2π in each variable, and $\beta_1, \cdots, \beta_m$ are rationally independent real numbers, the function $x = g(\beta_1 t, \cdots, \beta_m t)$ will be called a diagonal movement.

In Chapter 1 we shall investigate whether or not every continuous movement in $\mathfrak{M}$ can be approximated uniformly to any given accuracy by a diagonal movement. It will be proved that the answer is affirmative if $\mathfrak{M}$ is a complete space with the following property: To every compact subset $\mathfrak{C}$ of $\mathfrak{M}$ corresponds a number Δ, such that any two points $x, y \in \mathfrak{C}$ with $[x, y] \leqq \Delta$ can be connected in $\mathfrak{M}$ by a continuous curve which depends continuously on x, y and which reduces to the point x if $y = x$. A space with this property will be called continuously locally arcwise connected. We shall also prove that a uniformly continuous family $f(t; v)$ of almost periodic movements can be approximated uniformly to any given accuracy by a uniformly continuous family of diagonal movements. These results will be proved by methods very similar to those applied for ordinary almost periodic functions. On the other hand, it will be proved that certain almost periodic movements in the so-called solenoidal spaces introduced by D. van Dantzig [6] cannot be approximated uniformly by diagonal movements. The solenoidal spaces

are complete, compact and connected, but neither locally nor arcwise connected.

Two almost periodic movements $f_1(t)$ and $f_2(t)$ are called homotopic if there exists a uniformly continuous family $f(t; v)$, $0 \leqq v \leqq 1$, of almost periodic movements which contains them. If the space $\mathfrak{M}$ is complete and continuously locally arcwise connected, there exists an approximating family $g(\beta_1 t, \cdots, \beta_m t; v)$ such that $f_1(t)$ and $f_2(t)$ are homotopic to the approximating diagonal functions. Thus, the homotopy between $f_1(t)$ and $f_2(t)$ gives rise to a homotopy between two continuous, periodic functions $g_1(u_1, \cdots, u_m)$ and $g_2(u_1, \cdots, u_m)$, i. e. to a homotopy between two mappings of an m-dimensional torus into $\mathfrak{M}$.

In particular, it will be proved in Chapter 2 that every almost periodic movement in a complete, continuously locally arcwise connected space is homotopic to a periodic movement if and only if every continuous mapping of a torus (of any dimension) into $\mathfrak{M}$ is homotopic to a mapping of a torus into a closed curve in $\mathfrak{M}$.

W. Fenchel and B. Jessen [7] proved that every almost periodic movement on a complete surface admitting a hyperbolic non-Euclidean metric is homotopic to a periodic movement. (In their paper it was further assumed that there are no arbitrarily short closed paths on the surface. This restriction can, however, be removed on account of the fact that an almost periodic movement is confined to a compact part of the surface.) The only topological types of surfaces which cannot be metrized in this way are the orientable and the non-orientable tori, the sphere, and the projective plane. W. Fenchel and B. Jessen proved that the result does not hold for a torus, even if deformations are allowed which do not preserve almost periodicity. For the sphere the statement is obviously true in this weaker sense, but W. Fenchel and B. Jessen considered it unlikely that every almost periodic movement on a sphere is homotopic (in the strict sense defined above) to a periodic movement. We shall prove that this conjecture is true.

The general theory of homotopy between almost periodic movements in a metric space $\mathfrak{M}$ is reduced to the theory of homotopy between mappings of m-dimensional tori into $\mathfrak{M}$.

R. H. Fox ([8], p. 509) has briefly indicated a method to put this theory into a group-theoretical form, but so far the relations between the general torus homotopy groups introduced in his paper and the ordinary homotopy groups seem not to have been investigated. It is possible, by means of the results obtained by R. H. Fox to prove that every almost periodic movement in $\mathfrak{M}$ is homotopic to a periodic movement if the ordinary homotopy groups of $\mathfrak{M}$ are trivial and all Abelian subgroups of the fundamental group of $\mathfrak{M}$ are cyclic.

Chapter 1.

Almost Periodic Movements in Metric Spaces.

1. Uniformly Continuous Families of Almost Periodic Movements.

In the following we shall consider a metric space $\mathfrak{M}$ consisting of points $x, y, z, \cdots$ and with a distance $[x, y]$ satisfying the conditions

1. $[x, x] = 0$; $[x, y] > 0$ when $y \neq x$.
2. $[x, y] = [y, x]$.
3. $[x, y] \leq [x, z] + [z, x]$.

Definition 1. *A continuous movement in $\mathfrak{M}$ is a continuous function $x = f(t)$, $-\infty < t < \infty$, $x \in \mathfrak{M}$. A number $\tau = \tau_f(\varepsilon)$ is called a translation number of $f(t)$ corresponding to $\varepsilon > 0$ (or an ε-translation number) if the condition $[f(t), f(t+\tau)] \leq \varepsilon$ is satisfied for all real values of t. The movement $x = f(t)$ is called almost periodic if the set $\{\tau_f(\varepsilon)\}$ of translation numbers of $f(t)$ corresponding to ε is relatively dense for every $\varepsilon > 0$.*

Definition 2. *Let $\mathfrak{R}$ denote a compact S-space ([12], p. 40—45) consisting of points denoted $v, v_1, \cdots$ and with neighbourhoods $U(v)$. A function $x = f(t; v)$, $-\infty < t < \infty$, $v \in \mathfrak{R}$; $x \in \mathfrak{M}$ is called a uniformly continuous family of almost periodic movements when*

1. *The function* $f(t; v)$ *is almost periodic for every fixed* $v \in \mathfrak{R}$.
2. *To* $\varepsilon > 0$ *and* $v_0 \in \mathfrak{R}$ *corresponds a neighbourhood* $U_\varepsilon(v_0)$ *such that* $[f(t; v_0), f(t; v)] \leqq \varepsilon$ *when* $-\infty < t < \infty$, $v \in U_\varepsilon(v_0)$.

Definition 3. *A set* $\{f(t)\}$ *of almost periodic movements in* $\mathfrak{M}$ *is called a uniformity set if there exists a real-valued almost periodic function* $g(t)$ *such that the set of common* ε*-translation numbers of all functions* $f(t)$ *of the set for* $\varepsilon > 0$ *contains the set of* ε*-translation numbers of* $g(t)$.

We shall now prove some elementary theorems on uniformly continuous families of almost periodic functions.

Lemma 1. *An almost periodic movement* $x = f(t)$ *is bounded and uniformly continuous.*

Proof. That $f(t)$ is bounded means that $x = f(t)$ remains in a bounded subdomain of $\mathfrak{M}$, i. e. that there exists a positive number K such that $[f(t_1), f(t_2)] \leqq K$ for all real values of t_1 and t_2. But the function $\varphi(t) = [f(0), f(t)]$ satisfies the condition

$$|\varphi(t_2) - \varphi(t_1)| = |[f(0), f(t_2)] - [f(0), f(t_1)]| \leqq [f(t_1), f(t_2)].$$

This implies that $\varphi(t)$ is almost periodic. Hence we may choose K such that $[f(0), f(t)] \leqq \frac{K}{2}$, for all values of t, and the first part of the lemma is proved. The proof of the second part of Lemma 1 can be copied from the proof of the corresponding property for ordinary almost periodic functions ([3] p. 30).

Lemma 2. *Let* $x = f(t; v)$, $-\infty < t < \infty$, $v \in \mathfrak{R}$; $x \in \mathfrak{M}$ *be a uniformly continuous family of almost periodic movements. There exists a constant* K *such that* $[f(t_1; v), f(t_2; v)] \leqq K$ *for all real values of* t_1 *and* t_2 *and for every* $v \in \mathfrak{R}$.

Proof. From Lemma 1 and from 2 in Definition 2 it follows immediately that there exists a constant $K(v_0)$ such that $[f(t_1; v), f(t_2; v)] \leqq K(v_0)$ when $v \in U_\varepsilon(v_0)$. But the compact S-space $\mathfrak{R}$ is covered by the neighbourhoods $U_\varepsilon(v)$, hence we can find a finite number of points $v_1, \cdots, v_n \in \mathfrak{R}$, such that $\mathfrak{R}$ is covered by $U_\varepsilon(v_1), \cdots, U_\varepsilon(v_n)$. We have then $[f(t_1; v), f(t_2; v)] \leqq \operatorname{Max}_\nu K(v_\nu)$ for every $v \in \mathfrak{R}$. This completes the proof.

Definition 4. *The function*

$$e(\tau; v) = \underset{t}{\text{l. u. b.}} [f(t; v), f(t+\tau; v)]$$

is called the translation function of the uniformly continuous family of almost periodic movements $f(t; v)$ *and*

$$e(\tau) = \underset{v}{\text{l. u. b.}}\ e(\tau; v)$$

is called the translation majorant of $f(t; v)$.

The translation functions were introduced in the case of ordinary almost periodic functions by H. Bohr ([3], p. 37), and S. Bochner ([2], p. 136—146) used them for the study of the translation properties of almost periodic functions. Our generalized translation functions will enable us to apply theorems concerning translation numbers of ordinary almost periodic functions to almost periodic movements in $\mathfrak{M}$.

Lemma 3. *The translation function* $e(\tau; v)$ *and the translation majorant* $e(\tau)$ *are real, non-negative, and bounded.*

Proof. This lemma follows immediately from Lemma 2.

Lemma 4. *The translation function* $e(\tau; v)$ *and the translation majorant* $e(\tau)$ *satisfy the conditions*

$$e(0; u) = e(0) = 0;\ e(\tau; v) = e(-\tau; v);\ e(v) = e(-v);$$

$$e(\tau_1 + \tau_2; v) \leqq e(\tau_1; v) + e(\tau_2; v);\ e(\tau_1 + \tau_2) \leqq e(\tau_1) + e(\tau_2).$$

Proof. The properties in the first line are immediate consequences of Definition 4. We have further

$$e(\tau_1 + \tau_2; v) = \underset{t}{\text{l. u. b.}} [f(t; v), f(t + \tau_1 + \tau_2; v)] \leqq$$

$$\underset{t}{\text{l. u. b.}} [f(t; v), f(t+\tau_1; v)] + \underset{t}{\text{l. u. b.}} [f(t+\tau_1; v), f(t+\tau_1+\tau_2; v)]$$

$$= e(\tau_1; v) + e(\tau_2; v)$$

and

$$e(\tau_1 + \tau_2) = \underset{v}{\text{l. u. b.}}\ e(\tau_1 + \tau_2; v) \leqq$$

$$\underset{v}{\text{l. u. b.}} (e(\tau_1; v) + e(\tau_2; v)) \leqq e(\tau_1) + e(\tau_2).$$

In the following it will be convenient to write $e(t; v)$ and $e(t)$ instead of $e(\tau; v)$ and $e(\tau)$.

Lemma 5. *Let v be a fixed point of $\mathfrak{R}$. The translation function $e(t; v)$ is almost periodic and the sets of translation numbers of $f(t; v)$ and of $e(t; v)$ corresponding to $\varepsilon > 0$ are identical to each other and to the set of numbers τ for which $e(\tau; v) \leqq \varepsilon$.*

Proof. A number τ is a translation number of $f(t; v)$ corresponding to $\varepsilon > 0$ if and only if $e(\tau; v) = \text{l. u. b.}\,[f(t; v), f(t + \tau; v)] \leqq \varepsilon$. On the other hand we have by Lemma 4

$$(1)\quad \begin{cases} e(t + \tau; v) - e(t; v) \leqq e(\tau; v), \\ e(t; v) - e(t + \tau; v) \leqq e(-\tau; v) = e(\tau; v), \end{cases}$$

hence

$$(2)\quad e(\tau; v) = e(\tau; v) - e(0; v) = \text{l. u. b.}_{t} \left| e(t + \tau; v) - e(t; v) \right|.$$

It follows that $\left| e(t + \tau; v) - e(t; v) \right| \leqq \varepsilon$ for every t if and only if $e(\tau; v) \leqq \varepsilon$. We have thus proved that the set of translation numbers of $e(\tau; v)$ corresponding to ε is relatively dense. From Lemma 1 it follows that the set of translation numbers contains an interval about zero. Hence $e(\tau; v)$ is continuous. This completes the proof.

Lemma 6. *The translation majorant $e(t)$ is almost periodic.*

Proof. This theorem is far from trivial and its proof must be based on some deeper theorem on ordinary almost periodic functions. We prefer to make use of the theorem that a finite number of almost periodic functions have an almost periodic sum. ([3] p. 31—32).

The inequalities (1) and (2) are true for every $e(t; v)$, and also for $e(t)$. Hence, τ is a translation number of $e(t)$ corresponding to $\varepsilon > 0$ if and only if $e(\tau) \leqq \varepsilon$, i. e. if and only if $e(\tau; v) \leqq \varepsilon$ for every v in $\mathfrak{R}$. Let v be an arbitrary point of $\mathfrak{R}$ and $U_{\frac{\varepsilon}{3}}(v)$ the corresponding neighbourhood introduced in Definition 2. As $\mathfrak{R}$ is a compact S-space, we may choose a finite number of points $v_1, \cdots, v_n$ such that $\mathfrak{R} \subset U_{\frac{\varepsilon}{3}}(v_1) \cup \cdots \cup U_{\frac{\varepsilon}{3}}(v_n)$. The sum $e(t; v_1) + \cdots + e(t; v_n) = E(t)$ is an almost periodic function. If v is an arbitrary point of $\mathfrak{R}$, we can choose v_ν such that $v \in U_{\frac{\varepsilon}{3}}(v_\nu)$. It follows that

$$e(\tau; v) = \text{l. u. b.}_t [f(t; v), f(t+\tau; v)] \leqq$$

$$\text{l. u. b.}_t ([f(t; v), f(t; v_\nu)] + [f(t; v_\nu), f(t+\tau; v_\nu)] +$$

$$[f(t+\tau; v_\nu), f(t+\tau; v)]) \leqq \text{l. u. b.}_t \left(\frac{\varepsilon}{3} + [f(t; v_\nu), f(t+\tau; v_\nu)] + \frac{\varepsilon}{3}\right) =$$

$$e(\tau; v_\nu) + \frac{2\varepsilon}{3} \leqq E(\tau) + \frac{2\varepsilon}{3}.$$

We have thus proved that $e(\tau) \leqq \varepsilon$ if $E(\tau) \leqq \frac{\varepsilon}{3}$. But as $E(\tau) = E(\tau) - E(0)$, this condition is satisfied when τ is a translation number of the almost periodic function $E(t)$ corresponding to $\frac{\varepsilon}{3}$. This set of translation numbers is relatively dense, and it contains an interval about zero. This completes the proof.

Lemma 7. *The set of translation numbers of the translation majorant $e(t)$ corresponding to $\varepsilon > 0$ is identical to the set of common translation numbers of the almost periodic functions of the family $f(t; v)$ corresponding to the same ε.*

Proof. The set of common translation numbers of the functions $f(t; v)$ corresponding to ε is, according to Lemma 5, identical to the set of numbers τ satisfying $e(\tau; v) \leqq \varepsilon$ for every $v \in \mathfrak{N}$; but this condition is equivalent to the condition $e(\tau) \leqq \varepsilon$; and the set of numbers τ satisfying the last condition is exactly the set of translation numbers of $e(t)$ corresponding to ε.

Theorem 1. *The set of almost periodic functions belonging to a uniformly continuous family of almost periodic movements is a uniformity set.*

Proof. The theorem is an immediate corollary of Lemmas 6 and 7.

Definition 5. *For $\delta > 0$ and real numbers $\lambda_1, \cdots, \lambda_m$ the set of real numbers τ satisfying the conditions $|\lambda_\nu \tau| < \delta \pmod{2\pi}$, $\nu = 1, \cdots, m$ is called a $(\delta; \lambda)$-neighbourhood of zero.*

Our proof of the approximation theorem for almost periodic functions will be based on the following theorem.

Theorem 2. *The set of common ε-translation numbers of all almost periodic functions belonging to a uniformly continuous family of almost periodic movements contains a $(\delta; \lambda)$-neighbourhood of zero.*

Proof. According to Lemma 7 the set of common translation numbers is identical to the set $\{\tau_e(\varepsilon)\}$ and, according to a well-known property of ordinary almost periodic functions ([4], p. 110, Satz 2) this set contains a $(\delta; \lambda)$-neighbourhood of zero.

2. On Operations with Almost Periodic Movements in Complete Metric Spaces.

In this section we shall consider a finite number of complete metric spaces $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ and for $\nu = 1, \cdots, n$ an almost periodic movement $f_\nu(t)$ in $\mathfrak{M}_\nu$. The movement $f_\nu(t)$ can be considered as a uniformly continuous family of almost periodic movements (Definition 2) in the particular case where $\mathfrak{N}$ contains only one element. Hence, we can apply the results of Section 1.

We shall consider the topological product $\mathfrak{M} = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$. A point of $\mathfrak{M}$ is an ordered set $\boldsymbol{x} = (x_1, \cdots, x_n)$, $x_\nu \in M_\nu$, $\nu = 1, \cdots, n$. If $\boldsymbol{y} = (y_1, \cdots, y_n)$ is another point of $\mathfrak{M}$, the distance $[\boldsymbol{x}, \boldsymbol{y}]$ is defined by

$$[\boldsymbol{x}, \boldsymbol{y}] = \sqrt{[x_1, y_1]_1^2 + \cdots + [x_n, y_n]_n^2}$$

where $[x_\nu, y_\nu]_\nu$ is the distance between x_ν and y_ν in the metric of $\mathfrak{M}_\nu$. The space $\mathfrak{M}$ is complete.

Lemma 8. *The function* $\boldsymbol{f}(t) = (f_1(t), \cdots, f_n(t))$ *is an almost periodic movement in* $\mathfrak{M}$.

Proof. Let $e_\nu(\tau) = \underset{t}{\text{l. u. b.}}\, [f_\nu(t), f_\nu(t+\tau)]$ be the translation function of $f_\nu(t)$. According to Lemma 5 the set of translation numbers of $\boldsymbol{f}(t)$ corresponding to $\varepsilon > 0$ contains the set of all numbers τ satisfying $e_\nu(\tau) \leqq \frac{\varepsilon}{\sqrt{n}}$, $\nu = 1, \cdots, n$. The function $E(t) = e_1(t) + \cdots + e_n(t)$, however, is almost periodic and $E(0) = 0$. It follows that the set of numbers τ for which $E(\tau) \leqq \frac{\varepsilon}{\sqrt{n}}$, is relatively dense and contains an interval about zero. This proves the lemma.

Let $\mathfrak{M}$ be a complete metric space and $f(t)$ an almost periodic movement in $\mathfrak{M}$. The set of points of $\mathfrak{M}$ which are values of

$f(t)$ when t runs through all real numbers is called the range of $f(t)$. We shall prove the following lemma.

Lemma 9. *The closure of the range of an almost periodic movement $f(t)$ in a complete space $\mathfrak{M}$ is a compact set in $\mathfrak{M}$.*

Proof. Let $\varepsilon > 0$ be given. As $\mathfrak{M}$ is complete, we need only prove the existence of a finite number of points $x_1, \cdots, x_q$ of $\mathfrak{M}$ such that every point of the closure $\mathfrak{A}$ of the range $\mathfrak{B}$ of $f(t)$ is within ε-distance of at least one of the points x_ν. Let l be chosen such that every interval of length l contains a translation number of $f(t)$ corresponding to $\frac{\varepsilon}{3}$. According to Lemma 1 we can choose δ such that $[f(t_2), f(t_1)] \leqq \frac{\varepsilon}{3}$ when $|t_2 - t_1| \leqq \delta$. We put $x_\nu = f(\nu\delta)$, $\nu = 1, \cdots, \left[\frac{l}{\delta}\right]$. Let x be a point of $\mathfrak{A}$. We can find x' in $\mathfrak{B}$ such that $[x', x] < \frac{\varepsilon}{3}$. There exists a real number t' such that $x' = f(t')$ and a real number t'' such that $t' - t'' = \tau_f\left(\frac{\varepsilon}{3}\right)$ and $0 \leqq t'' \leqq l$. We have then $[f(t''), x'] \leqq \frac{\varepsilon}{3}$. Finally, we can choose ν such that $1 \leqq \nu \leqq \left[\frac{l}{\delta}\right]$ and $|t'' - \nu\delta| \leqq \delta$. It follows that $[f(\nu\delta), f(t'')] \leqq \frac{\varepsilon}{3}$. We have thus proved that $[x_\nu, x] < \varepsilon$ and this completes the proof.

Lemma 10. *Let $y = F(x)$, $x \in \mathfrak{M}$, $y \in \mathfrak{M}^*$, where $\mathfrak{M}$ and $\mathfrak{M}^*$ are metric spaces and $\mathfrak{M}$ is complete, be a function continuous in the closure of the range of an almost periodic movement $x = f(t)$. Then $F(f(t))$ is an almost periodic movement.*

Proof. Let $\varepsilon > 0$ be given. In consequence of Lemma 9 the function $F(x)$ is uniformly continuous in the closure $\mathfrak{A}$ of the range of $f(t)$. Hence, we can find a number $\delta > 0$ such that $[F(x_1), F(x_2)]_2 \leqq \varepsilon$ when $[x_1, x_2]_1 \leqq \delta$; $x_1, x_2 \in \mathfrak{A}$. It follows that the set $\{\tau_{F(f)}(\varepsilon)\}$ contains the set $\tau_f(\delta)$ and this completes the proof.

Theorem 3. *Let $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ be complete metric spaces and let $\mathfrak{M}$ denote their topological product. Let $f_\nu(t)$ for $\nu = 1, \cdots, n$ denote an almost periodic movement in $\mathfrak{M}_\nu$ and let $\mathfrak{A}$ denote the closure of the range of the almost periodic movement $\boldsymbol{f}(t) = (f_1(t), \cdots, f_n(t))$ in $\mathfrak{M}$. Let $y = F(x_1, \cdots, x_n)$ where y belongs to a metric space $\mathfrak{M}^*$, be a function continuous when $(x_1, \cdots,$*

$x_n) \in \mathfrak{A}$. *Then* $F(f_1(t), \cdots, f_n(f))$ *is an almost periodic movement in* $\mathfrak{M}^*$.

This follows immediately from Lemmas 8 and 10. The function $F(x_1, \cdots, x_n)$ may be considered as a composition rule in $\mathfrak{M}$. For example if we have some algebraic operation defined in $\mathfrak{M}$ and this operation is continuous, the set of all almost periodic movements in $\mathfrak{M}$ will be mapped into itself by this operation. If $\mathfrak{M}$ is a metric group, almost periodicity will be preserved by group multiplication.

The complex number sphere is metrized by the usual distance in the 3-dimensional Euclidean space and the ordinary addition and multiplication form an algebra on the sphere except at the point at infinity. At this point the operations are discontinuous. Therefore, it cannot be expected that addition or multiplication of almost periodic movements on the sphere will lead to almost periodic sums and products when the original movements get arbitrarily close to infinity. For more details on this question including explicit examples we refer to R. Norgil [11].

3. On the Spatial Extension of a Uniformly Continuous Family of Almost Periodic Movements in a Complete Metric Space.

We shall now study functions $x = G(u_1, u_2, \cdots)$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $x \in \mathfrak{M}$ depending on an infinite sequence of variables. We shall use the vectorial notation $\boldsymbol{u} = (u_1, u_2, \cdots)$, $\boldsymbol{u}' = (u_1', u_2', \cdots)$, $G(\boldsymbol{u}) = G(u_1, u_2, \cdots)$ and the linear operations $\boldsymbol{u}t = (u_1 t, u_2 t, \cdots)$, $\boldsymbol{u} + \boldsymbol{u}' = (u_1 + u_1', u_2 + u_2', \cdots)$. A neighbourhood of the vector $\boldsymbol{u}$ is defined as the set of vectors $\boldsymbol{u}'$ satisfying the inequalities

$$|u_\mu' - u_\mu| < \delta, \ \mu = 1, \cdots, m,$$

where m is a positive integer and $\delta > 0$.

Definition 6. *A function* $x = G(\boldsymbol{u})$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$, $x \in \mathfrak{M}$ *is called limit periodic with the limit period* 2π *in each variable if it satisfies the following condition: To* $\varepsilon > 0$

correspond $\delta > 0$ *and positive integers* m *and* N *such that*

$$[G(\boldsymbol{u}'),\ G(\boldsymbol{u}'')] \leqq \varepsilon$$

when

$$|u_\mu'' - u_\mu'| \leqq \delta \pmod{2N\pi},\ \mu = 1, \cdots, m. \tag{3}$$

It follows immediately that we have the following lemma.

Lemma 11. *A function* $G(\boldsymbol{u})$ *with the limit period* 2π *in each variable is uniformly continuous,* i. e. *to* $\varepsilon > 0$ *correspond* $\delta > 0$ *and a positive integer* m *such that* $[G(\boldsymbol{u}'),\ G(\boldsymbol{u}'')] \leqq \varepsilon$, *when* $|u_\mu'' - u_\mu'| \leqq \delta$, $\mu = 1, \cdots, m$.

The theorem follows immediately from Definition 10 since the conditions $|u_\mu'' - u_\mu'| \leqq \delta$, $\mu = 1, \cdots, m$, imply (3).

Definition 7. *Let* $\mathfrak{R}$ *denote a compact S-space consisting of points denoted* $v, v_1, \cdots$ *and with neighbourhoods* $U(v)$. *A function* $x = G(\boldsymbol{u}; v) = G(u_1, u_2, \cdots; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $v \in \mathfrak{R}$; $x \in \mathfrak{M}$, *is called a uniformly continuous family of limit periodic functions when*

1. *The function* $G(\boldsymbol{u}; v)$ *has the limit period* 2π *in each variable* u_μ *for every fixed* $v \in \mathfrak{R}$.
2. *To* $\varepsilon > 0$ *and* $v_0 \in \mathfrak{R}$ *corresponds a neighbourhood* $U_\varepsilon(v_0)$ *such that* $[G(\boldsymbol{u}; v_0),\ G(\boldsymbol{u}; v)] \leqq \varepsilon$ *when* $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $v \in U_\varepsilon(v_0)$.

Lemma 12. *A uniformly continuous family* $G(\boldsymbol{u}; v)$ *of limit periodic functions has the following property: To* $\varepsilon > 0$ *correspond* $\delta > 0$ *and positive integers* m *and* N *such that*

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when

$$|u_\mu'' - u_\mu'| \leqq \delta \pmod{2N\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{R}.$$

Proof. We can choose $\delta(v_0) > 0$ and positive integers $m(v_0)$ and $N(v_0)$ such that

$$[G(\boldsymbol{u}'; v_0),\ G(\boldsymbol{u}''; v_0)] \leqq \frac{\varepsilon}{3}$$

when

$$|u_\mu'' - u_\mu'| \leqq \delta(v_0) \pmod{2\pi N(v_0))},\ \mu = 1, \cdots, m(v_0).$$

By condition 2 of Definition 7 it follows that

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when

$$|u''_\mu - u'_\mu| \leqq \delta(v_0) \pmod{2\pi N(v_0)},\ \mu = 1, \cdots, m(v_0);\ v \in U_{\frac{\varepsilon}{3}}(v_0).$$

As $\mathfrak{R}$ is a compact S-space, we may choose $v_1, \cdots, v_n \in \mathfrak{R}$ such that $\mathfrak{R} \subset U_{\frac{\varepsilon}{3}}(v_1) \cup \cdots \cup U_{\frac{\varepsilon}{3}}(v_n)$. It follows that the statement in Lemma 12 holds if we choose

$$\delta = \operatorname{Min} \delta(v_\nu),\ m = \operatorname{Max} m(v_\nu),\ N = N(v_1) \cdots N(v_n).$$

This completes the proof.

A connection between uniformly continuous families of limit periodic functions and uniformly continuous families of almost periodic movements is indicated by the following theorem.

Theorem 4. *Let $G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots)$ be a constant real vector. The function $f(t; v) = G(\boldsymbol{\lambda} t; v) = G(\lambda_1 t, \lambda_2 t, \cdots; v)$ is then a uniformly continuous family of almost periodic movements.*

Proof. With the notations of Lemma 12 we have that a real number τ is a translation number of every function $f(t; v_0)$, $v_0 \in \mathfrak{R}$, corresponding to $\varepsilon > 0$ if the inequalities

$$|\lambda_\mu \tau| \leqq \delta \pmod{2N\pi},\ \mu = 1, \cdots, m,$$

are satisfied. It follows from Bohl-Wennberg's theorem that these inequalities are satisfied for a relatively dense set of numbers τ.

Hence condition 1 of Definition 2 is satisfied. Condition 2 of Definition 2 is an immediate consequence of condition 2 of Definition 7. This completes the proof.

Lemma 13. *Let $x_1 = G_1(\boldsymbol{u})$, $x_2 = G_2(\boldsymbol{u})$; $x_1, x_2 \in \mathfrak{M}$ be two functions with the limit period 2π and let $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots)$ be a real vector with rationally independent coordinates. If the almost periodic movements $G_1(\boldsymbol{\beta} t)$ and $G_2(\boldsymbol{\beta} t)$ are identical, the functions $G_1(\boldsymbol{u})$ and $G_2(\boldsymbol{u})$ are also identical.*

Proof. According to Definition 6 we can choose the numbers

δ_1, m_1, and N_1 corresponding to $G_1(\boldsymbol{u})$ and $\frac{\varepsilon}{2} > 0$ and similarly δ_2, m_2, and N_2 corresponding to $G_2(\boldsymbol{u})$ and $\frac{\varepsilon}{2}$. We put

$$\delta = \text{Min}(\delta_1, \delta_2),\ m = \text{Max}(m_1, m_2),\ N = N_1 N_2,$$

and it follows that we have

$$[G_1(\boldsymbol{u}),\ G_2(\boldsymbol{u})] \leqq \varepsilon \tag{4}$$

when there exists a real number t such that

$$|\beta_\mu t - u_\mu| \leqq \delta \pmod{2N\pi},\ \mu = 1, \cdots, m.$$

But it follows from KRONECKER's theorem that such a number t can be chosen for all possible values of $u_1, \cdots, u_m$. Hence (4) holds everywhere, and ε was an arbitrary positive number. Hence $[G_1(\boldsymbol{u}),\ G_2(\boldsymbol{u})] = 0$ for every $\boldsymbol{u}$ and this completes the proof.

Definition 8. *Let $x = G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions and let $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots)$ be a real vector with rationally independent coordinates. The uniformly continuous family $x = f(t; v) = G(\boldsymbol{\beta} t; v)$ of almost periodic movements is called the diagonal family of $G(\boldsymbol{u}; v)$ corresponding to $\boldsymbol{\beta}$ and the family $x = G(\boldsymbol{u}; v)$ is called the spatial extension of $f(t; v)$ corresponding to $\boldsymbol{\beta}$.*

It follows from Lemma 13 that the spatial extension of a uniformly continuous family of almost periodic movements $x = f(t; v)$ corresponding to a vector $\boldsymbol{\beta}$ is uniquely determined by $f(t; v)$ and $\boldsymbol{\beta}$. On the other hand it does not always exist, but we shall prove that we can choose the vector $\boldsymbol{\beta}$ such that a spatial extension of $f(t; v)$ corresponding to $\boldsymbol{\beta}$ exists. For the proof we need the notion of a rational basis of a uniformly continuous family of almost periodic movements.

Definition 9. *A sequence $\beta_1, \beta_2, \cdots$ of rationally independent numbers is called a basis of $x = f(t; v)$ if the following condition is satisfied: To $\varepsilon > 0$ correspond $\eta > 0$ and positive integers m and N such that every real number τ satisfying the conditions*

$$|\beta_\mu \tau| \leqq \eta \pmod{2N\pi},\ \mu = 1, \cdots, m \tag{5}$$

is a common translation number corresponding to ε of all almost periodic functions of the family $x = f(t; v)$.

We shall first prove the existence of a basis.

Theorem 5. *Every uniformly continuous family of almost periodic movements in $\mathfrak{M}$ has a basis.*

Proof. Let $f(t; v)$ be the given family. To every $\varepsilon > 0$ corresponds, according to Theorem 2, a $(\delta; \lambda)$-neighbourhood of zero such that every real number in this neighbourhood is a common ε-translation number of the functions $f(t; v)$. Let us consider the particular case where $\varepsilon = \frac{1}{n}$, $n = 1, 2, \cdots$. We arrange all numbers λ occurring in the definitions of the corresponding $(\delta; \lambda)$-neighbourhoods in a single sequence $\lambda_1, \lambda_2, \cdots$ and we may then write the statement of Theorem 2 on the form: To $\varepsilon > 0$ corresponds $\delta > 0$ and an integer m such that every number τ satisfying

$$|\lambda_\mu \tau| \leqq \delta \pmod{2\pi}, \quad \mu = 1, \cdots, m \tag{6}$$

is a common ε-translation number.

According to a well-known theorem we can find a sequence $\beta_1, \beta_2, \cdots$ of rationally independent numbers such that every λ_μ has a representation

$$\lambda_\mu = r_{\mu 1}\beta_1 + \cdots + r_{\mu q_\mu}\beta_{q_\mu}$$

where the numbers $r_{\mu\nu}$ are rational. Let N denote the common denominator of the numbers $r_{\mu\nu}$, $\mu \leqq m$ and let η be chosen such that

$$0 < \eta < \operatorname*{Min}_{1 \leqq \mu \leqq m} \frac{\delta}{|r_{\mu 1}| + \cdots + |r_{\mu q_\mu}|}.$$

With this choice of the constants the inequalities (6) follow from (5). This proves that the sequence $\beta_1, \beta_2, \cdots$ is a basis of $f(t; v)$.

Theorem 6. *If the sequence $\beta_1, \beta_2, \cdots$ is a basis of a uniformly continuous family of almost periodic movements $f(t; v)$, there exists a spatial extension of $f(t; v)$ corresponding to $\beta = \beta_1, \beta_2, \cdots$.*

Proof. A similar proof for ordinary almost periodic functions was given by E. PEDERSEN [12]. Let $u_1, u_2, \cdots$ be a given se-

quence of real numbers. If n is a positive integer, it follows from KRONECKER's theorem that we can find a real number $t_n(\boldsymbol{u})$ such that

$$|\beta_\nu t_n(\boldsymbol{u}) - u_\nu| \leqq \frac{1}{n} \pmod{n!\,2\pi}, \quad \nu = 1, \cdots, n.$$

If p is a positive integer, it follows that

$$|\beta_\nu (t_{n+p}(\boldsymbol{u}) - t_n(\boldsymbol{u}))| \leqq \frac{2}{n} \pmod{n!\,2\pi}, \quad \nu = 1, \cdots, n.$$

By comparison with the conditions (5) we find that $t_{n+p}(\boldsymbol{u}) - t_n(\boldsymbol{u})$ is a common translation number of $f(t; v)$ corresponding to $\varepsilon > 0$ if n is large enough. Since $\mathfrak{M}$ is complete, it follows that there exists a limit function

$$G(\boldsymbol{u}; v) = \lim_{n \to \infty} f(t_n(\boldsymbol{u}); v).$$

The function $G(\boldsymbol{u}; v)$ is independent of the particular choice of the sequence of functions $t_n(\boldsymbol{u})$. It is not very difficult to prove this, but we find it more convenient to prove first that $G(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.

Let $\varepsilon > 0$ be given. According to Definition 9 we can find a corresponding $\eta > 0$ and positive integers m and N such that every real τ satisfying (5) is a common ε-translation number. If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are two vectors satisfying

$$(7) \qquad |u_\nu'' - u_\nu'| \leqq \frac{\eta}{3} \pmod{2N\pi}, \quad \nu = 1, \cdots, m,$$

we have for $n \geqq \operatorname{Max}(m, N)$ that the following inequalities hold mod $2N\pi$ when $\nu = 1, \cdots, m$.

$$|\beta_\nu (t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}'))| \leqq$$

$$|\beta_\nu t_n(\boldsymbol{u}'') - u_\nu''| + |u_\nu'' - u_\nu'| + |u_\nu' - \beta_\nu t_n(\boldsymbol{u}')| \leqq \frac{\eta}{3} + \frac{2}{n},$$

and if we further choose $n \geqq \dfrac{6}{\eta}$, we obtain

$$|\beta_\nu (t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}'))| \leqq \eta \pmod{2N\pi}, \quad \nu = 1, \cdots, m.$$

Thus, the inequalities (5) are satisfied by $t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}')$ and we have

$$[f(t_n(\boldsymbol{u}'); v),\ f(t_n(\boldsymbol{u}''); v)] \leqq \varepsilon.$$

It follows that

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leqq \varepsilon \tag{8}$$

when (7) is satisfied and this proves that the first condition of Definition 7 is satisfied. If $v_0 \in \mathfrak{R}$ is given, it follows from Definition 2 that we can find a neighbourhood $U_\varepsilon(v_0)$ such that

$$[f(t; v_0),\ f(t; v)] \leqq \varepsilon \text{ when } -\infty < t < \infty,\ v \in U_\varepsilon(v_0).$$

We have then in particular

$$[f(t_n(\boldsymbol{u}); v_0),\ f(t_n(\boldsymbol{u}); v)] \leqq \varepsilon$$

for all $\boldsymbol{u}$ and n when $v \in U_\varepsilon(v_0)$, hence for $n \to \infty$

$$[G(\boldsymbol{u}; v_0),\ G(\boldsymbol{u}; v)] \leqq \varepsilon,\ v \in U_\varepsilon(v_0).$$

This proves that the second condition of Definition 7 is satisfied and the function $G(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.

This implies that $G(\boldsymbol{u}; v)$ is independent of the choice of the sequence $t_n(\boldsymbol{u})$. In fact, if we made another choice of the sequence $t_n(\boldsymbol{u}^*)$ for just one fixed $\boldsymbol{u}^*$ and thereby changed the value of $G(\boldsymbol{u}^*; v^*)$, this would certainly disturb the continuity of the limit periodic function $G(\boldsymbol{u}; v^*)$.

If, in particular, $\boldsymbol{u} = \boldsymbol{\beta} t$, we may always choose $t_n(\boldsymbol{u}) = t$ and we obtain $G(\boldsymbol{\beta} t; v) = f(t; v)$. This completes the proof of Theorem 6.

In some special cases the preceding result may be simplified. We shall first consider a function with a finite basis.

Definition 10. *If the number m in Definition 9 can be chosen independent of ε, the finite sequence $\beta_1, \cdots, \beta_m$ is called a finite basis of $f(t; v)$.*

In this case the numbers $\beta_{m+1}, \beta_{m+2}, \cdots$ may obviously be chosen at random, since they do not occur in the conditions (5).

Theorem 7. *If* $\beta_1, \cdots, \beta_m$ *in Theorem* 6 *is a finite basis of* $f(t; v)$, *the spatial extension* $G(\boldsymbol{u}; v)$ *is independent of* u_{m+1}, $u_{m+2}, \cdots$.

Proof. If the vectors $\boldsymbol{u}'$ and $\boldsymbol{u}''$ satisfy the condition $u_\nu' = u_\nu''$, $\nu = 1, \cdots, m$, the condition (7) is satisfied for every $\eta > 0$ and every positive integer N. Hence (8) is satisfied for every $\varepsilon > 0$. This completes the proof. In this case we write $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)$ and $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$.

Definition 11. *The basis* $\beta_1, \beta_2, \cdots$ *is called integral if we can always choose* $N = 1$ *in Definition* 7.

This happens if the coefficients $r_{\mu\nu}$ in the proof of Theorem 5 are integers.

Theorem 8. *If the basis* $\beta_1, \beta_2, \cdots$ *in Theorem* 6 *is integral, the spatial extension has the period* 2π *in each variable.*

Proof. When $N = 1$, the conditions (7) are obviously satisfied when

$$u_\nu'' = u_\nu' + 2 h_\nu \pi, \quad \nu = 1, 2, \cdots$$

where the numbers h_ν are integers. With this connection between $\boldsymbol{u}'$ and $\boldsymbol{u}''$ the relation (8) is then satisfied for every $\varepsilon > 0$. This completes the proof.

Finally, we shall prove the following lemmas on families of limit periodic functions.

Lemma 14. *To a uniformly continuous family* $G(\boldsymbol{u}; v)$ *of limit periodic functions and a given* $\varepsilon > 0$ *corresponds a uniformly continuous family* $G^*(\boldsymbol{u}; v) = G^*(u_1, \cdots, u_m; v)$ *of limit periodic functions depending only on a finite number of variables and satisfying*

$$[G^*(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leq \varepsilon.$$

Proof. From Lemma 12 follows immediately that

$$[G(u_1, \cdots, u_m, 0, 0, \cdots; v), G(u_1, u_2, \cdots; v)] \leq \varepsilon,$$

which proves the lemma.

Lemma 15. *The closure of the range of a uniformly continuous family of limit periodic functions is a compact set.*

Proof. With the notations of Lemma 12 the range of the function

$x = G(u_1, \cdots, u_m, 0, 0, \cdots; v), 0 \leqq u_\mu \leqq 2N\pi, \mu = 1, \cdots, m; v \in K$

is a compact set, which comes within ε-distance of every point of the closure of the range of $G(\boldsymbol{u}; v)$. As ε is arbitrary, this implies Lemma 15.

4. The Approximation Theorem.

It will now be necessary to restrict the investigations to a much more special type of metric spaces. The conditions which we are going to impose upon the space $\mathfrak{M}$ are stronger than necessary for the validity of the approximation theorem, but they are rather simple and the proof will not be too difficult.

Definition 12. *The metric space $\mathfrak{M}$ is called continuously locally arcwise connected when it satisfies the following condition: To every compact subset $\mathfrak{C}$ of $\mathfrak{M}$ corresponds a positive number Δ and a continuous function $z = \varphi(x; t; y)$, $0 \leqq t \leqq 1$, $x, y \in \mathfrak{C}$, $[x, y] \leqq \Delta$; $z \in \mathfrak{M}$ satisfying the conditions*

$$\varphi(x; 0; y) = x, \ \varphi(x; 1; y) = y; \ \varphi(x; t; x) = x.$$

This condition is satisfied if any two points x and y of $\mathfrak{M}$ whose distance remains below a certain number can be connected by a geodetic arc which depends continuously on x and y.

Lemma 16. *Let $\mathfrak{C}$ be a compact subset of $\mathfrak{M}$. To $\varepsilon > 0$ corresponds $\delta > 0$ such that $[x, \varphi(x; t; y)] \leqq \varepsilon$ when $0 \leqq t \leqq 1$, $x, y \in \mathfrak{C}$, $[x, y] \leqq \delta$.*

Proof. As $\varphi(x; t; y)$ is uniformly continuous when $0 \leqq t \leqq 1$, $x, y \in \mathfrak{C}$, we can determine δ such that $[\varphi(x; t; x), \varphi(x; t; y)] \leqq \varepsilon$ when $0 \leqq t \leqq 1$, $x, y \in \mathfrak{C}$, $[x, y] \leqq \delta$. Since $x = \varphi(x; t; x)$, this proves the lemma.

We observe that the conditions in Definition 12 do not imply that $\mathfrak{M}$ is complete. In fact, an open segment of a straight line with the ordinary metric satisfies the conditions, but it is not complete.

We are going to prove that a uniformly continuous family of limit periodic functions can be approximated with any given

accuracy by another family depending only on a finite number of variables and continuous and periodic in each of these. This is an elementary consequence of the conditions in Definition 12, but the proof presents some technical difficulties. To make the proof more perspicuous we shall make use of some notions introduced by the following definition.

Definition 13. *Let $G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions with values from $\mathfrak{M}$. If we give the variable u_ν a fixed value a, we obtain a function, which we shall denote $G_{\nu;a}(\boldsymbol{u}; v)$. Let N be a positive integer and d a positive number such that $[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leqq \Delta$ (Definition 12), when $|u_\nu'' - u_\nu'| \leqq d \pmod{2N\pi}$ and $u_\mu' = u_\mu''$, $\mu \neq \nu$. We define $H(\boldsymbol{u}; v) = T_{\nu;N,d} G(\boldsymbol{u}; v)$ in the following way.*

$$H(\boldsymbol{u}; v) = G(\boldsymbol{u}; v) \text{ when } 0 \leqq u_\nu \leqq 2N\pi - d.$$

$$H(\boldsymbol{u}; v) = \varphi\left(G_{\nu;2N\pi-d}(\boldsymbol{u}; v); \frac{u_\nu - 2N\pi + d}{d}; G_{\nu;0}(\boldsymbol{u}; v)\right)$$
$$\text{when } 2N\pi - d \leqq u_\nu \leqq 2N\pi.$$

$H(\boldsymbol{u}; v)$ is periodic in u_ν with the period $2N\pi$.

The function $H(\boldsymbol{u}; v) = T_{\nu;N,d} G(\boldsymbol{u}; v)$ is called a periodification of $G(\boldsymbol{u}; v)$ with respect to the variable u_ν.

We observe that it follows from the properties of $\varphi(x; t; y)$ that the definition of $H(\boldsymbol{u}; v)$ is not ambiguous when $u_\nu = 2N\pi - d$ or when u_ν is a multiple of $2N\pi$. We shall first prove the following lemma.

Lemma 17. *The periodification $H(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.*

According to Lemma 15 there exists a compact set $\mathfrak{S} \subset \mathfrak{M}$ which contains all values assumed by $G(\boldsymbol{u}; v)$. The function $\varphi(x; t; y)$ is uniformly continuous when $0 \leqq t \leqq 1$, $x, y \in \mathfrak{S}$, $[x, y] \geqq \Delta$. Let $\varepsilon > 0$ be given. We can then choose two positive numbers η_1 and η_2 such that

$$[\varphi(x; t_1; y), \varphi(x; t_2; y)] \leqq \frac{\varepsilon}{4} \text{ when } |t_2 - t_1| \leqq \eta_1$$

and

$$[\varphi(x_1; t; y_1), \varphi(x_2; t; y_2)] \leqq \frac{\varepsilon}{2} \text{ when } [x_1, x_2] \leqq \eta_2, [y_1, y_2] \leqq \eta_2.$$

In both cases we must, of course, assume that each x and y belongs to $\mathfrak{C}$ and that every t belongs to the interval $0 \leqq t \leqq 1$.

We choose a positive number $\delta_1 \leqq \eta_1 d$ such that

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{4} \text{ when } |u_\nu'' - u_\nu'| \leqq \delta_1;\ u_\mu' = u_\mu'',\ \mu \neq \nu;\ v \in \mathfrak{R}$$

and it follows immediately that

$$[H(\boldsymbol{u}'; v),\ H(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{2}$$

when

$$|u_\nu'' - u_\nu'| \leqq \delta_1 \pmod{2N\pi};\ u_\mu' = u_\mu'',\ \mu \neq \nu;\ v \in \mathfrak{R}.$$

We choose a positive number δ_2 and positive integers N^* and $m \geqq \nu$ such that

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leqq \text{Min}\left(\frac{\varepsilon}{2}, \eta_2\right)$$

when

$$|u_\mu'' - u_\mu'| \leqq \delta_2 \pmod{2N^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{R}.$$

It follows immediately that

$$[H(\boldsymbol{u}'; v),\ H(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{2}$$

when

$$u_\nu' = u_\nu'';\ |u_\mu'' - u_\mu'| \leqq \delta_2 \pmod{2N^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{R}.$$

If we combine our two results, we obtain

$$[H(\boldsymbol{u}'; v),\ H(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when

$$|u_\mu'' - u_\mu'| \leqq \text{Min}(\delta_1, \delta_2) \pmod{2NN^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{R}.$$

Let $v_0 \in \mathfrak{R}$ be given. We choose a neighbourhood $U(v_0)$ such that $[G(\boldsymbol{u}; v_0),\ G(\boldsymbol{u}; v)] \leqq \text{Min}(\varepsilon, \eta_2)$ when $v \in U(v_0)$. It follows that

$$[H(\boldsymbol{u}; v_0),\ H(\boldsymbol{u}; v)] \leqq \varepsilon \text{ when } v \in U(v_0).$$

This completes the proof of Lemma 17.

Lemma 18. *If the family $G(\boldsymbol{u}; v)$ of Definition 13 is periodic with the period $2N_1\pi$ in the variable u_μ, the function $H(\boldsymbol{u}; v)$ also has the period $2N_1\pi$ in the variable u_μ. If $G(\boldsymbol{u}; v)$ is independent of u_μ, the function $H(\boldsymbol{u}; v)$ is also independent of u_μ.*

Proof. The theorem follows immediately from the definition of $H(\boldsymbol{u}; v)$.

Lemma 19. *Let $\varepsilon > 0$ be given. We can choose $d > 0$ and a positive integer N such that the family $H(\boldsymbol{u}; v)$ of Definition 13 satisfies the condition $[H(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leqq \varepsilon$ for every vector $\boldsymbol{u}$ and every $v \in \mathfrak{R}$.*

Proof. According to Lemmas 15 and 16 we can choose $\eta > 0$ such that $[x, \varphi(x; t; y)] \leqq \frac{\varepsilon}{2}$ when $0 \leqq t \leqq 1$ and x and y are values assumed by $G(\boldsymbol{u}; v)$ and satisfying $[x, y] \leqq \eta$. We can then, according to Lemma 12, choose $d \geqq 0$ and a positive integer N such that

$$[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leqq \operatorname{Min}\left(\frac{\varepsilon}{2}, \eta\right)$$

when

$$u'_\mu = u''_\mu,\ \mu \neq \nu;\ |u''_\nu - u'_\nu| \leqq d \pmod{2N\pi};\ v \in \mathfrak{R}.$$

Let $\boldsymbol{u}$ be arbitrary. There exists a vector $\boldsymbol{u}^*$ and an integer q satisfying

$$u^*_\mu = u_\mu,\ \mu \neq \nu;\ u_\nu - u^*_\nu = 2qN\pi;\ 0 \leqq u^*_\nu \leqq 2N\pi.$$

We have

$$H(\boldsymbol{u}; v) = H(\boldsymbol{u}^*; v);\ [G(\boldsymbol{u}; v), G(\boldsymbol{u}^*; v)] \leqq \frac{\varepsilon}{2}.$$

If $0 \leqq u^*_\nu \leqq 2N\pi - d$, we have $H(\boldsymbol{u}^*; v) = G(\boldsymbol{u}^*; v)$, hence

$$[H(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leqq \frac{\varepsilon}{2}.$$

If $2N\pi - d < u^*_\nu \leqq 2N\pi$, we have

$$[G_{\nu; 2N\pi - d}(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leqq \frac{\varepsilon}{2}$$

and

$$[G_{\nu;2N\pi-d}(\boldsymbol{u};v),\ H(\boldsymbol{u};v)] =$$

$$\left[G_{\nu;2N\pi-d}(\boldsymbol{u};v),\ \varphi\left(G_{\nu;2N\pi-d}(\boldsymbol{u};v);\ \frac{u_\nu^*-2N\pi+d}{d};\ G_{\nu;0}(\boldsymbol{u};v)\right)\right]\leqq\frac{\varepsilon}{2}$$

since $[G_{\nu;2N\pi-d}(\boldsymbol{u};v),\ G_{\nu;0}(\boldsymbol{u};v]\leqq\eta$. Hence, we have in this case $[H(\boldsymbol{u};v),\ G(\boldsymbol{u};v)]\leqq\varepsilon$ and this completes the proof.

We can now prove the approximation theorem for uniformly continuous families of limit periodic functions.

Theorem 9. *Let $G(\boldsymbol{u};v)$ be a uniformly continuous family of limit periodic functions with values from a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let ε be a positive number. There exist two positive integers m and N and a continuous function $g(\boldsymbol{u};v)=g(u_1,\cdots,u_m;v)$ with the period $2N\pi$ in each variable, satisfying $[g(\boldsymbol{u};v),\ G(\boldsymbol{u};v)]\leqq\varepsilon$ for every vector $\boldsymbol{u}$ and every $v\in\mathfrak{N}$.*

Proof. From Lemma 14 follows that we can find an integer m and a uniformly continuous family $g_0(\boldsymbol{u};v)=g_0(u_1,\cdots,u_m;v)$ of limit periodic functions satisfying

$$[g_0(\boldsymbol{u};v),\ G(\boldsymbol{u};v)]\leqq\frac{\varepsilon}{2}.$$

To prove the theorem we need only construct a finite sequence $g_0(\boldsymbol{u};v),\cdots,g_m(\boldsymbol{u};v)$ of uniformly continuous families of limit periodic functions satisfying the conditions

1). $[g_\nu(\boldsymbol{u};v),\ g_{\nu-1}(\boldsymbol{u};v)]\leqq\dfrac{\varepsilon}{2m},\ \nu=1,\cdots,m.$

2). $g_\nu(\boldsymbol{u};v)$ is independent of the variables $u_{m+1},u_{m+2},\cdots.$

3). There exists a sequence $N_1,\cdots,N_m$ of positive integers such that $g_\nu(\boldsymbol{u};v)$ has the periods $2N_1\pi,\cdots,2N_\nu\pi$ in the variables $u_1,\cdots,u_\nu$.

In fact, $g_m(\boldsymbol{u};v)$ depends only on $u_1,\cdots,u_m$, it has the period $2N_1\cdots N_m\pi$ in each of these variables, and it satisfies the condition

$$[g_m(\boldsymbol{u};v),\ G(\boldsymbol{u};v]\leqq$$

$$\sum_{\nu=1}^{m}[g_\nu(\boldsymbol{u};v),\ g_{\nu-1}(\boldsymbol{u};v)]+[g_0(\boldsymbol{u};v),\ G(\boldsymbol{u};v)]\leqq m\frac{\varepsilon}{2m}+\frac{\varepsilon}{2}=\varepsilon.$$

Let us now assume that we have already constructed the functions $g_0(\boldsymbol{u}; v), \cdots, g_{\nu-1}(\boldsymbol{u}; v)$ satisfying 1)—3). According to Lemma 19 we can choose a periodification $g_\nu(\boldsymbol{u}; v) = T_{\nu; N_\nu, d}\, g_{\nu-1}(\boldsymbol{u}; v)$ such that 1) is satisfied. From Lemma 17 follows that $g_\nu(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions, and from Lemma 18 follows that the conditions 2) and 3) are satisfied. This completes the proof.

If we now combine Theorems 5, 6, and 9, we obtain the following approximation theorem.

Theorem 10. *Let $\mathfrak{M}$ be a complete, continuously locally arcwise connected space. To a uniformly continuous family of almost periodic movements $x = f(t; v)$, $-\infty < t < \infty$, $v \in \mathfrak{R}$; $x \in \mathfrak{M}$ and a positive number ε correspond positive integers m and N, a continuous function $g(u_1, \cdots, u_m; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$; $v \in \mathfrak{R}$ with the period $2N\pi$ in each of the variables u_ν, and rationally independent numbers $\beta_1, \cdots, \beta_m$ such that*

$$[g(\beta_1 t, \cdots, \beta_m t; v), f(t; v)] \leqq \varepsilon, \quad -\infty < t < \infty, \quad v \in \mathfrak{R}.$$

We could have proved this theorem by a more direct method similar to a variant of Bogolioùboff's proof of the theorem for ordinary almost periodic functions (Cf. [10] p. 96—109). By this method the existence of the spatial extension is not proved and it is not necessary to assume that $\mathfrak{M}$ is complete. We have preferred the longer proof because it yields the theorem on the existence of the spatial extension more directly.

5. A Complete Metric Space, in which the Approximation Theorem for Almost Periodic Movements does not hold.

We have seen that the approximation theorem for almost periodic movements holds in every complete, continuously locally arcwise connected space. Although this property of the space is not necessary, we shall prove that it is not superfluous. To do this, we find a complete metric space in which the approximation theorem does not hold. We shall use one of the so-called solenoidal spaces, introduced and investigated by D. van Dantzig [6]. He uses the following construction.

In a half-plane p bounded by a straight line l a circular disk c_1 is situated at a positive distance h from l. A circular disk c_2 has its center at a fixed point of c_1, a circular disk c_3 has its center at a fixed point of c_2, and so on. The boundary of c_{n+1} shall belong to the interior of c_n and the center of c_n shall be outside c_{n+1}. Let $q_1, q_2, \cdots$ be a sequence of integers greater than 1. For every n the disk c_n with the center of c_{n+1} rigidly attached to it revolves in its plane around its center with the angular velocity $(q_1 \cdots q_n)^{-1}$, while the plane p rotates about the line l with the angular velocity 1. The boundary of the disk c_n then describes a closed tube T_n, which is wound $q_1 \cdots q_n$ times around l. The radius of c_n is chosen so small that the $q_1 \cdots q_n$ strands of T_n are entirely separate. The intersection of all the tubes T_n is called a solenoid. It is a perfect point set in 3-dimensional space. Considered as a relative space the solenoid is complete, compact, and connected.

At every fixed moment during the rotation, the half-plane p intersects the interior of the tube T_n in $q_1 \cdots q_n$ circles congruent to c_n. For every n we choose one of these circles c_n^* such that $c_1 = c_1^* \supset c_2^* \supset c_3^* \supset \cdots$. Exactly one point x of the half-plane p will be contained in all these circles. When p rotates, the circles of intersection c_n^* move continuously in the plane and the point x defined by them performs a continuous movement $x = g(t)$ in the solenoid. This movement is not periodic, but the movements performed by the centers of $c_1^*, c_2^*, \cdots$ are periodic and converge uniformly to $g(t)$. Hence, $g(t)$ is an almost periodic movement in the solenoid. The center of c_n^* does not belong to the solenoid, and we shall see that $g(t)$ cannot be approximated by any diagonal movement in the solenoid with an accuracy better than $2h$.

The curve $x = g(t)$ has no double points. VAN DANTZIG called it a pseudo-component of the solenoid, and he proved that two points of two differents pseudo-components cannot be connected by a continuous curve, contained in the solenoid ([6], p. 116). Let $x = f(\boldsymbol{u}) = f(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, where x is a variable point of the solenoid, be a continuous function. It follows that x belongs to one pseudo-component of the solenoid, and we can choose one of the points x_0, in which this pseudo-component intersects p at the time

$t = 0$. The position of x is then determined by the angle θ of rotation of p, which makes x_0 slide along the pseudo-component until it reaches the position of x. Thus, the function $x = f(\boldsymbol{u})$ determines a continuous function $\theta = \theta(\boldsymbol{u})$. If $f(\boldsymbol{u})$ has the period 2π in each variable, the function $\theta(\boldsymbol{u})$ also has the period 2π. To the diagonal function $f(\boldsymbol{\beta} t)$, where $\beta_1 \cdots, \beta_m$ are rationally independent numbers, corresponds the bounded function $\theta = \theta(\boldsymbol{\beta} t)$. The movement $g(t)$ introduced above corresponds to the function $\theta_1 = t$. For some value of t we have $\theta_1 - \theta = \pi$ and for this particular value the distance between $f(\boldsymbol{\beta} t)$ and $g(t)$ is $> 2h$. This proves that the approximation theorem for almost periodic movements does not hold in the solenoid.

CHAPTER 2.

Homotopic Almost Periodic Movements.

1. Preliminary Definitions and Investigations.

In the present chapter we shall study some simple questions in connection with the topology of almost periodic movements in a metric space $\mathfrak{M}$. From a topological point of view two almost periodic movements are considered essentially identical to one another if one of them can be changed in a uniformly continuous way into the other. It must be specified, however, whether or not the almost periodicity shall be preserved during the process of changing one movement into the other. If we require that the almost periodicity shall be preserved, we get a finer classification and we prefer this point of view. Hence, the following definition.

Definition 14. *Two almost periodic movements* $x_1 = f_1(t)$, $x_2 = f_2(t)$, $-\infty < t < \infty$; $x_1, x_2 \in \mathfrak{M}$ *are called (almost periodically) homotopic to one another if there exists a uniformly continuous family* $x = f(t; v)$, $-\infty < t < \infty$, v *real,* $\alpha \leq v \leq \beta$, $x \in \mathfrak{M}$ *satisfying* $f(t; \alpha) = f_1(t)$, $f(t; \beta) = f_2(t)$.

The relation of almost periodic homotopy is an equivalence relation in the set of almost periodic movements in $\mathfrak{M}$ and it leads to a division of this set into almost periodic homotopy classes. A first object of a topology of almost periodic movements

will be a method to attach to every almost periodic movement certain constants characteristic for the class of the movement. In the present paper we shall solve only a small fraction of this problem. We shall find a property characteristic for the metric spaces in which every homotopy class contains a periodic movement. We shall always assume that the space $\mathfrak{M}$ is complete and continuously locally arcwise connected.

Lemma 20. *Every almost periodic movement* $x_1 = f_1(t)$ *in* $\mathfrak{M}$ *is homotopic to a certain diagonal movement* $x_2 = g(\boldsymbol{\beta} t) = g(\beta_1 t, \cdots, \beta_m t)$ *where* $g(\boldsymbol{u})$ *has the period* 2π *in each variable and* $\beta_1, \cdots, \beta_m$ *are rationally independent numbers.*

Proof. Let Δ be the positive number introduced in Definition 12. According to Theorem 10 we can choose $g(\boldsymbol{u})$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)$ such that $[g(\boldsymbol{\beta} t), f_1(t)] \leqq \Delta$, $-\infty < t < \infty$. We define

$$f(t; v) = \varphi(f_1(t); v; g(\boldsymbol{\beta} t)), \quad 0 \leqq v \leqq 1$$

where φ is the function introduced in Definition 16. Let $\varepsilon > 0$ be given. As $f_1(t)$ and $g(\beta_1 t, \cdots, \beta_m t)$, according to Lemma 9, are contained in a compact subset of $\mathfrak{M}$, we can find $\delta > 0$ such that

$$[f(t; v), f(t + \tau; v)] \leqq \varepsilon$$

when τ is a common translation number of $f_1(t)$ and $g(\boldsymbol{\beta} t)$ corresponding to δ. However, it follows from Lemma 8 that the set of these numbers is relatively dense. Hence, the first condition in Definition 2 is satisfied. The second condition follows immediately from the uniform continuity of φ. From Definition 12 follows finally

$$f(t; 0) = f_1(t), \quad f(t; 1) = g(\boldsymbol{\beta} t).$$

This completes the proof.

Remark. *The movement* $x_2 = g(\boldsymbol{\beta} t)$ *in Lemma 20 is just an arbitrary diagonal function, which approximates* $f_1(t)$ *with the accuracy* Δ.

2. On Reducible Almost Periodic Movements.

We shall prove the following lemma.

Lemma 21. *Let* $g(\boldsymbol{u}) = g(u_1, \cdots, u_m)$ *be a continuous function with the period* 2π *in each variable and let* $\beta_1, \cdots, \beta_m$ *be*

rationally independent numbers. The diagonal movement $f(t) = g(\boldsymbol{\beta} t) = g(\beta_1 t, \cdots, \beta_m t)$ *is periodic if and only if* $g(\boldsymbol{u})$ *has the form* $h(n_1 u_1 + \cdots + n_m u_m)$ *where* $n_1, \cdots, n_m$ *are integers and* $h(u)$ *has the period* 2π.

Proof. The function $g(n_1\beta_1 t + \cdots + n_m\beta_m t)$ is obviously periodic with the period $2\pi(n_1\beta_1 + \cdots + n_m\beta_m)^{-1}$. This proves the sufficiency. On the other hand, if $f(t)$ has the period p, a vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_m)$ is a period vector of $g(\boldsymbol{u})$ if the inequalities

$$|\beta_\mu p n - \xi_\mu| \leqq \varepsilon \pmod{2\pi}, \quad \mu = 1, \cdots, m$$

can be solved with respect to n for every $\varepsilon > 0$. This condition is satisfied if and only if the inequalities

$$(9) \quad \begin{cases} |\beta_\mu p \tau - \xi_\mu| \leqq \varepsilon \pmod{2\pi}, \ \mu = 1, \cdots, m, \\ |2\pi\tau| \leqq \varepsilon \pmod{2\pi} \end{cases}$$

can be solved with respect to τ for every $\varepsilon > 0$.

If the numbers $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ are rationally independent, the inequalities (9) have solutions for every vector $\boldsymbol{\xi}$ and every $\varepsilon > 0$, according to Kronecker's theorem; but this implies that $g(\boldsymbol{u})$ is constant.

If $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ are not rationally independent, they satisfy a relation

$$(10) \qquad N \cdot \frac{2\pi}{p} = n_1\beta_1 + \cdots + n_m\beta_m$$

where $N, n_1, \cdots, n_m$ are integers, some of which are different from zero. All possible linear relations with rational coefficients satisfied by the numbers $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$, are obtained from (10) by multiplication with an arbitrary rational constant. In fact, if $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ satisfied two linear relations of the form (10) and not proportional to one another, we could, by elimination of $\dfrac{2\pi}{p}$, obtain a non-trivial linear relation with rational coef-

ficients between $\beta_1, \cdots, \beta_m$ in contradiction to our assumption that $\beta_1, \cdots, \beta_m$ are rationally independent. In this case Kronecker's theorem states that the inequalities (9) have solutions in τ if and only of $n_1\xi_1 + \cdots + n_m\xi_m$ is an integral multiple of 2π. This implies that $g(\boldsymbol{u})$ has the form $h(n_1u_1 + \cdots + n_mu_m)$ where $h(u)$ has the period 2π. This proves the lemma.

Definition 15. *An almost periodic movement in the metric space $\mathfrak{M}$ is called reducible if it is homotopic to a periodic movement. If it is not reducible, it is called irreducible.*

Definiton 16. *A continuous function $x = g(\boldsymbol{u}) = g(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, $x \in \mathfrak{M}$ with the period $2N\pi$, N integral, in each variable is called an m-dimensional torus mapping into $\mathfrak{M}$. A continuous function $x = g(\boldsymbol{u}; v) = g(u_1, \cdots, u_m; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, $\alpha \leqq v \leqq \beta$, $x \in \mathfrak{M}$, with the period $2N\pi$, N integral, in the variables $u_1, \cdots, u_m$ is called a family of torus mappings into $\mathfrak{M}$. Two torus mappings $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ into $\mathfrak{M}$ are called homotopic if there exists a family $g(\boldsymbol{u}; v)$, $0 \leqq v \leqq 1$, of torus mappings into $\mathfrak{M}$ such that $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. A torus mapping into $\mathfrak{M}$ is called reducible if it is homotopic to a torus mapping of the form $h(\chi(\boldsymbol{u}))$ where $\chi = \chi(\boldsymbol{u})$ is a real-valued, continuous function with the property that $e^{i\chi(\boldsymbol{u})}$ has the period $2N\pi$ in each variable, while $h(\chi)$ is a continuous function with the period 2π. A torus mapping which is not reducible is called irreducible.*

Briefly, a torus mapping is reducible if it can be contracted to a torus mapping into a closed curve in $\mathfrak{M}$. By the following lemma the question about reducibility of an almost periodic movement in $\mathfrak{M}$ is reduced to the question about reducibility of a torus mapping. The proof depends on results from Chapter 1.

Lemma 22. *Let $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ be two m-dimensional torus mappings into $\mathfrak{M}$ and let $\beta_1, \cdots, \beta_m$ denote rationally independent real numbers. The almost periodic movements $g_1(\boldsymbol{\beta}t)$ and $g_2(\boldsymbol{\beta}t)$ are homotopic if and only if $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ are homotopic torus mappings.*

Proof. If $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ are homotopic, there exists a family $g(\boldsymbol{u}; v)$, $0 \leqq v \leqq 1$ of torus mappings into $\mathfrak{M}$. The function $g(\boldsymbol{u}; v) = g(u_1, \cdots, u_m; v)$ has a period $2N\pi$ in each of the variables u_ν and it satisfies $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. According to Theorem 4, $g(\boldsymbol{\beta}t; v)$ is a uniformly con-

tinuous family of almost periodic functions; and, since $g(\boldsymbol{\beta} t; 0) = g_1(\boldsymbol{\beta} t)$ and $g(\boldsymbol{\beta} t; 1) = g_2(\boldsymbol{\beta} t)$, this proves that $g_1(\boldsymbol{\beta} t)$ and $g_2(\boldsymbol{\beta} t)$ are homotopic.

If, on the other hand, $g_1(\boldsymbol{\beta} t)$ and $g_2(\boldsymbol{\beta} t)$ are homotopic, there exists a uniformly continuous family $f(t; v)$, $0 \leqq v \leqq 1$, of almost periodic movements satisfying $f(t; 0) = g_1(\boldsymbol{\beta} t)$ and $f(t; 1) = g_2(\boldsymbol{\beta} t)$. According to Theorem 5 the family $f(t; v)$ has a basis, and we can obviously use $\beta_1, \cdots, \beta_m$ as the first numbers of this basis such that the basis is $\beta_1, \beta_2, \cdots$. According to Theorem 6 the family $f(t; v)$ has a spatial extension $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$ and $G(\boldsymbol{\beta} t; v) = f(t; v)$. From Lemma 13 it follows that $G(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $G(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. These properties of the family $G(\boldsymbol{u}; v)$ are preserved if we give $u_{m+1}, u_{m+2}, \cdots$ arbitrary constant values. We may therefore assume that $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$. Let Δ be the real number introduced in Definition 12. According to Theorem 9 there exist an integer N and a continuous function $h(\boldsymbol{u}; v) = h(u_1, \cdots, u_m; v)$, $0 \leqq v \leqq 1$, such that $[h(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leqq \Delta$ and the functions $h(\boldsymbol{u}; v)$, $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ have the period $2N\pi$ in each of the variables u_ν. By means of the function φ of Definition 12 we define

$$g(\boldsymbol{u}; v) = \varphi(g_1(\boldsymbol{u}); 3v; h(\boldsymbol{u}; 0)), \qquad 0 \leqq v \leqq \frac{1}{3}$$

$$g(\boldsymbol{u}; v) = h(\boldsymbol{u}; 3v - 1), \qquad \frac{1}{3} \leqq v \leqq \frac{2}{3}$$

$$g(\boldsymbol{u}; v) = \varphi(h(\boldsymbol{u}; 1); 3v - 2; g_2(\boldsymbol{u})), \qquad \frac{2}{3} \leqq v \leqq 1.$$

The function $g(\boldsymbol{u}; v)$ is continuous and it has the period $2N\pi$ in each of the variables u_ν. It satisfies the conditions $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$, $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. Hence $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ are homotopic. This completes the proof.

Lemma 23. *If a torus mapping $g_1(\boldsymbol{u}) = g_1(u_1, \cdots, u_m)$ is homotopic to a torus mapping $g_2(\boldsymbol{u}) = g_2(u_1, \cdots, u_{m-1})$, it is also homotopic to the torus mapping $g_1(u_1, \cdots, u_{m-1}, k)$ where k is an arbitrary constant.*

Proof. There exists a family $g(\boldsymbol{u}; v)$, $0 \leqq v \leqq 1$, of torus mappings satisfying $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$.

It follows that $g(u_1, \cdots, u_{m-1}, k; v)$ is a family of torus mappings, which proves that $g_1(u_1, \cdots, u_{m-1}, k)$ is homotopic to

$$g(u_1, \cdots, u_{m-1}, k; 1) = g_2(u_1, \cdots, u_m).$$

Theorem 11. *Let $f(t)$ be an almost periodic movement in a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let $\beta_1, \beta_2, \cdots$ be a basis of $f(t)$. There exists a finite subset $\beta_{n_1}, \cdots, \beta_{n_q}$ of the basis such that the following conditions are satisfied.*

1. *$f(t)$ is homotopic to a function $g(\beta_{n_1} t, \cdots, \beta_{n_q} t)$ where $g(u_1, \cdots, u_q)$ is continuous and has the period $2N\pi$ (N integral) in each variable.*

2. *If a subset B of the basis $\beta_1, \beta_2, \cdots$ is a basis for an almost periodic movement homotopic to $f(t)$, then B contains the numbers $\beta_{n_1}, \cdots, \beta_{n_q}$.*

Proof. According to Lemma 20 the movement $f(t)$ is homotopic to a diagonal movement $g_1(\beta_1 t, \cdots, \beta_m t)$ where $g_1(\boldsymbol{u})$ has a period $2N_1\pi$ in each variable. We cannot be certain that $g_1(\boldsymbol{u})$ has the period 2π, because the basis numbers $\beta_1, \cdots, \beta_m$ are given beforehand, but we know from Theorems 6 and 9 that $f(t)$ can be approximated to any given accuracy by a function $g^*\left(\frac{\beta_1}{N_1} t, \cdots, \frac{\beta_m}{N_1} t\right)$ where $g^*(\boldsymbol{u})$ has the period 2π in each variable. This justifies our use of Lemma 20. We shall call a variable u_ν inessential for $g_1(\boldsymbol{u})$ if $g_1(\boldsymbol{u})$ is homotopic to a torus mapping which is independent of u_ν. A variable u_μ is called essential for $g_1(\boldsymbol{u})$ if it is not inessential. Let $u_{n_1}, \cdots, u_{n_q}$ denote the essential variables. From Lemma 23 follows that $g_1(\boldsymbol{u})$ is homotopic to a function $g(u_{n_1}, \cdots, u_{n_q})$ depending only on the essential variables and with a period $2N\pi$ in each variable. By Lemma 22 this implies that $g_1(\boldsymbol{\beta} t)$ is homotopic to $g(\boldsymbol{\beta} t)$, hence, that $f(t)$ is homotopic to $g(\boldsymbol{\beta} t) = g(\beta_{n_1} t, \cdots, \beta_{n_q} t)$. If, on the other hand, $f(t)$ is homotopic to a movement with basis B, it is, according to Lemma 20, homotopic to a movement $g_2(\beta_{\nu_1} t, \cdots, \beta_{\nu_p} t)$ where $\beta_{\nu_1}, \cdots, \beta_{\nu_p}$ belong to B and $g_2(\boldsymbol{u}) = g_2(u_{\nu_1}, \cdots, u_{\nu_p})$ has a period $2N_2\pi$ in each variable. By Lemma 22 it follows that $g(u_{n_1}, \cdots, u_{n_q})$ is homotopic to

$g_2(u_{\nu_1}, \cdots, u_{\nu_p})$ and by Lemma 23 that $g(\boldsymbol{u})$ is homotopic to a function depending only on the variables u_{n_μ} occurring among the variables u_{ν_μ}. But the variables u_{n_μ} were essential. Hence every u_{n_μ} occurs among $u_{\nu_1}, \cdots, u_{\nu_p}$, i. e. every β_{n_μ} belongs to B. This completes the proof.

Lemma 24. *Let $g_1(\boldsymbol{u})$ be an m-dimensional torus mapping into $\mathfrak{M}$ and let $\beta_1, \cdots, \beta_m$ be rationally independent numbers. If the diagonal movement $f_1(t) = g_1(\boldsymbol{\beta} t)$ is homotopic to a movement $f_2(t)$ with the period p, and the numbers $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally independent, then $f_1(t)$ is homotopic to a constant.*

Proof. We choose the extra basis number $\beta_{m+1} = \frac{2\pi}{p}$. According to Lemma 13 $f_1(t)$ and $f_2(t)$ have $g_1(\boldsymbol{u})$ and $f_2\left(\frac{p}{2\pi} u_{m+1}\right)$ as spatial extensions. By Lemma 23 this implies that $g_1(\boldsymbol{u})$ is homotopic to a constant and the lemma follows from Lemma 22.

Theorem 12. *Let $g(\boldsymbol{u})$ be an m-dimensional torus mapping into a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let $\beta_1, \cdots, \beta_m$ be rationally independent numbers. The almost periodic movement $f(t) = g(\boldsymbol{\beta} t)$ is reducible if and only if the torus mapping $g(\boldsymbol{u})$ is reducible.*

Proof. If $f(t)$ is reducible, it is homotopic to a movement $f_1(t)$ with a period p. If $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally independent, it follows from Lemma 24 that $f(t)$ is homotopic to a constant. Hence, by Lemma 22, $g(\boldsymbol{u})$ is homotopic to a constant, i. e. reducible. If $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally dependent, we have a relation

$$n\frac{2\pi}{p} = n_1\beta_1 + \cdots + n_m\beta_m$$

where $n, n_1, \cdots, n_m$ are integers and $n \neq 0$. The numbers $\frac{\beta_1}{n}, \cdots, \frac{\beta_m}{n}$ form a finite integral basis (Definitions 10 and 11) of $f_1(t)$. According to the Theorems 7 and 8, the movement $f_1(t)$ has a spatial extension $g_1(\boldsymbol{u}) = g_1(u_1, \cdots, u_m)$ with the period 2π in each variable, and we have

$$f_1(t) = g_1\left(\frac{\beta_1}{n}t, \cdots, \frac{\beta_m}{n}t\right).$$

According to Lemma 22 the function $g(\boldsymbol{u})$ is homotopic to $g_1\left(\frac{u_1}{n}, \cdots, \frac{u_m}{n}\right)$. Since $g_1(\beta_1 t, \cdots, \beta_m t)$ has the period $\frac{n}{p}$, Lemma 21 implies that $g_1(\boldsymbol{u}) = h(n_1 u_1 + \cdots + n_m u_m)$ where $n_1, \cdots, n_m$ are integers and $h(u)$ has the period 2π. This proves that the torus mapping $g(\boldsymbol{u})$ is reducible.

If, on the other hand, $g(\boldsymbol{u})$ is reducible, it is homotopic to a torus mapping $h(\chi(\boldsymbol{u}))$ where $\chi(\boldsymbol{u})$ is a real-valued, continuous function with the property that $e^{i\chi(u)}$ has the period $2N\pi$, N integral, in each variable, while $h(\chi)$ is a continuous function with the period 2π.

Since $\chi(\boldsymbol{u})$ is the argument of the periodic function $e^{i\chi(u)}$, it has the form ([5])

$$\chi(\boldsymbol{u}) = \frac{n_1}{N}u_1 + \cdots + \frac{n_m}{N}u_m + \psi(\boldsymbol{u})$$

where $n_1, \cdots, n_m$ are integers, and $\psi(\boldsymbol{u})$ is a continuous function with the period $2N\pi$ in each variable. According to Lemma 22, $f(t)$ is homotopic to the almost periodic movement

$$h(\chi(\boldsymbol{\beta}t)) = h(\gamma t + \psi(\boldsymbol{\beta}t)) \tag{11}$$

where

$$\gamma = \frac{n_1\beta_1 + \cdots + n_m\beta_m}{N}.$$

The family

$$h(\gamma t + (1-v)\psi(\boldsymbol{\beta}t)),\ 0 \leqq v \leqq 1, \tag{12}$$

is obviously a uniformly continuous family of almost periodic movements. In fact, it is equal to $H(\boldsymbol{\beta}t; v)$ where

$$H(\boldsymbol{u}; v) = h\left(\frac{n_1}{N}u_1 + \cdots + \frac{n_m}{N}u_m + (1-v)\psi(\boldsymbol{u})\right)$$

is a continuous function with the period $2N\pi$ in each variable. The family (12) proves that the movement (11) is homotopic to

the function $h(\gamma t)$ which has the period $\frac{2\pi}{\gamma}$. This completes the proof of Theorem 12.

Theorem 13. *Every almost periodic movement in a complete and continuously locally arcwise connected space $\mathfrak{M}$ is reducible if and only if every torus mapping into $\mathfrak{M}$ is reducible.*

Proof. The theorem follows immediately from Lemma 20 and Theorem 12.

3. Almost Periodic Movements on a Sphere.

In this section we consider the m-dimensional sphere, i. e. the set of real vectors $\boldsymbol{x} = (x_0, \cdots, x_m)$ satisfying the condition

$$|\boldsymbol{x}|^2 = x_0^2 + \cdots + x_m^2 = 1.$$

We shall use the ordinary vector notations. If $\boldsymbol{y} = (y_0, \cdots, y_m)$ is another vector, we define $\boldsymbol{x} \pm \boldsymbol{y} = (x_0 \pm y_0, \cdots, x_m \pm y_m)$ and $\boldsymbol{xy} = x_0 y_0 + \cdots + x_m y_m$. If λ is a real number, we define further $\lambda \boldsymbol{x} = (\lambda x_0, \cdots, \lambda x_m)$. We consider the m-dimensional sphere as a metric space $\mathfrak{S}_m$ with the metric

$$[\boldsymbol{x}, \boldsymbol{y}] = |\boldsymbol{y} - \boldsymbol{x}| = \sqrt{(y_0 - x_0)^2 + \cdots + (y_m - x_m)^2}.$$

Obviously we have

Lemma 25. *The metric space $\mathfrak{S}_m$ is complete and continuously locally arcwise connected.*

Lemma 26. *There exists an irreducible torus mapping into $\mathfrak{S}_m$ if $m \geqq 2$.*

Proof. Let $\mathfrak{T}_m$ denote the parallellepiped $|u_\nu| \leqq \pi$, $\nu = 1, \cdots, m$. If we identify all boundary points of this parallellepiped with one another ([1] p. 64), it becomes an m-dimensional sphere and the identical mapping becomes a mapping of $\mathfrak{T}_m$ onto this sphere. This mapping can be extended to a continuous function with the period 2π in every u_ν, i. e. to a torus mapping. Since the mapping is one-to-one in the interior of D_m, its degree is ± 1. Consequently this mapping is irreducible.

Theorem 14. *There exists an irreducible almost periodic movement in $\mathfrak{S}_m$ if $m \geqq 2$.*

Proof. Immediate consequence of Theorem 13 and Lemma 26.

In the case $m = 2$ we shall give some examples of irreducible almost periodic movements by explicit expressions or by geometrical descriptions. In the proof of Lemma 26 we used a torus mapping which mapped the boundary of the parallellepiped $|u_\nu| \leqq \pi$, $\nu = 1, \cdots, m$, onto a single point of the sphere. In

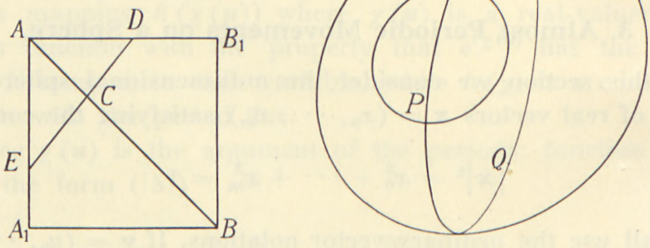

Fig. 1.

the case $m = 2$ we shall map the square $|u_1| \leqq \pi$, $|u_2| \leqq \pi$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ such that the boundary of the square is mapped onto one point of the sphere.

To accomplish this we map the diagonal AB (fig. 1) onto a great circle OPQ on the sphere such that A and B are mapped into O, and such that corresponding lengths are proportional. If γ is an irrational number, the square is covered by segments with the slope γ, as for instance DCE. We map the segment DCE into the sphere onto a circle perpendicular to OPQ. The images of D, C, and E are already chosen and each of the segments CD and CE is mapped onto a semicircle such that corresponding lengths are proportional. This yields a mapping with the desired properties. If we put $u_1 = t$, $u_2 = a + \gamma t$ where a is constant, we obtain a very simple irreducible almost periodic movement, which can be described in the following way: On a great circle (fig. 2) we choose a point O and an infinite sequence $\cdots, P_{-1}, P_0, P_1, \cdots$ of points such that the arcs $P_{\nu-1}P_\nu$ have the same length, and the ratio between this length and the circumference is irrational. The movement describes the circles with diameters $\cdots, OP_{-1}, OP_0, OP_1, \cdots$ in succession according to the indices.

The speed is constant on each semicircle, but it varies a little from one semicircle to another. However, it is not difficult to prove that the movement which runs through the circles with constant speed all the time is also almost periodic and irreducible.

The mapping of the square $|u_1| \leq \pi$, $|u_2| \leq \pi$ onto the sphere can also be carried out such that the side A_1B corresponds to

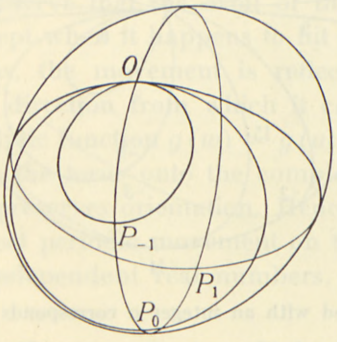

Fig. 2.

a great circle and the perpendiculars to A_1B correspond to small circles through a point O of the great circle and perpendicular to it. In this way we obtain the following movement

$$(13)\qquad \begin{cases} x_1 = \left| \sin \dfrac{\beta_1}{2} t \right| \sin \beta_2 t \\ x_2 = \sin \dfrac{\beta_1}{2} t \cos \dfrac{\beta_1}{2} t \, (1 - \cos \beta_2 t) \\ x_3 = 1 - \sin^2 \dfrac{\beta_1}{2} t \, (1 - \cos \beta_2 t) \end{cases}$$

where β_1 and β_2 are rationally independent. We have written $\frac{\beta_1}{2}$ instead of β_1, because (β_1, β_2) then is an integral basis. We shall omit the (not very interesting) proof of the formulas (13). Fig. 3 shows part of the projection on the (x_1, x_2)-plane of the orbit of the movement.

We can construct an irreducible mapping of the torus onto the sphere in a quite different way. We divide the square $|u_1| \leq \pi$,

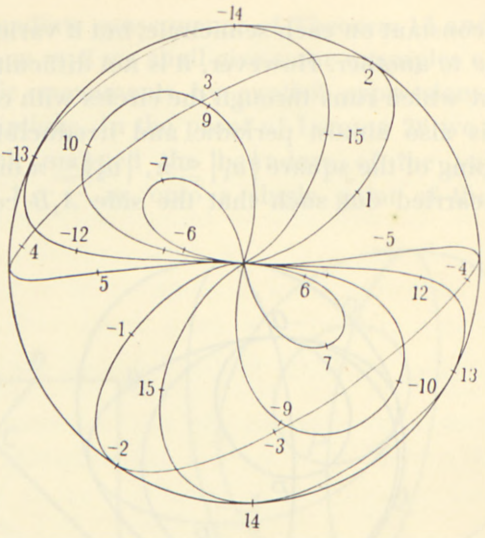

Fig. 3. The point marked with an integer n corresponds to $t = \frac{n\pi}{4}$. The values $n = -16, -8, 0, 8$, and 16 are in the center. The values $n = \pm 11$ are marked near the center.

$|u_2| \leqq \pi$ into $4pq$ congruent rectangles by $2p - 1$ lines parallel to one pair of its sides and $2q - 1$ lines parallel to the other pair. We fold the square along all these lines. Points that are brought together by the folding of the square are identified if

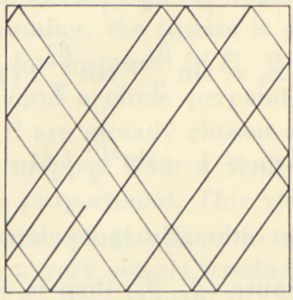

Fig. 4.

they are given the same colour when the $4pq$ rectangles are coloured alternately black and white like the squares on a chessboard. In this way we obtain a very simple irreducible mapping of the torus onto the sphere. In the simplest case, $p = q = 1$,

we have drawn the corresponding irreducible almost periodic movement in fig. 4. The "sphere" consists of two square sheets, joined along the edges. The movement follows the orbit of a billiard ball, started inside the square in a direction with irrational slope against the sides, and reflected from the sides according to the law of equal angles. By each reflexion the almost periodic movement passes into the other sheet. The movement has constant speed. We observe that the orbit of the movement never intersects itself except when it happens to hit one of the corners. When this happens, the movement is reflected into the same sheet and into the direction from which it came.

An arbitrary elliptic function $g(w) = g(u_1 + iu_2)$ with periods 2π and $2\pi i$ maps the torus onto the complex number sphere, and the mapping preserves orientation. Hence $g((\beta_1 + i\beta_2)t)$ is an irreducible almost periodic movement on the sphere if β_1 and β_2 are rationally independent real numbers.

4. Almost Periodic Movements in a Projective Space.

Let $\mathfrak{S}_m$ denote the m-dimensional sphere. The projective space $\mathfrak{P}_m$ with the elliptic non-Euclidean metric consists of all points $(\boldsymbol{x}, -\boldsymbol{x}) = (-\boldsymbol{x}, \boldsymbol{x})$ where $\boldsymbol{x}$ belongs to the sphere, the distance being defined by

$$[(\boldsymbol{x}, -\boldsymbol{x}), (\boldsymbol{y}, -\boldsymbol{y})] = \operatorname{Min}([\boldsymbol{x}, \boldsymbol{y}], [\boldsymbol{x}, -\boldsymbol{y}]).$$

It is well known that $\mathfrak{P}_m$ is a complete metric space. If

$$(\boldsymbol{x}, -\boldsymbol{x}) = (f(t), -f(t)) \tag{14}$$

is a continuous movement in $\mathfrak{P}_m$, we can obviously choose the representation (14) such that $f(t)$ is a continuous movement in $\mathfrak{S}_m$. If

$$(\boldsymbol{x}, -\boldsymbol{x}) = (f(t; v), -f(t; v))$$

is a uniformly continuous family of almost periodic movements in $\mathfrak{P}_m$, we can assume that $f(t; v)$ is continuous. If E denotes

the mapping $E(x) = -x$ of $\mathfrak{S}_m$ onto itself and τ is an ε-translation number of $(f(t; v), -f(t; v))$, we have

$$[f(t; v), E^{\nu} f(t + \tau; v)] \leqq \varepsilon$$

for some power ν. It follows that

$$[f(t; v), E^{2\nu} f(t + 2\tau; v)] \leqq 2\varepsilon.$$

But $E^{2\nu}$ is the identity. Hence $f(t; v)$ is a uniformly continuous family of almost periodic movements in $\mathfrak{S}_m$. If $f(t)$ is an irreducible almost periodic movement in $\mathfrak{S}_m$, it follows immediately that $(f(t), -f(t))$ is an irreducible almost periodic movement in $\mathfrak{P}_m$. We have thus proved the theorem

Theorem 15. *There exists an irreducible almost periodic movement in the projective space* $\mathfrak{P}_m$ *if* $m \geqq 2$.

5. Some Remarks Concerning the General Problem.

The question of homotopy between m-dimensional tori in a general metric space $\mathfrak{M}$ has never been investigated. If the investigations are restricted to tori which are topological products of a fixed $m-1$-dimensional torus in $\mathfrak{M}$ and a closed curve, it is possible, as R. H. Fox [9] pointed out, to introduce general torus homotopy groups analogous to the ordinary homotopy groups. The particular case, where the tori are "pinched", i. e. where the given $m-1$-dimensional torus reduces to a single point, was investigated in detail by R. H. Fox, who proved that the "pinched" tori homotopy groups can be built from the ordinary homotopy groups by means of Whitehead products. It follows in particular from his investigations that every "pinched" torus in $\mathfrak{M}$ is reducible if and only if every sphere in $\mathfrak{M}$ is trivial, i. e., homotopic to a point.

Let a and b denote two elements of the fundamental group of $\mathfrak{M}$ corresponding to a point P. They correspond to two curves A and B in $\mathfrak{M}$ starting and ending at P. If a and b are commutative, the curve composed of A, B, $-A$, and $-B$ can be contracted to a point and by this contraction it describes a

torus. If every torus in $\mathfrak{M}$ is reducible, we can conclude that some multiple of a is homotopic to some multiple of b. It follows that *every commutative subgroup of the fundamental group is cyclic.* This is a necessary but not a sufficient condition for reducibility of every torus in $\mathfrak{M}$. If, however, the further condition that every m-dimensional sphere in $\mathfrak{M}$ is trivial for $m > 1$, is satisfied, it is not hard to prove that every torus can be contracted to a "pinched" torus, and it follows then from the results of R. H. Fox that every torus in $\mathfrak{M}$ is reducible. Hence, we have the general theorem:

Every almost periodic movement in a complete and continuously arcwise locally connected space $\mathfrak{M}$ is reducible if the mth homotopy group of $\mathfrak{M}$ is trivial for $m > 1$, and every Abelian subgroup of the fundamental group of $\mathfrak{M}$ is cyclic.

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